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Quasi-Static Field Penetration Into a Two-Dimensional Rectangular Well in a Ground Plane

Lennart Marin
Dikewood Corporation, Westwood Research Branch
Los Angeles, California

Abstract

The penetration of quasi-static electric and magnetic fields into an indentation (a two-dimensional well) in a ground plane is studied with the aid of the Schwarz-Christoffel conformal mapping technique.

I. Introduction

For aerodynamical reasons many aircraft antennas are housed inside depressions in the aircraft exterior skin. The depressions can have many different shapes and will, of course, affect the antennas responses. It is the purpose of this note to study the effects of a depression on the antenna responses. The presence of these depressions will also somewhat modify the EMP coupling of the overall aircraft structure and this perturbation effect will be estimated.

In many cases the depression plays an important role in the way in which the antenna works within "in-band frequencies", e.g., a horn in the gigahertz region. For our present EMP calculations, however, all depressions are assumed electrically small, i.e., their linear dimensions are small in terms of important wavelengths. When calculating the EMP response of these antennas a quasi-static analysis is therefore sufficient.

The general problem of calculating the quasi-static electromagnetic fields inside an arbitrary-shaped depression is difficult. The present note is the first attempt addressing the depression problem and to make the problem more tractable we will solve a two-dimensional indentation problem. Some three-dimensional depressions of special shapes are now being studied and the results will be reported in a future Interaction Note. It should also be mentioned that the complementary problem, i.e., an ellipsoidal boss resting on a ground plane, has been investigated extensively in [1].

Both the electrostatic and the magnetostatic problems of a rectangular trough (or well) in a ground plane are formulated and solved in Section II with the aid of conformal mapping techniques. The conformal mapping is then used in Section III to calculate (1) the magnetic and electric field penetration into the indentation and (2) the electric and magnetic polarizability of the trough.

II. The Geometry and the Conformal Mapping

In this section we will use the method of conformal mapping to calculate the static electric and magnetic field penetration into an indentation in a ground plane. The shape of the indentation is that of an infinitely long trough with a rectangular cross section (see Fig. 1). The width of the trough is a and the depth is b. The incident fields are such that far away from the indentation the magnetic field $\underline{H}$ is homogeneous and parallel to the ground plane whereas the electric field $\underline{E}$ is taken to be homogeneous and normal to the ground plane. Since the problem under consideration is two dimensional there exist two potentials, $U(x,y)$ and $V(x,y)$ such that

$$\underline{H}(x,y) = H_o \nabla U(x,y), \qquad \nabla^2 U = 0$$

$$\underline{E}(x,y) = E_o \nabla V(x,y), \qquad \nabla^2 V = 0 \tag{1}$$

in the region inside and above the trough. The boundary conditions are

$$\frac{\partial U}{\partial n} = 0, \qquad V = 0 \tag{2}$$

on the ground plane and on the walls of the trough.

To find the functions $U(x,y)$ and $V(x,y)$ we make use of the following result from complex variable theory (a direct consequence of the Cauchy-Riemann relationships): since $U(x,y)$ and $V(x,y)$ are both harmonic functions of x and y there exists a complex-valued, holomorphic function $\phi(z)$, $z = x + iy$, such that

$$\phi(z) = U(x,y) + iV(x,y). \tag{3}$$

For large values of $|z|$ and $\mathrm{Im}\{z\} > 0$ we have asymptotically

$$\phi(z) \sim z. \tag{4}$$

The method of conformal mapping can then be invoked to find $\phi(z)$. The region of interest can be mapped into the upper half plane of the complex w-plane by using

