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EFFECTS OF RESISTIVITY ON THE CURRENT INDUCED IN AN INFINITE CIRCULAR CYLINDER BY A PLANE WAVE

by

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ABSTRACT

We find an expression for the axial conduction current induced in an infinitely long circular cylinder with small resistivity in free space by an incident electromagnetic plane wave. We study in particular the qualitative change in the late-time behavior of the response to a pulse and the limiting case for small angles between the axis of the cylinder and the direction of propagation of the wave when compared to the case of a perfectly conducting cylinder.

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CONTENTS

1. INTRODUCTION

A physical quantity of interest in the interaction of long antennas or cables with electromagnetic waves is the current induced in the conductors. It is straightforward to calculate this current for an infinitely long, perfectly conducting cylinder when it scatters a plane, monochromatic incident wave. This result is then used to find the current induced by an arbitrary plane-wave pulse by integrating the response due to the frequency spectrum of the incident pulse. This has been done by Barnes[1] in some specific cases, including a pulse that starts at $t = 0$ and decays exponentially with time.

A remarkable property of the solution to this problem is the late-time behavior of the induced current which tends to zero as $1/\log t$ as t tends to infinity. This temporal behavior is a property of the transfer function and is essentially independent of the precise shape of the incident pulse. The slow rate of decay can be understood by noting that currents travel unattenuated on a perfect conductor, and that the current at a fixed point for a late time arises from that induced far away by the front of the wave.

This late-time behavior is consequently not expected when the resistivity does not vanish, and the transfer function in the case of a cylinder with a finite conductivity leads to a different result. The current induced far away from the point of observation no longer determines the late-time behavior of the induced current pulse, but the latter is a local response to the incident wave and they both decay in the same manner.

Another result that is affected by the presence of a small resistivity is the limiting case of a small angle between the axis of the cylinder and the direction of propagation of the incident wave. For a perfectly conducting cylinder, the current tends to infinity as the angle tends to zero. This is clearly impossible in the presence of resistivity. Actually, the opposite occurs and the current tends to zero; this is a consequence of the attenuation of the current in the cylinder while the wave propagates parallel to it.

We use a notation close to that in Reference (1), but there are some differences that can be found in Section 2, where we present the functions that are needed to describe in cylindrical coordinates the propagation of waves in a conducting medium. In Section 3 we treat in detail the case of an incident plane wave propagating in a direction perpendicular to the axis of a finitely conducting cylinder. We find the current induced by monochromatic wave and we use this result to discuss the late-time behavior of an induced current pulse. In Section 4 we consider the more general case of an incident wave that propagates in an arbitrary direction. It is sufficient to consider separately the cases in which the incident wave is polarized either with the electric or the magnetic field perpendicular to the axis of the cylinder. Although the scattered wave is no longer linearly polarized, we retain the result, obtained for the perfect conductor, of no induced current in the case of the incident electric field perpendicular to the axis of the cylinder. In the other case, the properties of the induced current do not differ qualitatively from those found for perpendicular incidence. We use the general formulas to find the limiting case of a small angle between the direction of propagation and the cylinder. We state our conclusions in Section 5.

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[1] Barnes, P.R., EMP Interaction Notes, Note 64, March 1971, Air Force Weapons Laboratory, Kirtland AFB, New Mexico.

2. MONOCHROMATIC WAVES IN CONDUCTING MEDIA

The solution of our problem is based on Maxwell's equations in a homogeneous, isotropic medium. They are (^ denotes vector product)

$$\nabla \cdot \vec{D} = \rho \tag{2.1}$$

$$\nabla \wedge \vec{E} = - \partial \vec{B} / \partial t \tag{2.2}$$

$$\nabla \cdot \vec{B} = 0 \tag{2.3}$$

$$\nabla \wedge \vec{H} = \vec{j} + \partial \vec{D} / \partial t \tag{2.4}$$

For a conducting medium, we assume that the permittivity and permeability do not differ significantly from that for free space, that there is no free net charge and that the current density is proportional to the electric field. Hence

$$\vec{D} = \varepsilon_0 \vec{E} \tag{2.5}$$

$$\vec{H} = \vec{B} / \mu_0 \tag{2.6}$$

$$\rho = 0 \tag{2.7}$$

$$\vec{j} = \sigma \vec{E} \, . \tag{2.8}$$

In free space, the conductivity and, consequently, also the current density vanish. In the usual manner, we obtain from these equations the wave (or telegrapher's) equation

$$\nabla^2 \vec{E} - \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} - \sigma \mu_0 \frac{\partial \vec{E}}{\partial t} = 0 \tag{2.9}$$

for the electric field, and the magnetic field obeys the same equation. It is simpler to solve this equation for the Fourier transform of the field,

$$\vec{E} (\vec{x},t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \vec{E} (\vec{x},\omega) \exp[-i\omega t] \, d\omega. \tag{2.10}$$

