ELECTROMAGNETIC SCATTERING BY APERTURES VII.

THE ANNULARLY SLOTTED SPHERE THEORY

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ABSTRACT: We present in this report the formal derivation of two sets of pairs of coupled integral equations. These arise in the scattering of an incident plane wave by a perfectly conducting sphere with an annular slot defined by means of two angles. The incident radiation is assumed for simplicity to be along the symmetry axis of the slotted sphere. Simple but quite remarkable relations between the sets of coupled equations are indicated. We further present the formal solution for the complementary geometry, namely the perfectly conducting spherical ribbon.

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INTRODUCTION

This paper is a continuation of a series of reports on investigations into the effect of apertures on the scattering of electromagnetic radiation by perfectly conducting shields. The previous papers in the series were concerned with a slot aperture in an infinite conducting plane,[1,2] or with an axial slot in an infinite conducting cylinder[3,4] or with a slot aperture in a general open, infinite, two dimensional perfect conductor.[5] We present here the first of a series of reports on scattering by finite conducting shields containing apertures. For this we have chosen to study the thin perfectly conducting sphere containing an annular aperture or, in other words, an aperture formed by cutting the sphere with two parallel planes and removing the enclosed conducting spherical surface area. Here we limit the discussion to the formal derivation of the coupled integral equations that must be solved. In papers that will appear subsequently we shall present the results of solution of these equations by means of numerical approximations and some further analytic studies we have made for the slotted sphere. It should be noted before continuing the discussion that we shall also consider the scattering by the spherical ribbon, which is merely the portion of the sphere we removed initially to form the annular aperture.

We consider here only the case of incidence along the symmetry-axis which simplifies the analyses somewhat. The geometry of the problem is illustrated in Figure 1. As we note, and develop in detail, as follows in this paper we shall approach the scattering problem via the Debye potentials, treating the scattering contribution from the aperture essentially as a perturbation contribution added to the standard solution for the closed conducting sphere. The incident linearly polarized plane wave is assumed to propagate in the negative Z-direction and for convenience we take the direction of polarization along the Y-axis. Thus for the incident fields we have

$$\vec{E}^i(\vec{r},t) = E^i(\vec{r},t)\vec{e}_y \tag{1}$$

$$\vec{H}^i(\vec{r},t) = H^i(\vec{r},t)\vec{e}_x \tag{2}$$

where of course the time dependence, $e^{i\omega t}$, is suppressed as usual and, consequently

$$E^i(\vec{r}) = E_o e^{ikr\cos\theta} = E_o \sum_{n=o}^{\infty} i^n(2n+1)j_n(kr)P_n(\cos\theta) \tag{3}$$

$$H^i(\vec{r}) = H_o e^{ikr\cos\theta} \tag{4}$$

and also

$$H_o = \sqrt{\frac{\varepsilon}{\mu}}\, E_o \tag{5}$$

We define the annular aperture in the sphere via the two colatitude angles θ_1 and θ_2 which specify respectively the upper and lower edges of the aperture. Alternatively these angles will define the spherical ribbon which we shall hereafter refer to as the complement to the slotted sphere.

At this point we note that if we take $\theta_1 \equiv 0$ we have the special case of a spherical shell with a capping hole or circular aperture in it. This problem has been considered by Chang and Senior[6] assuming arbitrary size holes and arbitrary frequency. Their method was essentially to approximate the scattered field by a finite number of terms from a general infinite series expression for the field. By applying the boundary conditions together with the method of least square error they computed the backscattering from spheres with large holes. Sancer and Varvatsis[7] also worked on this problem. They followed basically the same method but used the Debye potential approach. Enander[8] considered the case when the circular aperture is small. Their method of approximation is equivalent to one used by Morse and Feshbach[9] in a similar acoustic scattering problem by a sphere with a small hole. We shall compare our results for the special case of the sphere with a cap removed to those of all of these researchers in subsequent reports.

Further discussion of the results for the various asymptotic cases that arise in our geometry will also be deferred to the reports that shall be published shortly.

THE FIELDS AND DEBYE POTENTIALS FOR THE SLOTTED SPHERE

External to the spherical surface $r = a$ we shall assume the form of the Debye potentials to be simply the sum of the Debye potential in the absence of the aperture, which we denote by $u^o(r,\theta,\phi)$ or $v^o(r,\theta,\phi)$, and a potential due to the presence of the aperture. This latter potential function we denote by $u^s(r,\theta,\phi)$ or $v^s(r,\theta,\phi)$. We thus have outside the sphere

$$\left.\begin{aligned} u(r,\theta,\phi) &= u^o(r,\theta,\phi) + u^s(r,\theta,\phi) \\ v(r,\theta,\phi) &= v^o(r,\theta,\phi) + v^s(r,\theta,\phi) \end{aligned}\right\} \quad r > a \tag{6}$$

The Debye potentials u^o and v^o are well known.[10] They are precisely the following.

$$u^o(r,\theta,\phi) = \frac{E_o \sin\phi}{ik} \sum_{n=1}^{\infty} \frac{i^n(2n+1)}{n(n+1)} \left\{ j_n(kr) - h_n^{(2)}(kr) \left(\frac{d[rj_n(kr)]}{dr}\right)_{r=a} \Bigg/ \left(\frac{d[rh_n^{(2)}(kr)]}{dr}\right)_{r=a} \right\} P_n^1(\cos\theta) \tag{7a}$$

and

$$v^o(r,\theta,\phi) = \frac{E_o \cos\phi}{ik} \sqrt{\frac{\varepsilon}{\mu}} \sum_{n=1}^{\infty} \frac{i^n(2n+1)}{n(n+1)} \left\{ j_n(kr) - h_n^{(2)}(kr) \frac{j_n(ka)}{h_n^{(2)}(ka)} \right\} P_n^1(\cos\theta) \tag{7b}$$

Within the sphere the Debye potentials will be denoted by

$$\left.\begin{aligned} u(r,\theta,\phi) &= u^t(r,\theta,\phi) \\ v(r,\theta,\phi) &= v^t(r,\theta,\phi) \end{aligned}\right\} \quad r < a \tag{8}$$

It should be quite evident that the interior potentials arise solely from the presence of the aperture in the conducting sphere.

In the absence of the aperture the fields are expressed, of course, in terms of the corresponding Debye potentials of Equations (7) and are

$$\vec{E}^{o}(\vec{r}) = \vec{\nabla} \times \vec{\nabla} \times [\vec{r}\, u^{o}(\vec{r})] - i\, \omega\mu\, \vec{\nabla} \times [\vec{r}\, v^{o}(\vec{r})] \tag{9}$$

$$\vec{H}^{o}(\vec{r}) = i\, \omega\varepsilon\, \vec{\nabla} \times [\vec{r}\, u^{o}(\vec{r})] + \vec{\nabla} \times \vec{\nabla} \times [\vec{r}\, v^{o}(\vec{r})] \tag{10}$$

Outside the spherical surface r = a the fields are

$$\vec{E}(\vec{r}) = \vec{\nabla} \times \vec{\nabla} \times [\vec{r}\, u(r)] - i\, \omega\mu\, \vec{\nabla} \times [\vec{r}\, v(\vec{r})] \tag{11}$$

$$\vec{H}(\vec{r}) = i\, \omega\varepsilon\, \vec{\nabla} \times [\vec{r}\, u(\vec{r})] + \vec{\nabla} \times \vec{\nabla} \times [\vec{r}\, v(\vec{r})] \tag{12}$$

where the external Debye potentials are those of Equation (6).