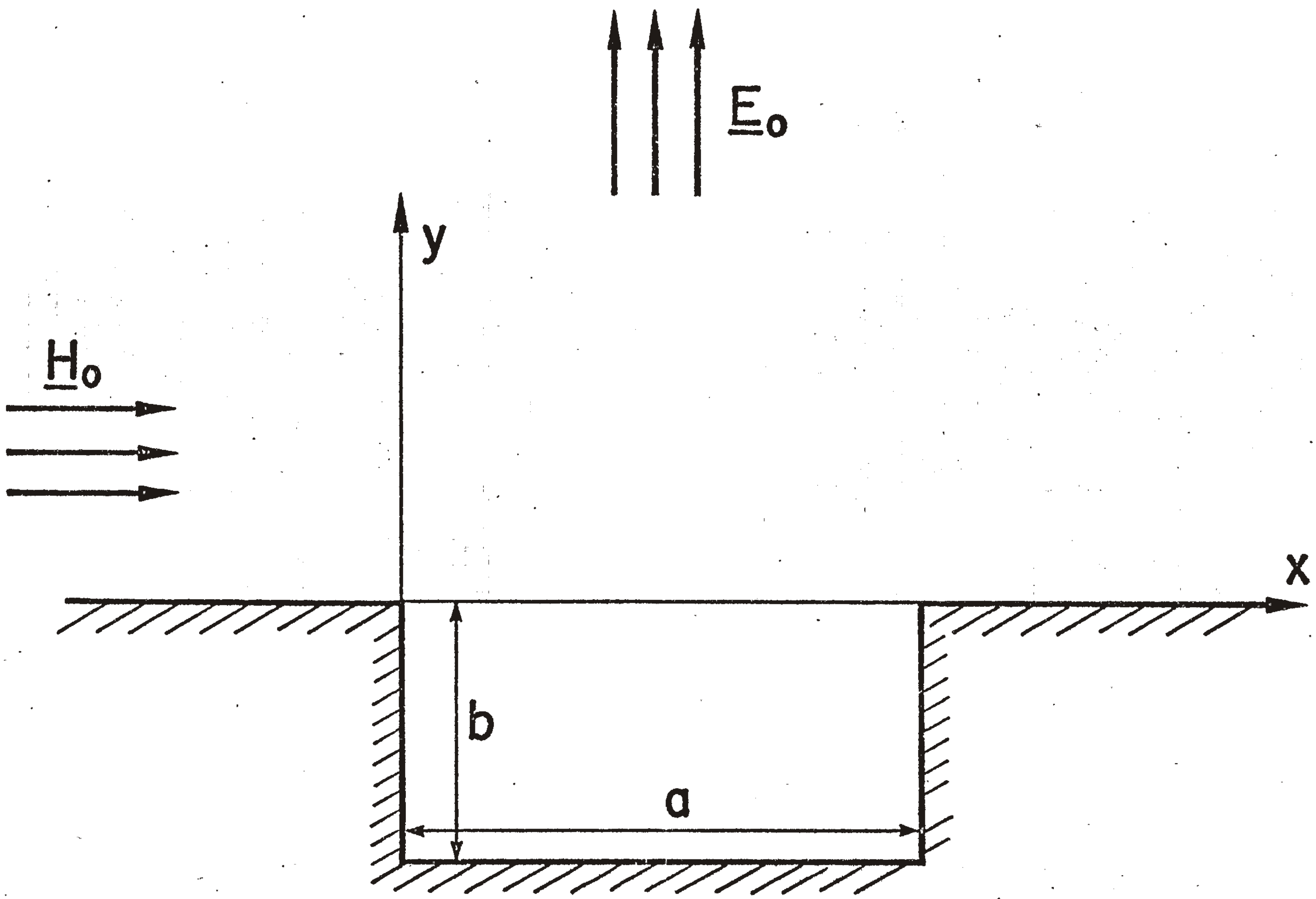

Figure 1. The geometry of the problem.

the Schwarz-Christoffel transformation [2,3],

$$\frac{dz}{dw} = C\sqrt{\frac{w^2-k^{-2}}{w^2-1}}\,, \qquad k \text{ real and } k \leq 1. \tag{5}$$

Integration of (5) gives

$$z = C\int_{w_1}^{w}\sqrt{\frac{\zeta^2-k^{-2}}{\zeta^2-1}}\,d\zeta + C_1. \tag{6}$$

This transformation, with suitable choices of k, w_1, C and C_1, maps the boundary of the region under consideration in the z-plane to the real axis in the w-plane, and vice versa. To determine the unknown constants we require that the following mapping among points holds true:

z-plane	w-plane
0	$-1/k$
$-ib$	-1
$a-ib$	1
a	$1/k$

This leads us to the following transformation

$$z = C\int_{-1/k}^{w}\sqrt{\frac{\zeta^2-k^{-2}}{\zeta^2-1}}\,d\zeta \tag{7}$$

where k and C are determined from the equations

$$-ib = C\int_{-1/k}^{-1}\sqrt{\frac{t^2-k^{-2}}{t^2-1}}\,dt$$

$$a = C\int_{-1/k}^{1/k}\sqrt{\frac{t^2-k^{-2}}{t^2-1}}\,dt \tag{8}$$

and since a, b are positive we define $\sqrt{t^2 - k^{-2}} = -i\sqrt{k^{-2} - t^2}$ for $|t| < 1/k$. The integrals in (6) can be expressed in terms of the elliptic integrals $E(k)$, $K(k)$[4]

$$\frac{a}{c} = 2E(k), \qquad \frac{b}{c} = K(k') - E(k') \tag{9}$$

where

$$c = C/k, \qquad k' = \sqrt{1 - k^2}.$$

The constant k can therefore be determined from the transcendental equation

$$\frac{b}{a} = \frac{K(k')-E(k')}{2E(k)} \tag{10}$$

and the constant c is then given by

$$c = \frac{a}{2E(k)} . \tag{11}$$

Let us now see how to use the transformation (7) to determine $\phi(z)$. Since the Laplacian equation and the boundary conditions remain invariant under the conformal mapping the potential $\psi(w) = \phi(z(w))$ is given by

$$\psi(w) = \psi_o w \tag{12}$$

where ψ_o is a constant. To determine the constant ψ_o we observe that for large values of w we have from (4) and (7)

$$z \sim Cw = kcw \tag{13}$$

and

$$\phi(z) \sim z \sim kcw. \tag{14}$$

Comparing (12) and (14) we see that

$$\psi_o = kc \tag{15}$$

so that

$$w = \psi(w)/(kc) = \phi(z)/(kc). \tag{16}$$

Thus, from (7) and (16) we have the following implicit equation for $\phi(z)$,

$$z = c \int_{-1/k}^{\phi/(kc)} \sqrt{\frac{k^2\zeta^2-1}{\zeta^2-1}}\, d\zeta. \tag{17}$$

In the next section we will use (17) to calculate the electric and magnetic fields inside the trough.

III. The Magnetic and Electric Fields

In this section we will use the conformal transformation derived in the previous section to calculate the electrostatic and magnetostatic fields inside the trough. From (1) we get

$$\underline{H} = H_o\left[\frac{\partial U}{\partial x}\hat{x} + \frac{\partial U}{\partial y}\hat{y}\right]$$

$$\underline{E} = E_o\left[\frac{\partial V}{\partial x}\hat{x} + \frac{\partial V}{\partial y}\hat{y}\right] = E_o\left[-\frac{\partial U}{\partial y}\hat{x} + \frac{\partial U}{\partial x}\hat{x}\right] \tag{18}$$

and from complex variable theory we have

$$\frac{\partial U}{\partial x} - i\frac{\partial U}{\partial y} = \frac{\partial V}{\partial y} + i\frac{\partial V}{\partial x} = \frac{d\phi}{dz} . \tag{19}$$

Differentiation of (17) gives us $d\phi/dz$,

$$\frac{d\phi}{dz} = \sqrt{\frac{\phi^2-k^2c^2}{\phi^2-c^2}} \tag{20}$$

and from (17) through (20) we can determine the electric and magnetic fields as a function of position.