The wave equation now reduces to the Helmholtz equation

$$(\nabla^2 + K^2) \vec{E} (\vec{x},\omega) = 0, \tag{2.11}$$

where K is the complex wave number given by

$$K^2 = \frac{\omega^2}{c^2} \left(1 + i \frac{\sigma}{\varepsilon_0 \omega} \right) = k^2 \left(1 + i \frac{\sigma}{\varepsilon_0 \omega} \right) . \tag{2.12}$$

This is equivalent to the consideration of a monochromatic wave, for which the electric field and all related quantities have a simple sinusoidal time dependence, that is

$$\vec{E}(\vec{x},t) = \vec{E}(\vec{x}) \exp[-i\omega t] \ , \tag{2.13}$$

where $\vec{E}(\vec{x})$ can be complex and it is understood that we take the only real part of the right-hand side of the equation. Equations (2.2) and (2.4) reduce to

$$\vec{B} = -\frac{i}{\omega} \nabla\wedge\vec{E} \ , \tag{2.14}$$

$$\vec{E} = \frac{ic^2}{\omega(1+i\sigma/\varepsilon_o\omega)} \nabla\wedge\vec{B} = \frac{i\omega}{K^2} \nabla\wedge\vec{B} \ . \tag{2.15}$$

The solution of the vector Helmholtz equation can be generated from that of the scalar equation. This equation is separable in cylindrical coordinates. These are convenient in our case because of the presence of the infinite cylinder. The general solution then is a linear combination of functions of the form

$$T^{(\ell)}(K,\ k_3,\ m;\ r,\ \phi,\ z) = F_m^{(\ell)}(\kappa r) \exp[i(m\phi+k_3 z)], \tag{2.16}$$

where

$$\kappa^2 = K^2 - k_3^2 \ , \tag{2.17}$$

and $F_m^{(\ell)}$ is one of the cylindrical Bessel functions, with $\ell = 1,2,3,4$, corresponding to J_m, Y_m, $H_m^{(1)}$ and $H_m^{(2)}$, respectively. Three independent solutions to the vector equation then are

$$\vec{L} = (1/k)\ \nabla T \ , \tag{2.18}$$

$$\vec{M} = (1/k)\ \nabla\wedge\ (T\hat{z}) \ , \tag{2.19}$$

$$\vec{N} = (1/k)\ \nabla\wedge\ \vec{M} \tag{2.20}$$

We include the factor $1/k$ only for dimensional reasons, and other authors have chosen $1/K$. In the present problem m is an integer and, as the fields are solenoidal, we use only $\vec{M}$ and $\vec{N}$. Their components are

$$\vec{M} = (1/k)\ [(im/r)\ F_m(\kappa r)\ \hat{r} - \kappa F_m'(\kappa r)\hat{\phi}] \ \exp[i(m\phi+k_3 z)] \tag{2.21}$$

$$\vec{N} = (1/k^2)[i\kappa\ k_3\ F_m'(\kappa r)\ \ \hat{r} + (m\ k_3/r)\ F_m(\kappa r)\ \ \hat{\phi}$$

$$+ \kappa^2\ F_m(\kappa r)\hat{z}]\ \exp[i(m\phi+k_3 z)] \ . \tag{2.22}$$

We also need the relations

$$\nabla\wedge\vec{M} = k\vec{N} \tag{2.23}$$

$$\nabla\wedge\vec{N} = (K^2/k)\ \vec{M} \tag{2.24}$$

A plane wave in free space linearly polarized in the y-direction can be expanded in terms of these solutions by means of the equation

$$\hat{y}\exp[i\vec{k}\cdot\vec{x}] = (k/\kappa_0)\sum_{m=-\infty}^{\infty} i^{m+1}\ \vec{M}^{(1)}(k,k_3,m;\ r,\phi,z), \tag{2.25}$$