Inside the sphere $r = \underline{a}$ the fields hereafter referred to as the transmitted fields $\vec{E}^{t}$ and $\vec{H}^{t}$ are similarly found from the internal Debye potentials of Equation (8) i.e.

$$\vec{E}^{t}(\vec{r}) = \vec{\nabla} \times \vec{\nabla} \times [\vec{r}\, u^{t}(\vec{r})] - i\, \omega\mu\, \vec{\nabla} \times [\vec{r}\, v^{t}(\vec{r})] \tag{13}$$

$$\vec{H}^{t}(\vec{r}) = i\, \omega\varepsilon\, \vec{\nabla} \times [\vec{r}\, u^{t}(\vec{r})] + \vec{\nabla} \times \vec{\nabla} \times [\vec{r}\, v^{t}(\vec{r})] \tag{14}$$

BOUNDARY CONDITIONS ON THE FIELDS AT THE SLOTTED SPHERE

Next we consider the boundary conditions at the spherical surface r = a. Clearly we must still have the normal component of the magnetic field $\vec{H}^o$ and the tangential component of $\vec{E}^o$ identically zero everywhere over the sphere r = a, i.e.

$$H_r^o(a,\theta,\phi) = E_\theta^o(a,\theta,\phi) = E_\phi^o(a,\theta,\phi) \equiv 0 \qquad \text{for all } \theta \text{ and } \phi \tag{15}$$

Now the normal component of the total external magnetic field and the tangential component of the total external electric field go identically to zero everywhere on sphere r = a except over the aperture. Thus

$$H_r(a,\theta,\phi) = E_\theta(a,\theta,\phi) = E_\phi(a,\theta,\phi) \equiv 0 \qquad \begin{cases} \text{for } \theta_2 < \theta < \theta_1 \\ 0 \leq \phi \leq 2\pi \end{cases} \tag{16}$$

The same requirement holds for the transmitted fields:

$$H_r^t(a,\theta,\phi) = E_\theta^t(a,\theta,\phi) = E_\phi^t(a,\theta,\phi) \equiv 0 \qquad \begin{cases} \theta_2 < \theta < \theta_1 \\ 0 \leq \phi \leq 2\pi \end{cases} \tag{17}$$

In addition to the boundary conditions at the conductor itself we further have the requirement at r = a that all field components must be continuous across the aperture, i.e.

$$\begin{aligned} H_{r,\theta,\phi}(a,\theta,\phi) &= H^t_{r,\theta,\phi}(a,\theta,\phi) \\ E_{r,\theta,\phi}(a,\theta,\phi) &= E^t_{r,\theta,\phi}(a,\theta,\phi) \end{aligned} \qquad \begin{cases} 0 \leq \theta_1 < \theta < \theta_2 \leq \pi \\ 0 \leq \phi \leq 2\pi \end{cases} \tag{18}$$

The fields exterior to the sphere r = a must satisfy the radiation condition as $r \to \infty$. The potentials $u^o(\vec{r})$ and $v^o(\vec{r})$ generate fields that satisfy this requirement. Symmetry conditions require that the polar and azimuthal dependence for u^s and v^s should be the same as for u^o and v^o respectively. We therefore assume the Debye potentials to be of the form

$$u^s(r,\theta,\phi) = \frac{E_o \sin\phi}{ik} \sum_{n=1}^{\infty} a_n^s h_n^{(2)}(kr)P_n^1(\cos\theta) \Bigg/ \left(\frac{d[r\, h_n^{(2)}(kr)]}{dr}\right)_{r=a} \tag{19}$$

and

$$v^s(r,\theta,\phi) = \frac{E_o \cos\phi}{ik} \sqrt{\frac{\varepsilon}{\mu}} \sum_{n=1}^{\infty} b_n^s P_n^1(\cos\theta) \frac{h_n^{(2)}(kr)}{h_n^{(2)}(ka)} \tag{20}$$

The expansion coefficients a_n^s and b_n^s will be determined by satisfying the boundary conditions at $r = a$.

For the interior fields we note first that the same symmetry requirements on u^t and v^t will establish their dependence on θ and ϕ. Next from the requirement that the interior fields be well behaved at the origin we take the interior Debye potentials to be of the form

$$u^t(r,\theta,\phi) = \frac{E \sin\phi}{ik} \sum_{n=1}^{\infty} a_n^t j_n(kr)P_n^1(\cos\theta) \Bigg/ \left(\frac{d[rj_n(kr)]}{dr}\right)_{r=a} \tag{21}$$

and

$$v^t(r,\theta,\phi) = \frac{E_o \cos\theta}{ik} \sqrt{\frac{\varepsilon}{\mu}} \sum_{n=1}^{\infty} b_n^t P_n^1(\cos\theta) \frac{j_n(kr)}{j_n(ka)} \tag{22}$$

where the interior expansion coefficients a_n^t and b_n^t will be determined by the boundary conditions at the spherical surface $r = a$.

As we shall see below it is convenient to introduce some new notation for the normal component of the magnetic field and the tangential components of the electric field at the aperture in the boundary sphere $r = a$. Let us then denote the radial magnetic aperture field as

$$\mathcal{H}_r(\theta,\phi) \qquad \text{for } \theta_1 < \theta < \theta_2, \quad 0 \leq \phi \leq 2\pi$$

and the θ and ϕ components of the electric aperture field as respectively

$$\mathcal{E}_\theta(\theta,\phi) \text{ and } \mathcal{E}_\phi(\theta,\phi) \qquad \text{for } \theta_1 < \theta < \theta_2, \quad 0 \leq \phi \leq 2\pi$$

Now let us consider the radial component of the magnetic field at $r = a$. We have for the exterior region as $r \to a$

$$H_r^s(a,\theta,\phi) = \begin{cases} \mathcal{H}_r(\theta,\phi) & 0 \leq \theta_1 < \theta < \theta_2 \leq \pi \\ 0 & \theta_1 > \theta > \theta_2 \end{cases} \; 0 \leq \phi \leq 2\pi \tag{23}$$

and similarly for the interior region as $r \to a$

$$H_r^t(a,\theta,\phi) \doteq \begin{cases} \mathcal{H}_r(\theta,\phi) & 0 \leq \theta_1 < \theta < \theta_2 \leq \pi \\ 0 & \theta_1 > \theta > \theta_2 \end{cases} \; 0 \leq \phi \leq 2\pi \tag{24}$$

Thus we observe

$$H_r^s(a,\theta,\phi) \equiv H_r^t(a,\theta,\phi) \qquad \text{for all } \theta \text{ and } \phi . \tag{25}$$

We then obtain from the radial components of Equations (12) and (14), and Equation (25)

$$b_n^s = b_n^t \equiv b_n \qquad n = 1, 2, 3, \ldots \tag{26}$$

where we drop the superscript for convenience.

Next we consider the θ-component of the electric field at r = a. The exterior θ-component of the electric field as r → a is given by

$$E_\theta^s(a,\theta,\phi) = \begin{cases} E_\theta(\theta,\phi) & 0 \le \theta_1 < \theta < \theta_2 \le \pi \\ 0 & \theta_2 < \theta < \theta_1 \end{cases} \quad 0 \le \phi \le 2\pi \tag{27}$$

similarly for the interior electric field as r → a

$$E_\theta^t(a,\theta,\phi) = \begin{cases} E_\theta(\theta,\phi) & 0 \le \theta_1 < \theta < \theta_2 < \pi \\ 0 & \theta_2 < \theta < \theta_1 \end{cases} \quad 0 \le \phi \le 2\pi \tag{28}$$

Since

$$E_\theta^s(a,\theta,\phi) = E_\theta^t(a,\theta,\phi) \tag{29}$$

we find from the θ-component of Equation (11)

$$E_\theta^s(a,\theta,\phi) = \frac{E_o \sin\phi}{ika} \sum_{n=1}^{\infty} a_n^s \frac{dP_n^1(\cos\theta)}{d\theta} + E_o \sin\phi \sum_{n=1}^{\infty} b_n^s \frac{P_n^1(\cos\theta)}{\sin\theta} \tag{30}$$

and the θ-component of the interior electric field as given by Eq. (13)

$$E_\theta^t(a,\theta,\phi) = \frac{E_o \sin\phi}{ika} \sum_{n=1}^{\infty} a_n^t \frac{dP_n^1(\cos\theta)}{d\theta}$$

$$+ E_o \sin\phi \sum_{n=1}^{\infty} b_n^t \frac{P_n^1(\cos\theta)}{\sin\theta} \qquad (31)$$