Of special interest is the y-component of the electric field and the x-component of the magnetic field on the symmetry line $x = a/2$, $y > -b$, the reason being that inside the trough there may be a loop for picking up the H-field perpendicular to the symmetry line or a probe for picking up the E-field along the symmetry line. From the symmetry of the problem it follows that we can choose $U(a/2, y) = 0$ so that

$$\frac{H_x}{H_o} = \frac{E_y}{E_o} = \frac{\partial U}{\partial x} = \sqrt{\frac{V^2+k^2c^2}{V^2+c^2}} , \qquad x = \frac{a}{2} , \quad y > -b . \tag{21}$$

A. <u>The Field at the Bottom of the Trough</u>

Let us first calculate H_x and E_y at $x = a/2$, $y = -b$. From (7) it follows that the point $z = a/2 - ib$ conformally maps to $w = 0$. It then follows from (16) that $\phi(a/2 - ib) = 0$ so that $V(a/2, -b) = 0$, and (21) gives

$$\frac{\partial U}{\partial x}\left(\frac{a}{2},-b\right) = k. \tag{22}$$

The normalized fields H_x/H_o and E_y/E_o at $x = a/2$, $y = -b$ are graphed versus b/a in Fig. 2. For large values of b/a (a deep trough) we have the following approximate solution of (10),

$$k \sim 4e^{-1}\exp(-\pi b/a) \tag{23}$$

and this approximate form is plotted with dashed line in Fig. 2. For small values of b/a (a shallow trough) equation (10) has the following approximate solution

$$k' \sim \sqrt{\frac{8b}{\pi a}}, \qquad k \sim 1 - \frac{4b}{\pi a} \tag{24}$$

and this asymptotic form which has a more limited validity, is also graphed in Fig. 2. The asymptotic form (24) for k is valid with an accuracy of 1% for $b/a \geq 0.5$.

In passing let us note that the field inside the well can also be obtained by using the method of separation of variables which gives

$$U(x,y) = \sum_{n=1}^{\infty} A_n \cos\left[n\pi\left(\frac{x}{a} - \frac{1}{2}\right)\right]\cosh\left[n\pi\left(\frac{y}{a} + \frac{b}{a}\right)\right], \qquad 0 < x < a, \ -b < y < 0. \tag{25}$$

The results from the conformal mapping technique show that, when calculating the field at bottom of the trough, only the first term in the series (25) has to be included for the case where $b/a \geq 0.5$. The constant A_n can be determined from the potential distribution in the opening of the trough. To find this distribution one can either use the conformal mapping method of this note or an integral equation technique. To see how an assumed field distribution in the mouth of the well affects the field at the bottom of the well we assume that $\partial U/\partial y = 1$ for $y = 0$, $0 < x < a$. For the electric field this means $E_x = 0$ and $E_y = E_o$ and for the magnetic field it means $H_x = H_o$ and $H_y = 0$ for $y = 0$, $0 < x < a$. With this assumption we get $A_1 = A_1'$

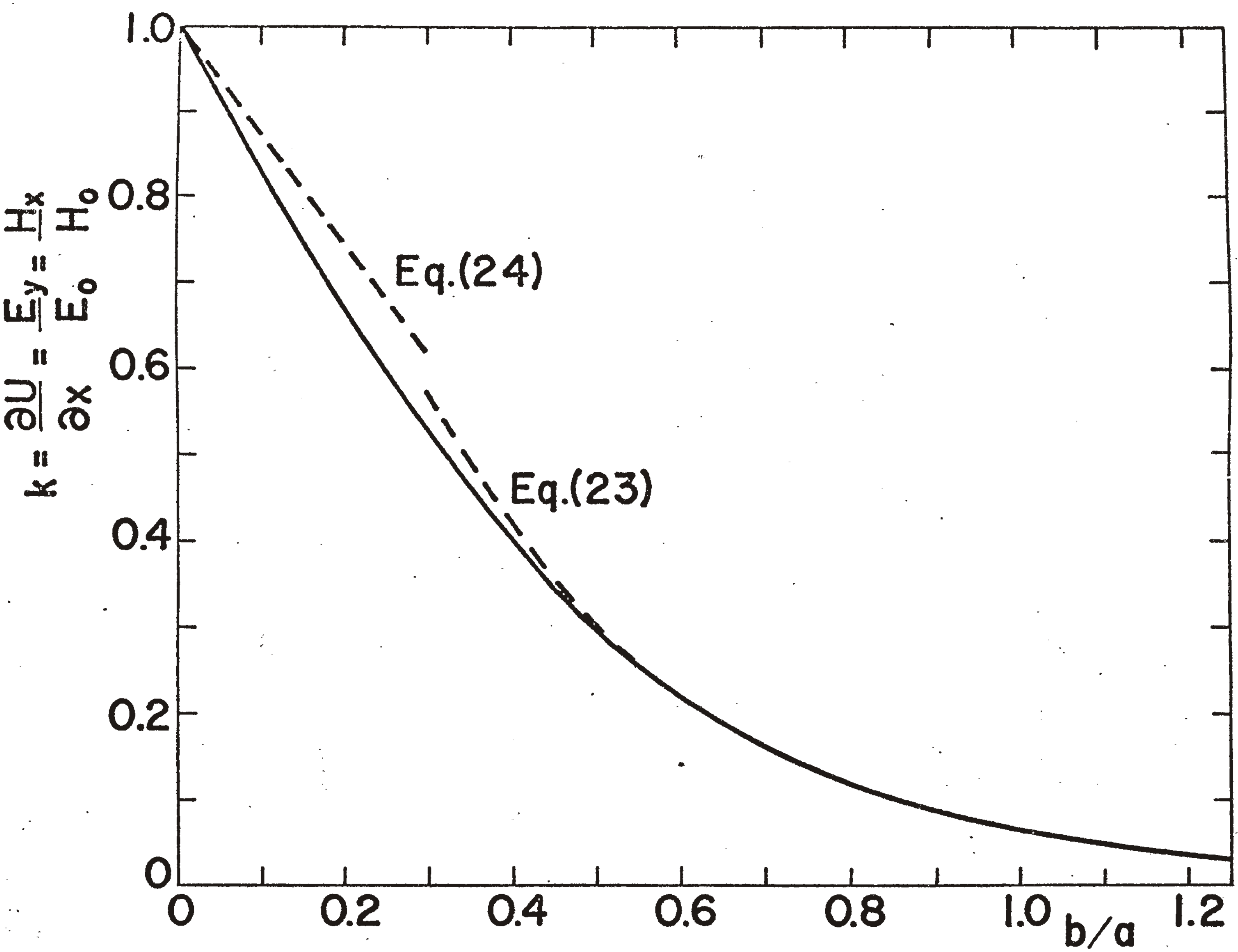

Figure 2. The normalized magnetic and electric fields at the bottom of the trough.

$$A_1' = 4a\pi^{-2}/\sinh(\pi b/a), \tag{26}$$

and for $b/a > 0.5$ we have approximately

$$A_1' \sim 8a\pi^{-2}\exp(-\pi b/a). \tag{27}$$

On the other hand, an exact analysis shows (see (23)) that

$$A_1 \sim 8a\pi^{-1}e^{-1}\exp(-\pi b/a), \qquad b/a > 0.5 \tag{28}$$

so that

$$\frac{A_1 - A_1'}{A_1} \sim \frac{\pi - e}{\pi} \approx 0.16, \qquad b/a > 0.5. \tag{29}$$

This result means that when estimating the field at the bottom of the trough the exact field distribution in the mouth is not very important. The attenuation of the field with the depth-to-width ratio is more important (the "Kamin-dämpfung"[5]). A result similar to (29) was obtained for a cylindrical hole[6].