where

$$\kappa_0^2 = k^2 - k_3^2 \tag{2.26}$$

At the boundary between media we have to satisfy the usual boundary conditions. For a perfect conductor, the normal component of $\vec{B}$ and the tangential component of $\vec{E}$ must vanish, while the tangential component of $\vec{H}$ determines the surface current density. If the conductivity is finite, the normal component of $\vec{B}$ and the tangential component of $\vec{E}$ have to be continuous. In addition, as no surface currents can exist in such a conductor, the tangential component of $\vec{H}$ also must be continuous. This gives us five scalar continuity conditions. These are reduced to four independent ones by equation (2.14), which can be used to show that the continuity of the tangential component of $\vec{E}$ implies that the normal component of $\vec{B}$ is continuous. The discontinuity of the normal component $\vec{D}$ gives the surface charge density.

3. NORMAL INCIDENCE

We first consider a plane wave whose direction of propagation is normal to the axis of a cylinder with finite conductivity.

If the wave is linearly polarized with the electric field perpendicular to the axis, by symmetry the field in the cylinder has no component along the axis and there is no induced axial current. This is also verified from the more general case in the next Section.

We thus need to consider only the plane monochromatic wave linearly polarized with the electric field parallel to the axis of the cylinder, and all electric fields have only a z-component, where the z-axis is the axis of the cylinder. The incident field is given by

$$E_z^{inc} = E_o\exp[ikr\cos\phi] = E_o\sum_{m=-\infty}^{\infty} i^m J_m(kr)\exp[im\phi]\ . \tag{3.1}$$

Similarly, we can expand the scattered field according to

$$E_z^{sc} = E_o\sum_{m=-\infty}^{\infty} a_m H_m^{(1)}(kr)\exp[im\phi]\ , \tag{3.2}$$

where a_m are coefficients to be determined. We have chosen the Hankel function $H_m^{(1)}(kr)$ because it corresponds to an outgoing wave, that at large distances behaves as $\exp[i(kr-\omega t)]/\sqrt{r}$ and satisfies the radiation condition. We drop the superscript from the Hankel function in most of the following expressions and we assume it is (1) unless otherwise specified. Inside the cylinder, the field is expanded in terms of the Bessel functions that are regular at the origin, and we set

$$E_z^{cyl} = E_o\sum_{m=-\infty}^{\infty} b_m J_m(Kr)\ \exp[im\phi]\ . \tag{3.3}$$

The continuity of the total electric field at the surface of a cylinder of radius $\underline{a}$ gives

$$E_z^{inc}(a) + E_z^{sc}(a) = E_z^{cyl}(a) , \tag{3.4}$$

and, as the functions of the angle are independent, we obtain

$$i^m J_m(ka) + a_m H_m(ka) = b_m J_m(Ka). \tag{3.5}$$

The continuity of the tangential component of $\vec{H}$ implies that

$$B_\phi^{inc}(a) + B_\phi^{sc}(a) = B_\phi^{cyl}(a), \tag{3.6}$$

and, as by equation (2.14)

$$B_\phi = \frac{i}{\omega} \frac{\partial E_z}{\partial r} , \tag{3.7}$$

we obtain

$$i^m k J_m'(ka) + a_m k H_m'(ka) = b_m K J_m'(Ka) . \tag{3.8}$$

The axial conduction current is obtained by integrating the current density over a cross section of the cylinder, and, by equation (2.8), we have

$$I = \int_0^{2\pi} d\phi \int_0^a r dr\, \sigma\, E_z^{cyl} . \tag{3.9}$$

The field is given by equation (3.3), and, as the integral over the angle vanishes for $m \neq 0$, we derive

$$I = 2\pi\sigma\, E_o b_o \int_0^a J_o(Kr)\, r dr \tag{3.10}$$

As the recursion relations for Bessel functions give

$$(d/d\zeta)[\zeta\, J_1(\zeta)] = \zeta\, J_o(\zeta), \tag{3.11}$$

we can perform the integration over the radius to obtain

$$I = 2\pi\sigma E_o b_o a J_1(Ka)/K , \tag{3.12}$$

where b_o is determined from equations (3.5) and (3.8), for $m = 0$, to be

$$b_o = \frac{kH_o'(ka)J_o(ka) - kH_o(ka)J_o'(ka)}{kH_o'(ka)J_o(Ka) - KH_o(ka)J_o'(Ka)} . \tag{3.13}$$