Using Equation (26) we find then that

$$a_n^t = a_n^s \equiv a_n \qquad n = 1, 2, 3, \ldots \qquad (32)$$

where again for convenience we have dropped the superscript. The results embodied in Equations (26) and (32) are consistent over the aperture with

$$E_\phi^s(a,\theta,\phi) = E_\phi^t(a,\theta,\phi) \equiv E_\phi(a,\theta,\phi) = \begin{cases} E_\theta(\theta,\phi) & \theta_1 < \theta < \theta_2 \\ 0 & \theta_2 < \theta < \theta_1 \end{cases} \quad 0 \leq \phi \leq 2\pi \qquad (33)$$

where at r = a we have more explicitly

$$E_\phi(a,\theta,\phi) = \frac{E_o \cos\phi}{ika} \sum_{n=1}^{\infty} a_n \frac{P_n^1(\cos\theta)}{\sin\theta} + E_o \cos\phi \sum_{n=1}^{\infty} b_n \frac{dP_n^1(\cos\theta)}{d\theta} \qquad (34)$$

DETERMINATION OF THE EXPANSION COEFFICIENTS a_n AND b_n

At this point all that remains essentially is to determine the two infinite sets of expansion coefficients, i.e. the a_n and b_n for n = 1, 2, ... These unknowns can be expressed in terms of integrals of the theta and phi components of the aperture electric field. We now proceed to do this. First let us construct the following integrations from Equations (30) and (34)

$$\int_o^\pi d\theta\, E_\theta(a,\theta,\phi)\frac{dP_m^1(\cos\theta)}{d\theta}\sin\theta = \frac{E_o \sin\phi}{ika}\sum_{n=1}^{\infty} a_n \int_o^\pi d\theta \frac{dP_m^1(\cos\theta)}{d\theta}\frac{dP_n^1(\cos\theta)}{d\theta}\sin\theta$$

$$+ E_o \sin\phi \sum_{n=1}^{\infty} bn \int_o^\pi d\theta\, P_n^1(\cos\theta)\frac{dP_m^1(\cos\theta)}{d\theta} \tag{35}$$

and

$$\int_o^\pi d\theta\, E_\phi(a,\theta,\phi)P_m^1(\cos\theta) = \frac{E_o \cos\phi}{ika}\sum_{n=1}^{\infty} a_n \int_o^\pi d\theta \sin\theta \frac{P_m^1(\cos\theta)P_n^1(\cos\theta)}{\sin^2\theta}$$

$$+ E_o \cos\phi \sum_{n=1}^{\infty} b_n \int_o^\pi P_m^1(\cos\theta)\frac{dP_n^1(\cos\theta)}{d\theta} \tag{36}$$

Clearly Equations (35) and (36) render the ϕ-dependence of the aperture fields $\mathcal{E}_\theta$ and $\mathcal{E}_\phi$ quite transparent. Thus we can write

$$\mathcal{E}_\theta(\theta,\phi) \equiv E_o f_\theta(\theta) \quad \sin\phi \qquad \theta_1 < \theta < \theta_2, \quad 0 \le \phi \le 2\pi \tag{37}$$

$$\mathcal{E}_\phi(\theta,\phi) \equiv E_o f_\phi(\theta) \quad \cos\phi \tag{38}$$

With the aid of Equations (37) and (38) we can remove the ϕ-dependence in Equations (35) and (36). Adding the resulting expressions we obtain

$$\int_{\theta_1}^{\theta_2} d\theta \, \sin\theta \, f_\theta(\theta) \frac{dP_m^1(\cos\theta)}{d\theta} + \int_{\theta_1}^{\theta_2} d\theta \, f_\phi(\theta) P_m^1(\cos\theta) =$$

$$[1/(ika)] \sum_{n=1}^{\infty} a_n \int_0^\pi d\theta \, \sin\theta \left[\frac{dP_m^1(\cos\theta)}{d\theta} \frac{dP_n^1(\cos\theta)}{d\theta} + \frac{P_m^1(\cos\theta) P_n^1(\cos\theta)}{\sin^2\theta} \right]$$

$$+ \sum_{n=1}^{\infty} b_n \int_0^\pi d\theta \left[P_m^1(\cos\theta) \frac{dP_n^1(\cos\theta)}{d\theta} + P_n^1(\cos\theta) \frac{dP_m^1(\cos\theta)}{d\theta} \right] \tag{39}$$

Since $P_m^1(\cos\pi) = P_m^1(\cos 0) = 0$ the integral in the second series on the right hand side vanishes for all n. Furthermore, we can use the orthogonality relation[11]

$$\int_0^\pi d\theta \, \sin\theta \left[\frac{dP_m^1(\cos\theta)}{d\theta} \frac{dP_n^1(\cos\theta)}{d\theta} + \frac{P_m^1(\cos\theta) P_n^1(\cos\theta)}{\sin^2\theta} \right] = \frac{2m(m+1)(m+1)!}{(2m+1)(m-1)!} \delta_{nm} \tag{40}$$

Using these results we obtain from Equation (39)

$$a_m \frac{1}{(ika)} \frac{2m(m+1)(m+1)!}{(2m+1)(m-1)!} = \int_{\theta_1}^{\theta_2} d\theta \, \sin\theta \, f_\theta(\theta) \frac{dP_m^1(\cos\theta)}{d\theta} + \int_{\theta_1}^{\theta_2} d\theta \, f_\theta(\theta) P_m^1(\cos\theta) \tag{41}$$

Now just as we did for Equations (35) and (36) we can also form the integrations

$$\int_0^\pi d\theta\ E_\theta(a,\theta,\phi)P_m^1(\cos\theta) = \frac{E_o\ \sin\phi}{ika}\sum_{n=1}^\infty a_n \int_0^\pi d\theta\ P_m^1(\cos\theta)\ \frac{dP_n^1(\cos\theta)}{d\theta} +$$

$$+\ E_o\ \sin\phi \sum_{n=1}^\infty b_n \int_0^\pi d\theta\ \sin\theta\ \frac{P_m^1(\cos\theta)P_n^1(\cos\theta)}{\sin^2\theta} \qquad (42a)$$

and

$$\int_0^\pi d\theta\ \sin\theta\ E_\phi(a,\theta,\phi)\ \frac{dP_m^1(\cos\theta)}{d\theta} = \frac{E_o\ \cos\phi}{ika}\sum_{n=1}^\infty a_n \int_0^\pi d\theta\ P_n^1(\cos\theta)\frac{dP_m^1(\cos\theta)}{d\theta}$$

$$+\ E_o\ \cos\phi \sum_{n=1}^\infty b_n \int_0^\pi d\theta\ \sin\theta\ \frac{dP_m^1(\cos\theta)}{d\theta}\ \frac{dP_n^1(\cos\theta)}{d\theta} \qquad (42b)$$

Using Equations (37) and (38) to suppress the phi dependence and proceeding as before we obtain

$$\int_0^\pi d\theta\ f_\theta(\theta)P_m^1(\cos\theta) + \int_0^\pi d\theta\ \sin\theta\ f_\phi(\theta)\frac{dP_m^1(\cos\theta)}{d\theta}$$

$$= \frac{1}{ika}\sum_{n=1}^\infty a_n \int_0^\pi d\theta\ \frac{d}{d\theta}[P_m^1(\cos\theta)P_n^1(\cos\theta)]$$

$$+ \sum_{n=1}^\infty b_n \int_0^\pi d\theta\ \sin\theta\left[\frac{dP_m^1(\cos\theta)}{d\theta}\ \frac{dP_n^1(\cos\theta)}{d\theta} + \frac{P_m^1(\cos\theta)P_n^1(\cos\theta)}{\sin^2\theta}\right]$$

(43)

as before

$$\int_0^{\pi} d\theta \frac{d}{d\theta} P_m^1 P_n^1 \equiv 0$$

Then using the orthogonality relation Equation (40) we get the remaining set of expansion coefficients

$$b_m \frac{2m(m+1)(m+1)!}{(2m+1)(m-1)!} = \int_{\theta_1}^{\theta_2} d\theta\, f_\theta(\theta) P_m^1(\cos\theta) + \int_{\theta_1}^{\theta_2} d\theta\, \sin\theta\, f_\phi(\theta) \frac{dP_m^1(\cos\theta)}{d\theta} \tag{44}$$

Since examination of Equations (41) and (44) reveals the fact that the expansion coefficients are given in terms of integrations of the tangential electric fields over the aperture we see that our problem becomes one of finding these aperture fields explicitly.