B. The Field Along the Center Line

The x-component of the magnetic field and the y-component of the electric field along the center line can be calculated from the expression (c.f. (21));

$$\frac{\partial U}{\partial x} = \sqrt{\frac{V^2+k^2c^2}{V^2+c^2}}, \qquad x = \frac{a}{2}, \; y > -b \tag{21'}$$

where V is determined from (c.f. (17))

$$\frac{a}{2} + iy = c\int_{-1/k}^{iV/(kc)} \sqrt{\frac{k^2\zeta^2-1}{\zeta^2-1}}\, d\zeta \tag{30}$$

which can be reduced to the transcendental equation

$$2\left(\frac{y+b}{a}\right)E(k) = F(\varphi\backslash\alpha) - E(\varphi\backslash\alpha) + \frac{V}{c}\sqrt{\frac{V^2+c^2}{V^2+c^2k^2}} \tag{31}$$

where $\varphi = \arctan(V/kc)$, $\alpha = \arcsin k'$ and $F(\varphi\backslash\alpha)$, $E(\varphi\backslash\alpha)$ are the incomplete elliptic integrals[7]. For $b/a \geq 0.5$ equation (31) can be replaced by the following somewhat simpler but approximate equation which yields an accuracy of about 2% for V,

$$V = a\xi/\pi \tag{32}$$

where ξ is the solution of

$$\frac{\pi y}{a} = \sqrt{1 + \xi^2} + \text{arcsinh}\,\frac{1}{\xi}\,. \tag{33}$$

For a shallow hole (31) has the following approximate solution

$$V \sim y + b. \tag{34}$$

The normalized electric and magnetic fields obtained from (21) and (31) are graphed in Fig. 3 for $0 \leq y/b \leq 1$ with different values of a/b.

C. The Integral of the Field Along the Axis

Two common devices for picking up an electromagnetic signal are the loop and the stub. The output voltage from the loop is proportional to the magnetic flux passing through the loop whereas the output voltage from the stub is proportional to the voltage drop along the stub. The magnetic flux per unit length between the bottom of the trough and an arbitrary point on the center line is

$$\Phi(y) = \int_{-b}^{y} H_x dy = H_o \int_{-b}^{y} \frac{\partial U}{\partial x}\, dy = H_o V(\frac{a}{2}\,, y). \tag{35}$$

In the electric-field case the potential distribution along the center line is $E_o V(a/2, y)$. The quantity of interest is therefore $V(a/2, y)$ which can be found from (31). In Fig. 4 we graph the normalized quantity