The Wronskian of J_o and $H_o^{(1)}$ is

$$J_o(\zeta)\, H_o'(\zeta) - J_o'(\zeta)\, H_o(\zeta) = 2i/(\pi\zeta), \tag{3.14}$$

and we use it to simplify the numerator in equation (3.13). The axial current in equation (3.12) can then be written in the form

$$I = \frac{4iE_0}{\omega\mu_0(\varepsilon_0\omega/\sigma + i)H_0(ka)} \left[1 - \frac{kH_1(ka)J_0(Ka)}{KH_0(ka)J_1(Ka)} \right]^{-1} , \tag{3.15}$$

as

$$J_0' = -J_1 \quad , H_0' = - H_1 . \tag{3.16}$$

Equation (2.12) shows that, when $\sigma \to \infty$, the real and imaginary parts of κ tend to infinity and the argument of K tends to $\pi/4$. The asymptotic forms of the Hankel functions for large ζ are

$$H_0^{(1,2)}(\zeta) \sim \left(\frac{2}{\pi\zeta}\right)^{1/2} \exp[\pm i(\zeta-\pi/4)] , \tag{3.17}$$

$$H_1^{(1,2)}(\zeta) \sim \left(\frac{2}{\pi\zeta}\right)^{1/2} \exp[\pm i(\zeta-3\pi/4)] , \tag{3.18}$$

and, because

$$J_m(\zeta) = \frac{1}{2} [H_m^{(1)}(\zeta) + H_m^{(2)}(\zeta)] , \tag{3.19}$$

the contribution of $H_m^{(2)}(Ka)$ dominates and

$$\lim_{\sigma\to\infty} \frac{J_0(Ka)}{J_1(Ka)} = - i . \tag{3.20}$$

The additional factor K in the denominator of the second term in the bracket in equation (3.15) causes this term to tend to zero, and

$$\lim_{\sigma\to\infty} I = \frac{4E_0}{\omega\mu_0 H_0(ka)} , \tag{3.21}$$

which is precisely the expression obtained in a direct approach such as Barnes' for the perfectly conducting cylinder.

The current induced by an arbitrary pulse can then be represented by

$$I(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} I(\omega) \exp[-i\omega t] \, d\omega \tag{3.22}$$

or

$$I(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E_0(\omega) \, T(\omega) \exp[-i\omega t] \, d\omega , \tag{3.23}$$

where the transfer function is

$$T(\omega) = \frac{4i}{\omega\mu_0(\varepsilon_0\omega/\sigma+i)H_0(ka)} \left[1 - \frac{kH_1(ka)J_0(Ka)}{KH_0(ka)J_1(Ka)} \right]^{-1} \tag{3.24}$$

By the convolution theorem,

$$I(t) = \int_{-\infty}^{\infty} E\,(t-t')\,F\,(t')\,dt' \,, \tag{3.25}$$

where the transfer function in time is

$$F(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} T(\omega)\,\exp[-i\omega t]\,d\omega \tag{3.26}$$

The integrand in equations (3.23) and (3.26) has a singular point at the origin in the ω-plane, where the Hankel function has a logarithmic branch point. As for sufficiently large negative times we can close the contour around the upper-half plane without changing the value of the integral, we have to deform the contour along the real axis and pass above the origin to obtain a zero current for these times, as demanded by causality. For the same reason, or from the properties of the Hankel functions, we find that

$$F(t) = 0 \quad \text{for} \quad t < -a/c \,, \tag{3.27}$$

the time when the pulse reaches the cylinder (assuming it reaches the axis at $t = 0$). Thus, equation (3.25) becomes

$$I(t) = \int_{-a/c}^{\infty} E_o(t-t')\,F\,(t')\,dt' \tag{3.28}$$

and, changing the variable of integration to

$$\tau' = ct'/a + 1 \tag{3.29}$$

we find, as $E_o(t)$ vanishes for $t < 0$,

$$I(t) = \frac{c}{a} \int_{o}^{\tau} E_o[t - a(\tau'-1)/c]F[a(\tau'-1)/c]\,d\tau' \,, \tag{3.30}$$

where

$$\tau = ct/a + 1. \tag{3.31}$$

The behavior of $I(t)$ for large t depends on the properties of $F(t')$ for large t' when $E_o(t)$ goes to zero sufficiently rapidly with increasing t. If this is so, the main contribution to the integral in equation (3.28) comes from values of $t - t'$ that are small compared to t, that is, for relatively large t'. These considerations can be made more precise in a given case, such as that of the exponentially decaying incident pulse, analyzed by Barnes for a perfect conductor.