We shall present in a subsequent report methods of approximately solving for these fields and consequently solving in turn for the fields everywhere. Integral equations for the tangential component of the aperture electric field will next be derived.

INTEGRAL EQUATIONS FOR THE APERTURE FIELDS $\mathcal{E}_\theta$ AND $\mathcal{E}_\phi$

At the aperture in the boundary sphere r = a we must have the tangential components of the magnetic field continuous across. Thus we have for the θ-component

$$H_\theta^t(a,\theta,\phi) = H_\theta^o(a,\theta,\phi) + H_\theta^s(a,\theta,\phi); \qquad \theta_1 < \theta < \theta_2,\ 0 \leq \phi \leq 2\pi$$

or rearranging to a more useful form

$$H_\theta^s(a,\theta,\phi) - H_\theta^t(a,\theta,\phi) = -H_\theta^o(a,\theta,\phi)$$

In the absence of the aperture the surface current on the whole sphere is given by the relation

$$\vec{K}^o(a,\theta,\phi) = \vec{e}_r(\theta,\phi) \times \vec{H}^o(a,\theta,\phi) \qquad \text{for all } \theta \text{ and } \phi$$

Extracting the φ-component of this equation:

$$K_\phi^o(a,\theta,\phi) = \vec{e}_\phi(\phi)\cdot\vec{e}_r(\theta,\phi) \times \vec{H}^o(a,\theta,\phi)$$

$$= \vec{e}_\phi(\phi)\cdot[\vec{e}_r(\theta,\phi) \times \vec{e}_\theta(\theta,\phi)]H_\theta^o(a,\theta,\phi)$$

$$K_\phi^o(a,\theta,\phi) = H_\theta^o(a,\theta,\phi) \tag{46}$$

From this result we see that continuity of the θ-component of the magnetic field across the aperture can be expressed in terms of the φ-component of the current density in the absence of the aperture. Thus

$$H_\theta^s(a,\theta,\phi) - H_\theta^t(a,\theta,\phi) = -K_\phi^o(a,\theta,\phi); \quad \theta_1 < \theta < \theta_2,\ \theta < \phi < 2\pi \tag{47}$$

The magnetic field components in Equation (47) can be found by substituting from Equations (19), (20), (21) and (22) into Equations (12) and (14). Doing this we obtain finally at r = a (with the help of Equations (26) and (32))

$$H_\theta^s(a,\theta,\phi) = \frac{E_o \cos\phi}{ika} \sqrt{\frac{\varepsilon}{\mu}} \sum_{n=1}^{\infty} b_n \left(\frac{d[rh_n^{(2)}(kr)]}{dr}\right)_{r=a} \frac{dP_m^1(\cos\theta)}{d\theta} \Big/ h_n^{(2)}(ka)$$

$$+ E_o \cos\phi \sqrt{\frac{\varepsilon}{\mu}} \sum_{n=1}^{\infty} a_n [P_n^1(\cos\theta)/\sin\theta] h_n^{(2)}(ka) \Big/ \left(\frac{d[rh_n^{(2)}(kr)]}{dr}\right)_{r=a}$$

and

$$H_\theta^t(a,\theta,\phi) = \frac{E_o \cos\phi}{ika} \sqrt{\frac{\varepsilon}{\mu}} \sum_{n=1}^{\infty} b_n \left(\frac{d[rj_n(kr)]}{dr}\right)_{r=a} \frac{dP_m^1(\cos\theta)}{d\theta} \Big/ j_n(ka)$$

$$+ E_o \cos\phi \sqrt{\frac{\varepsilon}{\mu}} \sum_{n=1}^{\infty} a_n [P_n^1(\cos\theta)/\sin\theta] j_n(ka) \Big/ \left(\frac{d[rj_n(kr)]}{dr}\right)_{r=a}$$

Substituting these into the aperture equation, i.e. Equation (47), we obtain an expression which can be made somewhat more compact by using the Wronskian for j_n and $h_n^{(2)}$, that is

$$j_n(kr)\frac{d}{dr} h_n^{(2)}(kr) - h_n^{(2)}(kr)\frac{d}{dr} j_n(kr) = \frac{1}{ikr^2}$$

From this Wronskian we can construct the helpful identity

$$\frac{1}{h_n^{(2)}(kr)} \frac{d}{dr}[rh_n^{(2)}(kr)] - \frac{1}{j_n(kr)} \frac{d}{dr}[rj_n(kr)] = \frac{1}{ikr\ j_n(kr)h_n^{(2)}(kr)} \tag{48}$$

In precisely the same manner we can derive from the Wronskian the additional relation

$$h_n^{(2)}(kr)/\frac{d[rh_n^{(2)}(kr)]}{dr} - j_n(kr)/\frac{d[rj_n(kr)]}{dr} = \frac{-1}{ikr\ \frac{d[rj_n(kr)]}{dr}\ \frac{d[rh_n^{(2)}(kr)]}{dr}} \tag{49}$$

The more compact expression for Eq. (47) using the results of Eq. (48) and (49) is then

$$E_o \cos\phi \sqrt{\frac{\varepsilon}{\mu}} \left\{ \frac{1}{(ka)^2} \sum_{n=1}^{\infty} \frac{b_n}{j_n(ka)h_n^{(2)}(ka)} \frac{dP_n^1(\cos\theta)}{d\theta} + \right.$$

$$\left. + \frac{1}{ika} \sum_{n=1}^{\infty} \left(a_n \Big/ \left\{ \frac{d[rj_n(kr)]}{dr} \right\}_{r=a} \left\{ \frac{d[rh_n^{(2)}(kr)]}{dr} \right\}_{r=a} \right) \frac{P_n^1(\cos\theta)}{\sin\theta} \right\} =$$

$$= K_\phi^o(\theta,\phi) \equiv E_o \cos\phi \sqrt{\frac{\varepsilon}{\mu}}\ I_\phi^o(\theta) \quad ; \quad \theta_1 < \theta < \theta_2, \quad 0 \le \phi \le 2\pi \tag{50}$$