$$v(y) = \frac{V(a/2,y)}{E_o a/2} = \frac{\Phi(y)}{H_o a/2} \tag{36}$$

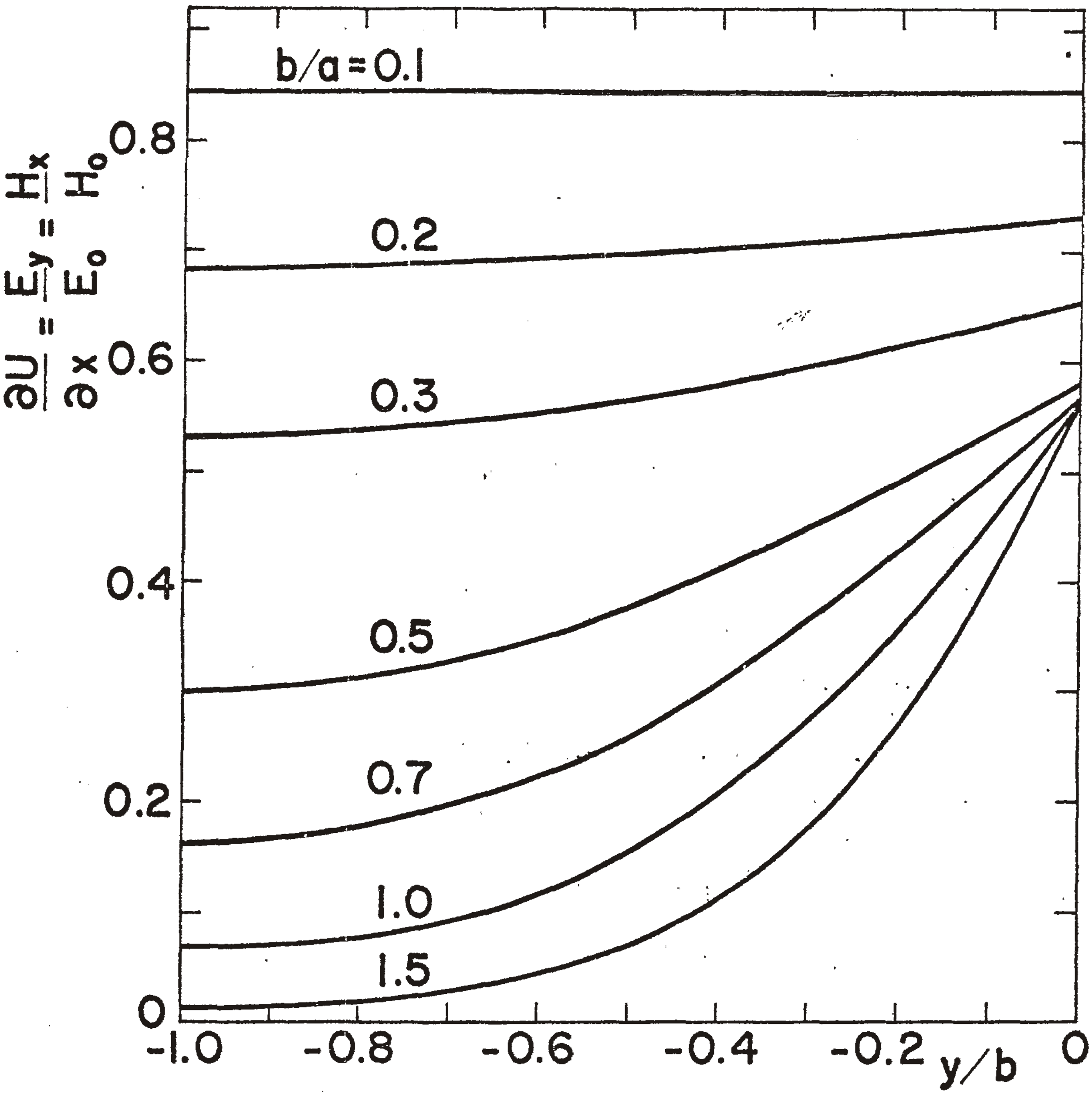

Figure 3. The normalized magnetic and electric fields along the center line of the trough.

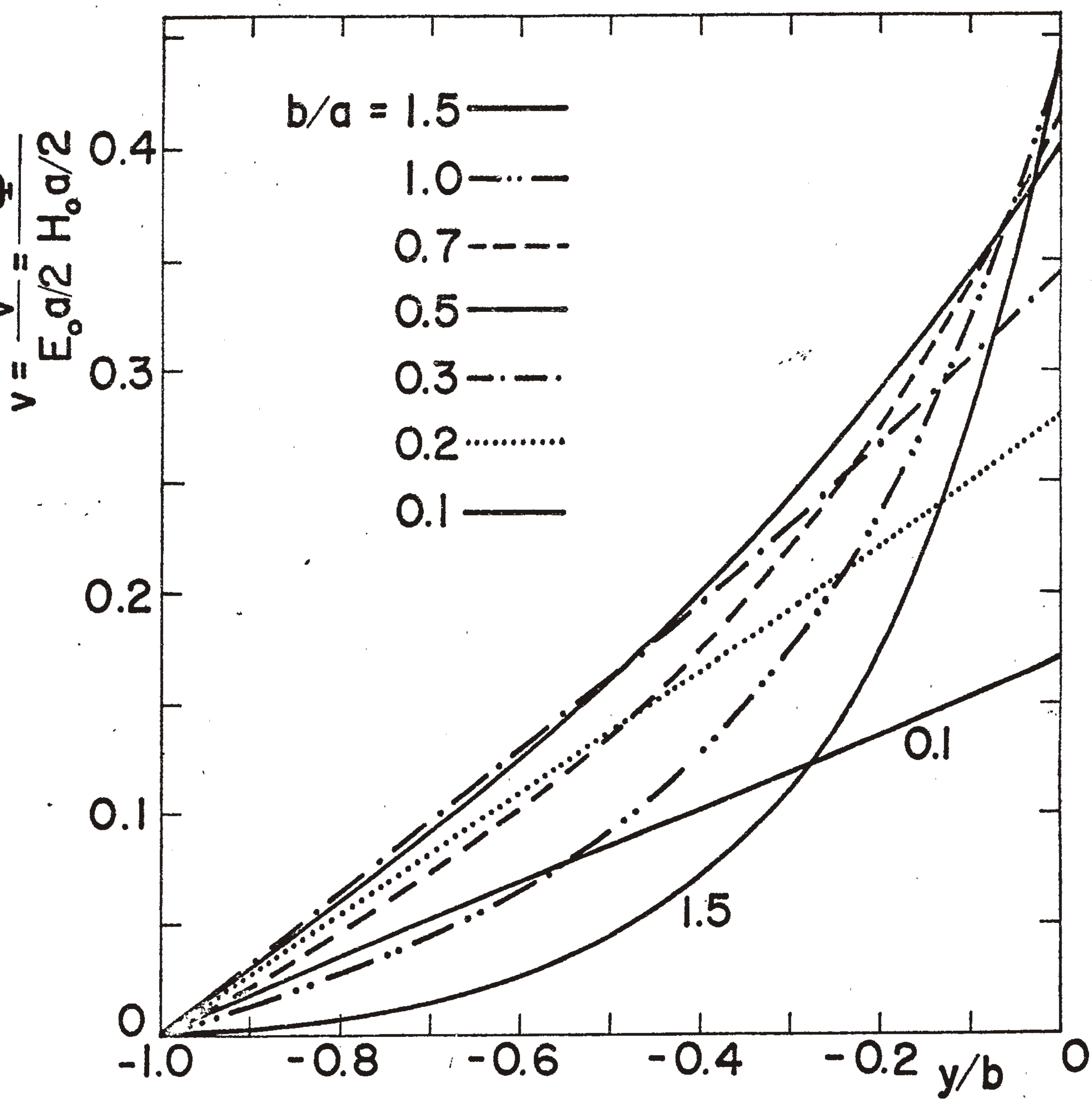

Figure 4a. The normalized magnetic flux and the normalized electrostatic potential along the center line of the trough.