Furthermore, the values of $F(t')$ for large t' depend on those of $T(\omega)$ for small ω. This is made plausible and can be seen if we change the variable in equation (3.26) to

$$u = \omega t, \tag{3.32}$$

so that

$$F(t) = \frac{1}{2\pi t} \int_{-\infty}^{\infty} T\left(\frac{u}{t}\right) \exp[-iu]\,du \,. \tag{3.33}$$

In the case of a perfect conductor, we obtain $T(\omega)$ from equation (3.21), and Barnes shows that the late-time behavior of the induced current is proportional to $1/\log t$.

When the conductivity is finite, the second term in the bracket in equation (3.24), which vanished for a perfect conductor, has a modulus large compared to 1 for sufficiently small ω. When

$$\omega << \sigma/\varepsilon_0 , \tag{3.34}$$

equation (2.12) reduces to

$$K^2 \approx i\, \mu_0 \sigma \omega , \tag{3.35}$$

whence, for

$$\omega << 1/(\mu_0 \sigma a^2), \tag{3.36}$$

$$|Ka| << 1 , \tag{3.37}$$

and also

$$k^2/|K|^2 \approx \varepsilon_0 \omega/\sigma << 1 . \tag{3.38}$$

Consequently, we need the expansions of the Bessel functions for small arguments,

$$J_0(\zeta) \sim 1 , \tag{3.39}$$

$$J_1(\zeta) \sim \frac{1}{2}\zeta \tag{3.40}$$

$$Y_0(\zeta) \sim (2/\pi)[\log(\zeta/2) + \gamma] \tag{3.41}$$

$$Y_1(\zeta) \sim -2/(\pi\zeta) , \tag{3.42}$$

where γ is Euler's constant

$$\gamma = 0.577... \tag{3.43}$$

Thus,

$$\frac{kH_1(ka)J_0(Ka)}{KH_0(ka)J_1(Ka)} \approx -\frac{4i}{\pi(Ka)^2 H_0(ka)} \tag{3.44}$$

and, as

$$\lim_{\zeta \to 0} \zeta \log \zeta = 0 , \tag{3.45}$$

equations (3.35) and (3.41) show that the modulus of this term becomes large compared to 1. We neglect the 1 in the bracket and equation (3.15) becomes

$$I(\omega) \approx \pi a^2\, \sigma\, E_0(\omega) . \tag{3.46}$$

This result means that, for sufficiently low frequency, the axial current induced in the cylinder is produced directly by the incident field; that is, the scattered field becomes negligible. Because the field inside a perfect conductor must vanish, the scattered field can never be negligible in that limit. As a consequence of equation (3.39), the induced current at large times is proportional to the incident field at those times, a more reasonable result than a $1/\log t$ dependence.

In order to get some idea of the order of magnitude of the quantities involved, we can choose a cylinder with $\sigma = 10^8\ \mathrm{ohm}^{-1}\ \mathrm{m}^{-1}$ and $a = 3 \times 10^{-3}$ m. Then

$$1/(\mu_0 \sigma a^2) \approx 10^3\ \mathrm{s}^{-1} \tag{3.47}$$

$$\sigma/\varepsilon_0 \approx 10^{11}\ \mathrm{s}^{-1} \tag{3.48}$$

and, for $\omega = 10\ \mathrm{s}^{-1}$,

$$K^2 a^2 \approx 10^{-2} i \tag{3.49}$$

$$ka = 10^{-10} \tag{3.50}$$

$$H_0(ka) \approx 1 - 15i \tag{3.51}$$

There is another region of the frequency spectrum where

$$|Ka| \gg 1 \tag{3.52}$$

and

$$ka \ll 1, \tag{3.53}$$

or, in this example, a frequency range such that

$$10^3 \ll \omega \ll 10^{11} \tag{3.54}$$

In such a region, equations (3.20) and (3.42) show that

$$\frac{kH_1(ka)J_0(Ka)}{KH_0(ka)J_1(Ka)} \approx -\frac{2}{\pi K a H_0(ka)} . \tag{3.55}$$

The modulus of this term is small compared to 1, and in this region we find the same behavior as that of the current induced in a perfect conductor in the small ω limit. This corresponds to the $1/\log t$ behavior of the current induced by a pulse.