Note that we have defined on the right hand side of Eq. (50) the phi-independent current $I^o_\phi(\theta)$. Substituting into Eq. (50) the expressions for a_n and b_n as given by Eqs. (41) and (44) we find the resulting relation

$$\int_{\theta_1}^{\theta_2} d\theta' \, f_\theta(\theta') \left\{ \sum_{n=1}^{\infty} \frac{(2n+1)(n-1)!}{2n(n+1)(n+1)!} \left[\frac{1}{(ka)^2 j_n(ka) h_n^{(2)}(ka)} \frac{dP_n^1(\cos\theta)}{d\theta} P_n^1(\cos\theta') \right.\right.$$

$$\left.\left. + \frac{1}{\left(\frac{d[rj_n(kr)]}{dr}\right)_{r=a} \left(\frac{d[rh_n^{(2)}(kr)}{dr}\right)_{r=a}} \frac{P_n^1(\cos\theta)}{\sin\theta} \frac{dP_n^1(\cos\theta')}{d\theta'} \sin\theta' \right] \right\} +$$

$$+ \int_{\theta_1}^{\theta_2} d\theta' \, f_\phi(\theta') \left\{ \sum_{n=1}^{\infty} \frac{(2n+1)(n-1)!}{2n(n+1)(n+1)!} \left[\frac{\sin\theta'}{(ka)^2 j_n(ka) h_n^{(2)}(ka)} \frac{dP_n^1(\cos\theta)}{d\theta} \frac{dP_n(\cos\theta')}{d\theta'} \right.\right.$$

$$\left.\left. + \frac{1}{\left(\frac{d[rj_n(kr)]}{dr}\right)_{r=a} \left(\frac{d[rh_n^{(2)}(kr)]}{dr}\right)_{r=a}} \frac{P_n^1(\cos\theta)}{\sin\theta} P_n^1(\cos\theta') \right] \right\} = I^o_\phi(\theta)$$

$$\text{for} \quad \theta_1 < \theta < \theta_2, \quad 0 \le \phi \le 2\pi \qquad (51)$$

A more convenient form of the ϕ-component of the current on the righthand side of Eq. (51) can be obtained in a straight forward manner. From Eqs. (46) and (10) we obtain

$$I^{o}_{\phi}(\theta) = \frac{1}{ika} \sum_{n=1}^{\infty} i^n \frac{(2n+1)}{n(n+1)} \left[\left(\frac{d[rj_n(kr)]}{dr} \right)_{r=a} - \frac{j_n(ka)}{h_n^{(2)}(ka)} \left(\frac{d[rh_n^{(2)}(kr)]}{dr} \right)_{r=a} \right] \frac{dP_n^1(\cos\theta)}{d\theta} +$$

$$+ \sum_{n=1}^{\infty} i^n \frac{(2n+1)}{n(n+1)} \left[j_n(ka) - h_n^{(2)}(ka) \left(\frac{d[rj_n(kr)]}{dr} \right)_{r=a} \Big/ \left(\frac{d[rh_n^{(2)}(kr)]}{dr} \right)_{r=a} \right]$$

$$\cdot \frac{P_n^1(\cos\theta)}{\sin\theta}$$

This can be simplified somewhat with the aid of Eqs. (48) and (49). From Eq. (48) we get after multiplying through by $j_n(kr)$ and evaluating at $r = a$:

$$\left(\frac{d[rj_n(kr)]}{dr} \right)_{r=a} - \frac{j_n(ka)}{h_n^{(2)}(ka)} \left(\frac{d[rh_n^{(2)}(kr)}{dr} \right)_{r=a} = \frac{-1}{ika\, h_n^{(2)}(ka)}$$

Similarly from Eq. (49) multiplying by $\frac{d[rj_n(kr)]}{dr}$ and evaluating at r=a gives

$$j_n(ka) - h_n^{(2)}(ka)\left(\frac{d[rj_n(kr)]}{dr}\right)_{r=a} \Bigg/ \left(\frac{d[rh_n^{(2)}(kr)]}{dr}\right)_{r=a} =$$

$$\frac{1}{ika\left(\frac{d[rh_n^{(2)}(kr)]}{dr}\right)_{r=a}}$$

Thus the ϕ-component of current $I_\phi^o(\theta)$ can be writtèn

$$I_\phi^o(\theta) = \frac{1}{(ka)^2}\sum_{n=1}^{\infty} i^n \frac{(2n+1)}{n(n+1)} \frac{1}{h_n^{(2)}(ka)} \frac{dP_n^1(\cos\theta)}{d\theta} +$$

$$+ \frac{1}{(ika)}\sum_{n=1}^{\infty} i^n \frac{(2n+1)}{n(n+1)} \frac{1}{\left(\frac{d[rh_n^{(2)}(kr)]}{dr}\right)_{r=a}} \frac{P_n^1(\cos\theta)}{\sin\theta} . \tag{52}$$

At this point in the format development we pause momentarily to note that Eqs. (51) and (52) constitute in essence one integral equation containing the two unknown functions f_θ, f_ϕ. These we recall are the tangential components of the aperture electric field. This integral equation was obtained from the requirement that H_θ be continuous across the aperture. In precisely the same fashion a second integral equation in $f_\theta(\theta)$ and $f_\phi(\theta)$ can be derived via the requirement that the azimuthal component of the magnetic field must also be continuous across the aperture. We next proceed to effect this derivation.

Over the aperture we must have

$$H_\phi^t(a,\theta,\phi) = H_\phi^s(a,\theta,\phi) + H_\phi^o(a,\theta,\phi) \quad ; \quad \theta_1 < \theta < \theta_2, \; 0 \leq \phi \leq 2\pi$$

which is readily rearranged to

$$H_\phi^t(a,\theta,\phi) - H_\phi^s(a,\theta,\phi) = H_\phi^o(a,\theta,\phi)$$

Now the θ-component of the current on the conducting sphere in the absence of the aperture is as usual

$$K_\theta^o(\theta,\phi) = \vec{e}_\theta(\theta,\phi)\cdot\{\vec{e}_r(\theta,\phi) \times \vec{H}^o(a,\theta,\phi)\}$$

$$= \vec{e}_\theta(\theta,\phi)\cdot[\vec{e}_r(\theta,\phi) \times \vec{e}_\phi(\phi)]H_\phi^o(a,\theta,\phi)$$

then we have

$$K_\theta^o(\theta,\phi) = - H_\phi^o(a,\theta,\phi) \tag{53}$$

Using this we write

$$H_\phi^t(a,\theta,\phi) - H_\phi^s(a,\theta,\phi) = - K_\theta^o(\theta,\phi) \qquad \theta_1 < \theta < \theta_2, \; 0 \leq \phi \leq 2\pi \tag{54}$$

We can then find H_ϕ^t and H_ϕ^s as we found the θ-components and obtain with the help of Eqs. (48) and (49)

$$E_o \sin\phi \sqrt{\frac{\varepsilon}{\mu}} \left\{ \frac{1}{(ka)^2} \sum_{n=1}^{\infty} \frac{b_n}{j_n(ka)h_n^{(2)}(ka)} \frac{P_n^1(\cos\theta)}{\sin\theta} + \right.$$

$$\left. + \frac{1}{ika} \sum_{n=1}^{\infty} \left[a_n \Big/ \left(\frac{d[rj_n(kr)]}{dr} \; \frac{d[rh_n^{(2)}(kr)]}{dr} \right) \right]_{r=a} \frac{dP_n^1(\cos\theta)}{d\theta} \right\}$$

$$= K_\theta^o(\theta,\phi) \equiv E_o \sqrt{\frac{\varepsilon}{\mu}} \sin\phi \quad I_\theta^o(\theta) \qquad \theta_1 < \theta < \theta_2,\; 0 \leq \phi \leq 2\pi \tag{55}$$

where we have defined the θ-dependent part of the θ-component of current on the sphere in the absence of the aperture. Upon substitution from Eqs. (41) and (44) into Eq. (55) we obtain the second integral equation in the aperture fields $f_\theta(\theta)$ and $f_\phi(\theta)$:

$$\int_{\theta_1}^{\theta_2} d\theta' \, f_\theta(\theta') \sum_{n=1}^{\infty} \frac{(2n+1)(n-1)!}{2n(n+1)(n+1)!} \left[\frac{1}{(ka)^2 j_n(ka)h_n^{(2)}(ka)} \frac{P_n^1(\cos\theta)P_n^1(\cos\theta')}{\sin\theta} \right.$$

$$\left. + \frac{1}{\left(\frac{d[rj_n(kr)]}{dr}\right)_{r=a} \left(\frac{d[rh_n^{(2)}(kr)]}{dr}\right)_{r=a}} \frac{dP_n^1(\cos\theta)}{d\theta} \frac{dP_n^1(\cos\theta')}{d\theta'} \sin\theta' \right] +$$

$$+ \int_{\theta_1}^{\theta_2} d\theta' \, f_\phi(\theta') \sum_{n=1}^{\infty} \frac{(2n+1)(n-1)!}{2n(n+1)(n+1)!} \left\{ \frac{1}{(ka)^2 j_n(ka)h_n^{(2)}(ka)} \frac{P_n^1(\cos\theta)}{\sin\theta} \frac{dP_n^1(\cos\theta')}{d\theta'} \sin\theta' \right.$$

$$\left. + \left(1 \Big/ \left[\left(\frac{d[rj_n(kr))}{dr}\right)_{r=a} \frac{d[rh_n^{(2)}(kr)]}{dr} \right]_{r=a} \right) \frac{dP_n^1(\cos\theta)}{d\theta} P_n^1(\cos\theta') \right\} =$$

$$= I_\theta^o(\theta) \qquad \theta_1 < \theta < \theta_2, \quad 0 \leq \phi \leq 2\pi \tag{56}$$