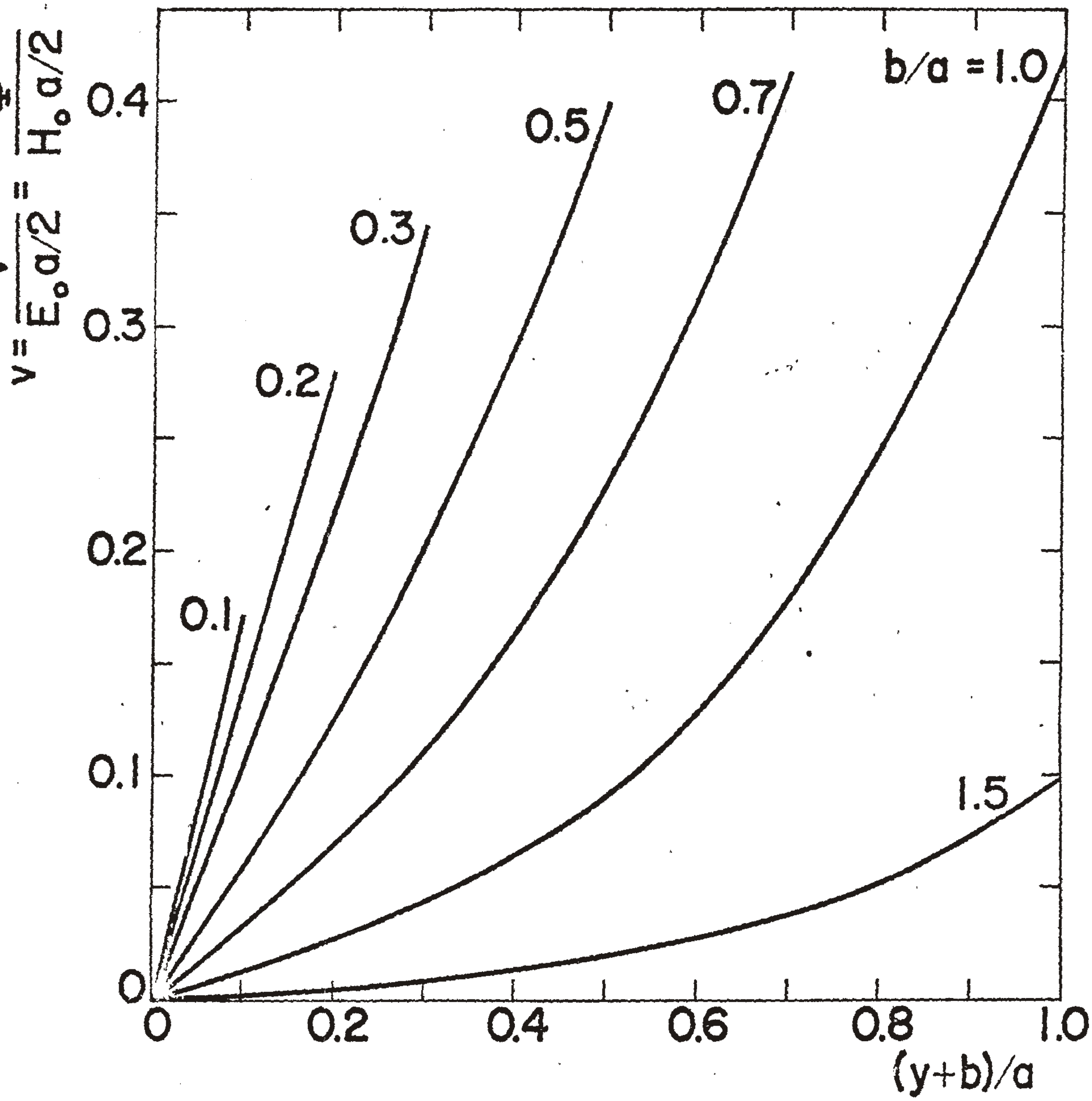

Figure 4b. The normalized magnetic flux and the normalized electrostatic potential along the center line of the trough.

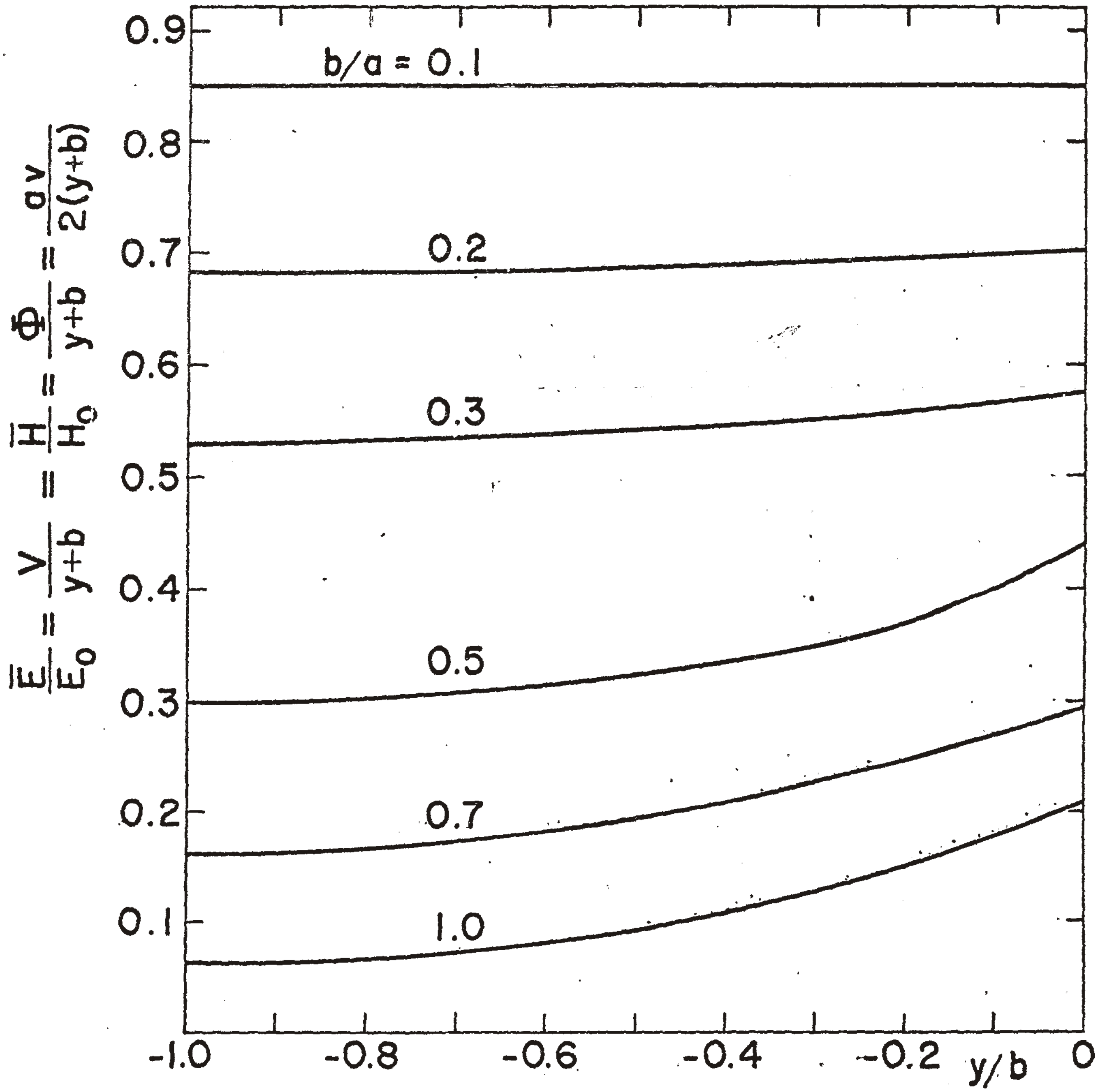

Figure 4c. The average magnetic and electric fields along the center line of the trough.

for $-b \leq y \leq 0$ with different values of b/a.