The current induced by a pulse in a good, but not perfect, conductor should initially be very close to that of a perfect conductor, including the $1/\log t$ behavior for moderately late times. For still later times, large compared to $\mu_0 \sigma a^2$, the resistivity attenuates the currents induced in the wire far away from the observation point and the only effect we still notice is the current set up in the cylinder by the late-time part of the incident wave.

4. OBLIQUE INCIDENCE

In order to formulate this problem in a compact manner, we use expansions of the field in the vector cylindrical harmonics discussed in Section 2.

It is sufficient to consider two cases of linearly polarized incident waves, although in the problem at hand the scattered and interior fields are no longer restricted, and both $\vec{M}$ and $\vec{N}$ should be used in the expansions.

The wave vector $\vec{k}$ of an incident, plane, monochromatic wave forms an angle θ with the axis of the cylinder, chosen as the z-axis of the coordinate system (in 1, it forms an angle θ with the negative z-axis, whence we show some discrepancies in signs), and equations (2.17) and (2.26) become

$$\kappa^2 = k^2 \ (\sin^2\theta + i\sigma/\varepsilon_o\omega), \tag{4.1}$$

$$\kappa_o^2 = k^2 \ \sin^2\theta \ . \tag{4.2}$$

We furthermore choose the x-axis in the plane defined by $\vec{k}$ and the axis of the cylinder.

We first consider the case of the incident electric field polarized in a direction perpendicular to the axis of the cylinder, that is,

$$\vec{E}^{inc} = E_o \ \exp[i\vec{k}\cdot\vec{r}] \ \hat{y}, \tag{4.3}$$

which can be expanded in the series

$$\vec{E}^{inc} = E_o \ (k/\kappa_o) \sum_{m=-\infty}^{\infty} i^{m+1} \ \vec{M}^{(1)}(k,k_3,m) \tag{4.4}$$

The scattered and interior fields must have the same frequency to match boundary conditions at all times, and they must have the same k_3 in order to match boundary conditions along the whole cylinder. Thus, we expand these fields in

$$\vec{E}^{sc} = E_o \sum_{m=-\infty}^{\infty} [a_m \vec{M}^{(3)}(k,k_3,m) + b_m \vec{N}^{(3)}(k,k_3,m)], \tag{4.5}$$

$$\vec{E}^{cyl} = E_o \sum_{m=-\infty}^{\infty} [c_m \vec{M}^{(1)}(K,k_3,m) + d_m \vec{N}^{(1)}(K,k_3,m)]. \tag{4.6}$$

From the continuity of the tangential components of $\vec{E}$ and $\vec{H}$ we obtain four linear equations for the coefficients with the same m. In matrix form, they are

$$A_m X_m = Y_m \ , \tag{4.7}$$

where, using equations (2.14) and (2.21) to (2.24), we find

$$A_m = \begin{pmatrix} 0 & \kappa_o^2 H_m(\kappa_o a) & 0 & -\kappa^2 J_m(\kappa a) \\ -a\,k\,\kappa_o H_m'(\kappa_o a) & m\,k_3 H_m(\kappa_o a) & a\,k\,\kappa\,J_m'(\kappa a) & -m\,k_3\,J_m(\kappa a) \\ \kappa_o^2 H_m(\kappa_o a) & 0 & -\kappa^2 J_m(\kappa a) & 0 \\ m\,k\,\kappa_o\,k_3\,H_m(\kappa_o a) & -a\,k^2\,\kappa_o^2\,H_m'(\kappa_o a) & -m\,k\,\kappa_o k_3\,J_m(\kappa a) & a\,K^2\kappa_o\kappa\,J_m'(\kappa a) \end{pmatrix}, \tag{4.8}$$

$$X_m = \begin{pmatrix} a_m \\ b_m \\ c_m \\ d_m \end{pmatrix} , \quad Y_m = i^{m+1} \begin{pmatrix} 0 \\ a\, k^2 J_m'(\kappa_o a) \\ -k\, \kappa_o\, J_m(\kappa_o a) \\ -\, m\, k^2\, k_3\, J_m(\kappa_o a) \end{pmatrix} \tag{4.9}$$

For normal incidence, k_3 vanishes and these equations show that in this case b_m and c_m also vanish, which verifies our symmetry arguments in Section 3. For a perfect conductor, terms in the third and fourth columns of A_m diverge, which indicates that c_m and d_m, and consequently the interior field vanish, as expected; the first equation then shows that the b_m also vanish, and the scattered field is dependent only on one set of parameters, as found by Barnes.