The θ-component of current on the right hand side is found from

$$K_\theta^o(\theta,\phi) = - H_\phi^o(a,\theta,\phi)$$

Cancelling off $E_o\sqrt{\frac{\varepsilon}{\mu}}\sin\phi$ gives us the desired relation

$$I_\theta^o(\theta) = \frac{1}{(ka)^2}\sum_{n=1}^{\infty}\frac{i^n(2n+1)}{n(n+1)}\frac{1}{h_n^{(2)}(ka)}\frac{P_n^1(\cos\theta)}{\sin\theta} + $$

$$+ \frac{1}{ika}\sum_{n=1}^{\infty}\frac{i^n(2n+1)}{n(n+1)}\left(\frac{d[rh_n^{(2)}(kr)]}{dr}\right)^{-1}_{r=a}\frac{dP_n^1(\cos\theta)}{d\theta} \qquad (57)$$

To solve these coupled integral equations analytically is a formidable task. We shall discuss the results of our analytic attempts in subsequent reports. An alternative approach to resolving this problem is via numerical methods of approximation. One such technique whose results we will report on in a later publication is basically a modified version of the method of moments. In this modified approach the edge effects are given special emphasis.

INTEGRAL EQUATION FOR THE COMPONENTS OF SURFACE CURRENT ON THE PERFECT CONDUCTOR

We next shall show that there exists a formulation of the problem we are treating which involves only the components of the surface current on the perfect conductor at r=a. This is in a sense complementary to the above formulation which involved only the tangential aperture electric fields E_θ and E_ϕ. From the requirement that at the perfect conductor at r=a the tangential components of the electric field must vanish we can construct a pair of coupled integral equations for the θ- and ϕ- components of the surface current. This will now be demonstrated. On the conducting portion of the spherical surface at r=a the surface current is given by

$$\vec{K}(\theta,\phi) = \vec{e}_r(\theta,\phi) \times \{[\vec{H}^s(a,\theta,\phi) + \vec{H}^o(a,\theta,\phi)] - \vec{H}^t(a,\theta,\phi)\}$$

$$\text{for } \theta_2 < \theta < \theta_1, \quad 0 \le \phi \le 2\pi \qquad (58)$$

Extracting the phi-component directly from this we have

$$K_\phi(\theta,\phi) = H^o_\theta(a,\theta,\phi) + [H^s_\theta(a,\theta,\phi) - H^t_\theta(a,\theta,\phi)] \qquad (59)$$

Equations (7) and (19) through (22) enable us to write Eq. (59) as

$$K_\phi(\theta,\phi) = E_o\sqrt{\frac{\varepsilon_o}{\mu_o}} \cos\phi \sum_{n=1}^{\infty} \{c_n \frac{P_n^1(\cos\theta)}{\sin\theta} + d_n \frac{dP_n^1(\cos\theta)}{d\theta}\} \qquad (60)$$

where the expansion coefficients are

$$c_n = \frac{1}{ika}\{\frac{i^n(2n+1)}{n(n+1)} / \left(\frac{d[rh_n^{(2)}(kr)]}{dr}\right)_{r=a} \quad -a_n/\left[\left(\frac{d[rj_n(kr)]}{dr}\right)_{r=a} \left(\frac{d[rh_n^{(2)}(kr)]}{dr}\right)_{r=a}\right]\}$$

$$(61)$$

and

$$d_n = [1/(ka)^2]\{\frac{i^n(2n+1)}{n(n+1)} \frac{1}{h_n^{(2)}(ka)} - \frac{b_n}{j_n(ka)h_n^{(2)}(ka)}\} \tag{62}$$

where a_n and b_n are the expansion coefficients introduced earlier in the Debye potentials.

Similarly we can extract the θ-component of current from Eq. (59) and rearrange to obtain

$$K_\theta(\theta,\phi) = - H_\phi^o(a,\theta,\phi) + [H_\phi^t(a,\theta,\phi) - H_\phi^s(a,\theta,\phi)] \tag{63}$$

and by Eqs. (7) and (19) through (22) this becomes

$$K_\theta(\theta,\phi) = E_o\sqrt{\frac{\varepsilon_o}{\mu_o}} \sin\phi \sum_{n=1}^{\infty} \{d_n \frac{P_n^1(\cos\theta)}{\sin\theta} + c_n \frac{dP_n^1(\cos\theta)}{d\theta}\} \tag{64}$$

For additional convenience we introduce the notation $\mathcal{K}_{\theta,\phi}(\theta,\phi)$ for the surface currents on the metallic conductor. We then can write

$$K_\phi(\theta,\phi) = \begin{cases} 0 & \theta_1 < \theta < \theta_2 \\ \mathcal{K}_\phi(\theta,\phi) & \theta_2 < \theta < \theta_1 \end{cases} \qquad 0 \le \phi \le 2\pi \tag{65}$$

$$K_\theta(\theta,\phi) = \begin{cases} 0 & \theta_1 < \theta < \theta_2 \\ \mathcal{K}_\theta(\theta,\phi) & \theta_2 < \theta < \theta_1 \end{cases} \qquad 0 \le \phi \le 2\pi \tag{66}$$

It will be further convenient to introduce, analogous to Eqs. (37) and (38), the definitions

$$K_\phi(\theta,\phi) = E_o\sqrt{\frac{\varepsilon_o}{\mu_o}}\ \cos\phi\ \ g_\phi(\theta); \qquad \theta_2 < \theta < \theta_1, \qquad 0 \le \phi \le 2\pi \qquad (67)$$

and

$$K_\theta(\theta,\phi) = E_o\sqrt{\frac{\varepsilon_o}{\mu_o}}\ \sin\phi\ \ g_\theta(\theta); \qquad \theta_2 < \theta < \theta_1, \qquad 0 \le \phi \le 2\pi \qquad (68)$$

We will next proceed to derive expressions for c_n and d_n in terms of g_θ, and g_ϕ analogous to those for a_n and b_n in Eqs. (41) and (44) above. Note that if in Eqs. (31), (34), (37), and (38) we make the following replacements

$$a_n/(ika) \rightarrow c_n$$

$$b_n \rightarrow d_n$$

$$f_\theta(\theta) \rightarrow g_\theta(\theta)$$

$$f_\phi(\theta) \rightarrow g_\phi(\theta)$$

$$0 \le \theta_1 < \theta < \theta_2 \le \pi \rightarrow \theta_2 < \theta < \theta_1$$

we find Eqs. (64), (60), (68) and (67) are essentially obtained. This enables us to write down immediately from Eqs. (41) and (44)

$$c_m\ \frac{2m(m+1)(m+1)!}{(2m+1)(m-1)!} = \oint d\theta'\ \sin\theta' g_\theta(\theta')\ \frac{dP_m^1(\cos\theta')}{d\theta'} + \oint d\theta' g_\phi(\theta')P_m^1(\cos\theta')$$

(69)

$$d_m \frac{2m(m+1)(m+1)!}{(2m+1)(m-1)!} = \not\oint d\theta' g_\theta(\theta') P_m^1(\cos\theta') + \not\oint d\theta' \sin\theta' g_\phi(\theta') \frac{dP_m^1(\cos\theta')}{d\theta'} \tag{70}$$

where by $\not\oint$ we mean integrate over the conducting portion of the sphere r=a, i.e. θ' from 0 to θ_1 and then from θ_2 to π.