A stub antenna with its base located at the bottom of the trough is exposed to an averaged electric field $\overline{E}$ given by

$$\frac{\overline{E}}{E_o} = \frac{V(a/2,y)}{y+b} = \frac{av(y)}{2(y+b)} \tag{37}$$

whereas the averaged magnetic field $\overline{H}$ passing through a loop with one side at the bottom of the trough is

$$\frac{\overline{H}}{H_o} = \frac{\Phi(y)}{y+b} = \frac{av(y)}{2(y+b)} . \tag{38}$$

In Fig. 4c we have graphed $\overline{H}/H_o$ and $\overline{E}/E_o$ versus y/b for different values of b/a.

D. <u>The Total Magnetic and Electric Flux Penetrating into the Trough</u>

A quantity measuring the overall effect of the indentation on the magnetic field is the total magnetic flux per unit length Φ penetrating into the trough

$$\Phi = -H_o \int_0^{a/2} \frac{\partial U}{\partial y} \cdot dx = H_o \int_0^{a/2} \frac{\partial V}{\partial x} dx = H_o V(\frac{a}{2},0). \tag{39}$$

In Fig. 5 we plot the normalized quantity

$$\frac{\Phi}{H_o a/2} = v(0) \tag{40}$$

for different values of b/a. We also, in this figure, graph two approximate forms of $v(0)$; one is obtained from the solution of the approximate equation (33) and the other from the limiting form (34).

The total electric flux per unit length Ψ penetrating into the trough is

$$\Psi = -2E_o \int_0^{a/2} \frac{\partial V}{\partial y} dx = 2E_o \int_0^{a/2} \frac{\partial U}{\partial x} dx = 2E_o U(\frac{a}{2},0) = aE_o/E(k). \tag{41}$$

In Fig. 5 we graph the normalized quantity

$$\frac{\Psi}{aE_o} = \frac{1}{E(k)} \tag{42}$$

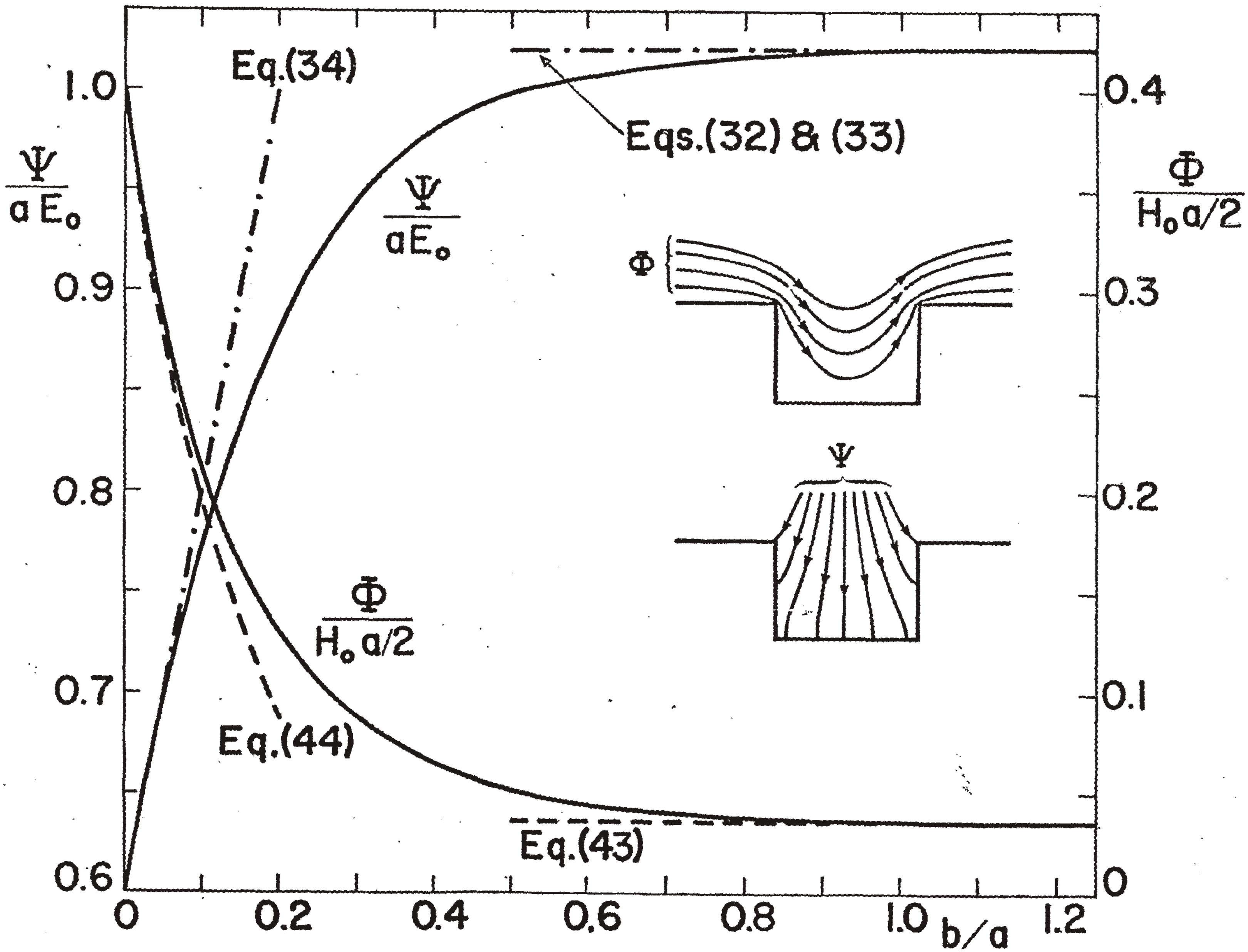

Figure 5. The total magnetic and electric flux penetrating into the trough.

for different values of b/a. We also include in this figure the approximate form of Ψ for a deep hole,

$$\frac{\Psi}{aE_o} \sim \frac{2}{\pi} . \tag{43}$$

For a shallow hole we can approximate the elliptic function to get

$$\frac{\Psi}{aE_o} \sim \frac{1}{2} [1 - \frac{2b}{\pi a} \ln \frac{2\pi a}{eb}] . \tag{44}$$

E. The Dipole Moments

Certain electromagnetic interaction problems of interest involve a structure whose surface can be approximated as a smooth one except for some surface anomalies such as indentations or protrusions. When solving these problems one can characterize the influences of these anamolies by their electric or magnetic dipole moments.

The electric line dipole moment $\underline{p}$ of the trough is[8]

$$\underline{p} = \varepsilon_o E_o \hat{y} \int_0^{a/2} V(x,0)dx = \varepsilon_o E_o \hat{y} \, \mathrm{Im}\left\{\int_0^{a/2} \phi(z)dz\right\} \tag{45}$$

and the magnetic line dipole moment $\underline{m}$ is[8]

$$\underline{m} = H_o \hat{x} \int_0^{a/2} x \frac{\partial U}{\partial y} dx = H_o \hat{x} \, \mathrm{Im}\left\{\int_0^{a/2} \phi(z)dz\right\} \tag{46}$$

where

$$\int_0^{a/2} \phi(z)dz = kc \int_0^{a/2} w dz = kc^2 \int_{-1/k}^{0} w\sqrt{\frac{k^2w^2-1}{w^2-1}} \, dw \tag{47}$$

from which it immediately follows that

$$\underline{p} = -\varepsilon_o E_o \alpha \hat{y}, \qquad \underline{m} = -H_o \alpha \hat{x}, \qquad \alpha = \frac{\pi(1-k^2)}{8E^2(k)} . \tag{48}$$

The quantity α can be interpreted as the normalized electric and magnetic dipole moment per unit length of the trough. For a deep hole we have approximately

$$\alpha \sim 1/(2\pi) \tag{49}$$

and for a shallow hole $(b \ll a)$ we have

$$\alpha \sim b/a. \tag{50}$$

In Fig. 6 we plot the normalized dipole moment α as a function of b/a together with the asymptotic forms (49) and (50).

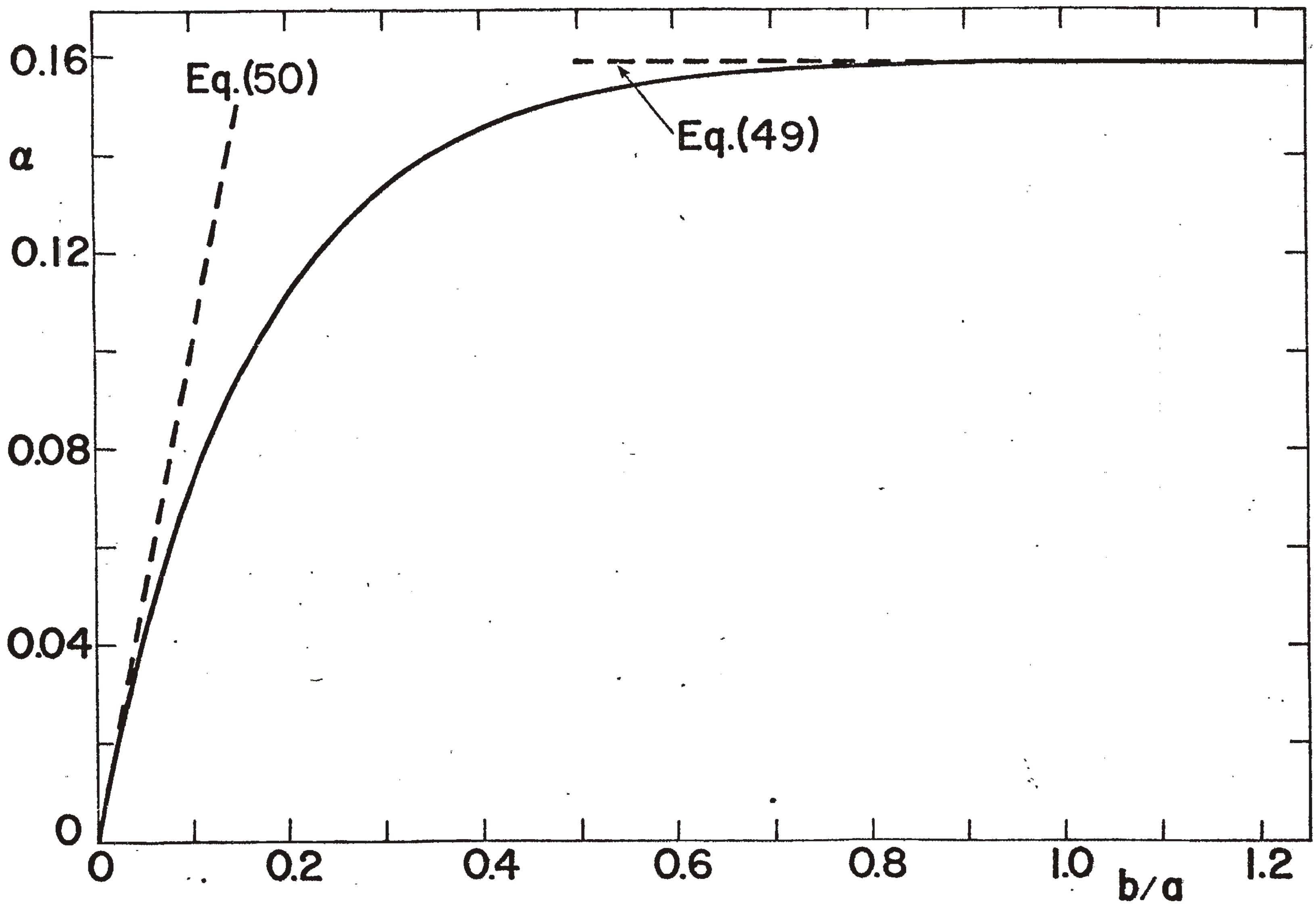

Figure 6. The normalized magnetic and electric line dipole moment.

Acknowledgment

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