At this point we observe that Eqs. (69) and (70) tell us that these expansion coefficients are given in terms of the current density on the conductor. Consequently our problem is also equivalent to solving for the current densities $g_\theta(\theta)$ and $g_\phi(\theta)$ everywhere on the conductor. The same method for approximate solution referred to in the previous section with regard to solving for the aperture tangential electric fields also can be used here. Thus we have two alternative but equivalent approaches to the solution of the slotted sphere scattering problem.

Next we shall derive a pair of coupled integral equations for g_θ and g_ϕ, the components of current density on the conductor. This is exactly analgous to the pair of coupled equations for the tangential components of the aperture electric field of the previous section.

On the conductor the θ-component of the electric field must vanish. Then Eq. (31) becomes

$$E_\theta(a,\ \theta_2 < \theta < \theta_1,\ \phi) = E_o \sin\phi \sum_{n=1}^{\infty} \left[\left(\frac{a_n}{ika}\right) \frac{dP_n^1(\cos\theta)}{d\theta} + \frac{b_n P_n^1(\cos\theta)}{\sin\theta} \right] = 0$$

from this we conclude that

$$\sum_{n=1}^{\infty} \left[\left(\frac{a_n}{ika}\right) \frac{dP_n^1(\cos\theta)}{d\theta} + b_n \frac{P_n^1(\cos\theta)}{\sin\theta} \right] = 0, \quad \text{for } \theta_2 < \theta < \theta_1 \tag{71}$$

Substituting for a_n and b_n in Eq. (71) from Eqs. (61), (62), (69) and (70) we obtain one of the integral equations for the surface currents, namely:

$$\oint d\theta' g_\theta(\theta') \left\{ \sum_{n=1}^{\infty} \frac{(2n+1)(n-1)!}{2n(n+1)(n+1)!} \left[(ka)^2 j_n(ka) h_n^{(2)}(ka) \frac{P_n^1(\cos\theta) P_n^1(\cos\theta')}{\sin\theta} + \right.\right.$$

$$\left.\left. + \left(\frac{d[r j_n(kr)]}{dr} \right)_{r=a} \left(\frac{d[r h_n^{(2)}(kr)]}{dr} \right)_{r=a} \frac{dP_n^1(\cos\theta)}{d\theta} \frac{dP_n^1(\cos\theta')}{d\theta} \sin\theta' \right] \right\} +$$

$$+ \oint d\theta' g_\phi(\theta') \left\{ \sum_{n=1}^{\infty} \frac{(2n+1)(n-1)!}{2n(n+1)(n+1)!} \left[(ka)^2 j_n(ka) h_n^{(2)}(ka) \frac{P_n^1(\cos\theta)}{\sin\theta} \frac{dP_n^1(\cos\theta')}{d\theta'} \cdot \right.\right.$$

$$\left.\left. \cdot \sin\theta' + \left(\frac{d[r j_n(kr)]}{dr} \right)_{r=a} \left(\frac{d[r h_n^{(2)}(kr)]}{dr} \right)_{r=a} \left(\frac{dP_n^1(\cos\theta)}{d\theta} P_n^1(\cos\theta') \right) \right] \right\} = V_\theta^i(\theta) \tag{72}$$

where we have defined

$$V_\theta^i(\theta) \equiv E_\theta^i(a,\theta,\phi)/E_o \sin\phi \tag{73}$$

The remaining coupled integral equation is obtained the same way. Since the ϕ-component of the electric field must also vanish on the perfect conductor we obtain from Eq. (34)

$$\sum_{n=1}^{\infty} \left[\left(\frac{a_n}{ika} \right) \frac{P_n^1(\cos\theta)}{\sin\theta} + b_n \frac{dP_n^1(\cos\theta)}{d\theta} \right] = 0, \quad \text{for } \theta_2 < \theta < \theta_1 \tag{74}$$

Again substituting for a_n and b_n in terms of c_n and d_n and then using Eqs. (69) and (70) we obtain

$$\oint d\theta' g_\phi(\theta') \left\{ \sum_{n=1}^{\infty} \frac{(2n+1)(n-1)!}{2n(n+1)(n+1)!} \left[(ka)^2 j_n(ka) h_n^{(2)}(ka) \frac{dP_n^1(\cos\theta)}{d\theta} P_n^1(\cos\theta') + \right.\right.$$

$$\left.\left. + \left(\frac{d[rj_n(kr)]}{dr} \right)_{r=a} \left(\frac{d[rh_n^{(2)}(kr)]}{dr} \right)_{r=a} \frac{P_n^1(\cos\theta)}{\sin\theta} \frac{dP_n^1(\cos\theta')}{d\theta} \sin\theta' \right] \right\} +$$

$$+ \oint d\theta' g_\phi(\theta') \left\{ \sum_{n=1}^{\infty} \frac{(2n+1)(n-1)!}{2n(n+1)(n+1)!} \left[(ka)^2 j_n(ka) h_n^{(2)}(ka) \frac{dP_n^1(\cos\theta)}{d\theta} \frac{dP_n^1(\cos\theta')}{d\theta'} \sin\theta' \right.\right.$$

$$\left.\left. + \left(\frac{d[rj_n(kr)]}{dr} \right)_{r=a} \left(\frac{d[rh_n^{(2)}(kr)]}{dr} \right)_{r=a} \frac{P_n^1(\cos\theta) P_n^1(\cos\theta')}{\sin\theta} \right] \right\} = V_\phi^i(\theta) \quad (75)$$

where this time we defined

$$V_\phi^i(\theta) \equiv E_\phi^i(a,\theta,\phi) / \left[E_o \cos\phi \right] \quad (76)$$

In Eq. (68) we used as the notation for the θ-component of the incident electric field at the conductor

$$E_\theta^i(a, \theta_2 < \theta < \theta_1, \phi) \equiv E_o \sin\phi \sum_{n=1}^{\infty} \frac{i^n(2n+1)}{n(n+1)} \left\{ \frac{1}{ika} \left(\frac{d[rj_n(kr)]}{dr} \right)_{r=a} \frac{dP_n^1(\cos\theta)}{d\theta} + \right.$$

$$\left. + j_n(ka) \frac{P_n^1(\cos\theta)}{\sin\theta} \right\} \quad (77)$$

Likewise in Eq. (75) we used as notation for the ϕ-component of the incident electric field at the conductor

$$E_\phi^i(a,\theta_2<\theta<\theta_1,\phi) \equiv E_o \cos\phi \sum_{n=1}^{\infty} \frac{i^n(2n+1)}{n(n+1)} \left\{ \frac{1}{ika} \left(\frac{d[rj_n(kr)]}{dr} \right)_{r=a} \frac{dP_n^1(\cos\theta)}{d\theta} + \right.$$

$$\left. + j_n(ka) \frac{P_n^1(\cos\theta)}{\sin\theta} \right\} .$$

Before continuing we pause momentarily at this point to emphasize that solution of the coupled integral Equations (72) and (75) for the tangential aperture electric fields is completely equivalent to solution of the coupled integral Equations (51) and (56) for the surface currents on the perfect conductor. Either set of quantities may be considered the sources of scattered and transmitted fields.

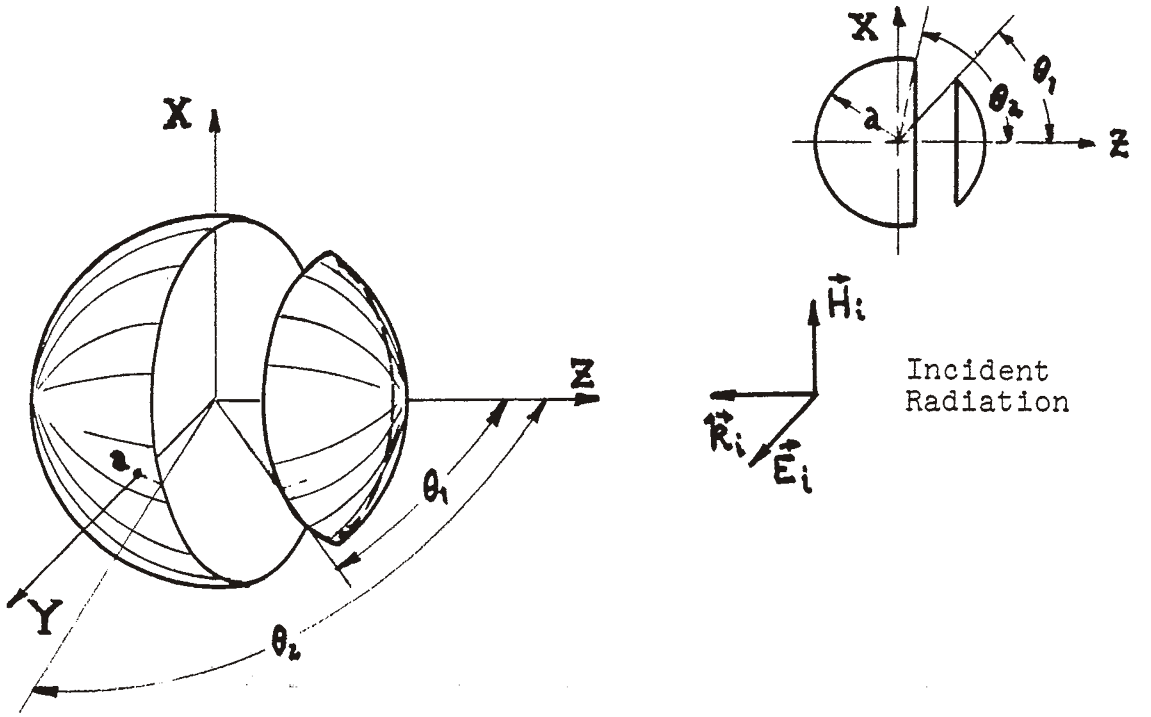
THE "COMPLEMENTARY" SCATTERING PROBLEM

Examination of the two pairs of coupled integral Equations, (51) and (56) for the θ-dependent part of the tangential aperture electric fields, and (72) and (75) for the θ-dependent part of the current density components on the perfect conductor, leads to an extremely interesting and very important observation. Note that the explicit functional dependence on ka in each term of the infinite sums that constitute the kernels in Equation (56) is simply the reciprocal of the corresponding term in the kernels in Equation (72). An identical relationship holds for the pair of Equations (51) and (75). Note further that in Equations (51) and (56) the integration ranges over the aperture. In addition the right hand side of the equations contains, in the first case, the ϕ-component of the current that would be present in the aperture region if it were filled with the perfect conductor and in the second case, the θ-component of the current under the same condition. On the other hand in Equations (75) and (72) the integration ranges over the perfect conductor. Now however the right hand sides of the respective equations contain the ϕ-component and θ-component of the incident field at the perfect conductor. As a consequence of these relations we observe that from either pair of the coupled integral equations derived for the annularly slotted sphere we can immediately deduce the other pair of coupled integral equations by simply making the appropriate substitutions.

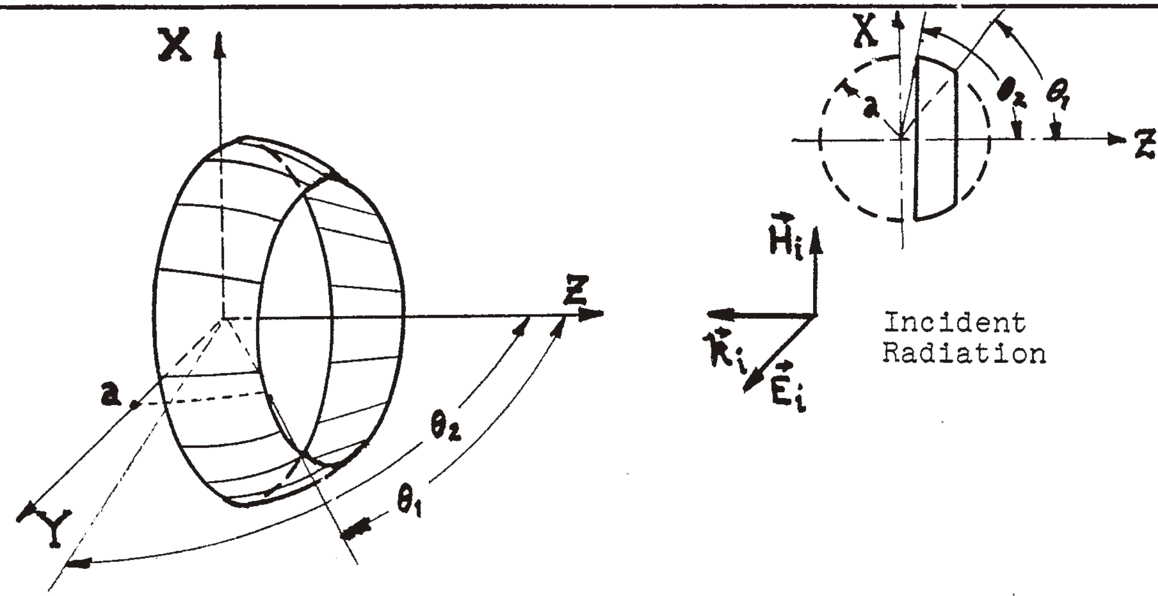
Clearly the corresponding pairs of coupled integral equations to be solved for the "complementary" problem (cf. figure 1(b)), namely, the scattering by a perfectly conducting sphereical ribbon, have identically the same form as those for the "direct" problem discussed here (cf. figure 1(a)). The only change necessary is to merely adjust the ranges of integration.

DISCUSSION OF RESULTS

The results exhibited in Equations (51), (56) and Equations (72), (75) can be readily shown to reduce properly in the limit as $\theta_1 \rightarrow 0$ and $\theta_2 \rightarrow \pi$. We have omitted this demonstration from this report for the sake of brevity. In the former pair of coupled equations we find the tangential aperture fields reduce correctly to the incident field at the aperture region. In a sequence of reports that follow this one we shall discuss these coupled integral equations and their solution. In the course of these discussions we shall have occasion to compare the solutions we obtain to those obtained by others in a number of special cases. We shall also present the solution for the more general situation and discuss a number of interesting results obtained, for example, those for the wide spherical ribbon which corresponds to the sphere with small circular apertures atthe top and at the bottom.



1a) THE ANNULARLY SLOTTED SPHERE



1b) THE COMPLEMENTARY PROBLEM = THE SPHERICAL RIBBON

FIGURE 1. THE SLOTTED SPHERE AND ITS COMPLEMENT

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We present in this report the formal derivation of two sets of pairs of coupled integral equations. These arise in the scattering of an incident plane wave by a perfectly conducting sphere with an annular slot defined by means of two angles. The incident radiation is assumed for simplicity to be along the symmetry axis of the slotted sphere. Simple but quite remarkable relations between the sets of coupled equations are indicated. We further present the formal solution for the complementary geometry namely the perfectly conducting spherical ribbon.

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