The axial current is found from the field inside the cylinder as in Section 3, and, corresponding to equation (3.12), we obtain the current at $z = 0$,

$$I = 2\pi\sigma\, E_o d_o\, \kappa a\, J_1(\kappa a)/k^2. \tag{4.10}$$

We only have to determine d_o, and from equation (4.7) for $m = 0$ we find

$$\kappa_o^2\, H_o(\kappa_o a)\; b_o - \kappa^2\, J_o(\kappa a) \quad d_c = 0 \;, \tag{4.11}$$

$$-k^2\, \kappa_o\, H_o'(\kappa_o a) \quad b_o + K^2\, \kappa\, J_o'(\kappa a) \quad d_o = 0 \;, \tag{4.12}$$

whence

$$d_o = 0 \;, \tag{4.13}$$

$$I = 0 \;, \tag{4.14}$$

the same result that was obtained for a perfect conductor.

We now consider the case of the incident magnetic field polarized in the y-direction, and set

$$\vec{B}^{inc} = -\, B_o \exp[i\vec{k}\cdot\vec{r}]\; \hat{y} \;, \tag{4.15}$$

which is expanded as

$$\vec{B}^{inc} = B_o(k/\kappa_o) \sum_{m=-\infty}^{\infty} i^{m-1}\, \vec{M}^{(1)}(k,k_3,m) \tag{4.16}$$

The scattered and interior fields are expanded as

$$\vec{B}^{sc} = B_o \sum_{m=-\infty}^{\infty} [a_m\, \vec{M}^{(3)}(k,k_3,m) \;\; + b_m\, \vec{N}^{(3)}(k,k_3,m)] \tag{4.17}$$

$$\vec{B}^{cyl} = B_o \sum_{m=-\infty}^{\infty} [c_m\, \vec{M}^{(1)}(K,k_3,m) \;\; + d_m\, \vec{N}^{(1)}(K,k_3,m)], \tag{4.18}$$

where the coefficients are unrelated to those of the first case presented above. Now we use equations (2.15) and (2.21) through (2.24) to find, from the boundary conditions,

$$
\Lambda_m = \left\{ \begin{array}{llll}
\kappa_0^2 H_m(\kappa_0 a) & 0 & -k^2 (\kappa^2/K^2) J_m(\kappa a) & 0 \\
m \kappa_0 k_3 H_m(\kappa_0 a) & -k \kappa_0 a H_m'(\kappa_0 a) & -m k^2 k_0 k_3 (1/K^2) J_m(\kappa a) & k \kappa_0 \kappa a J_m'(\kappa a) \\
0 & \kappa_0^2 H_m(\kappa_0 a) & 0 & -\kappa^2 J_m(\kappa a) \\
-k \kappa_0 a H_m'(\kappa_0 a) & m k_3 H_m(\kappa_0 a) & k \kappa a J_m'(\kappa a) & -m k_3 J_m(\kappa a)
\end{array} \right\} , \tag{4.19}
$$

$$
Y_m = i^{m-1} \begin{pmatrix} -k \kappa_0 J_m(\kappa_0 a) \\ -m k k_3 J_m(\kappa_0 a) \\ 0 \\ k^2 a J_m'(\kappa_0 a) \end{pmatrix} \tag{4.20}
$$

We find again that two sets of coefficients vanish when $k_3 = 0$, which verifies that the electric fields are along the z-axis for normal incidence. The induced current is given by

$$
I = 2\pi i \sigma E_0 c_0 \kappa a J_1(\kappa a)/K^2 , \tag{4.21}
$$

where

$$
E_0 = c B_0 . \tag{4.22}
$$

We compute the coefficient c_0 from

$$
\kappa_0^2 H_0(\kappa_0 a) \quad a_0 - k^2 (\kappa^2/K^2) J_0(\kappa a) \quad c_0 = ik \kappa_0 J_0(\kappa_0 a), \tag{4.23}
$$

$$
-\kappa_0 H_0'(\kappa_0 a) \quad a_0 + \kappa J_0'(\kappa a) \quad c_0 = -i k J_0'(\kappa_0 a), \tag{4.24}
$$

and find

$$
c_0 = \frac{ikK^2 \kappa_0 [J_0(\kappa_0 a) H_0'(\kappa_0 a) - J_0'(\kappa_0 a) H_0(\kappa_0 a)]}{\kappa[\kappa_0 K^2 H_0(\kappa_0 a) J_0'(\kappa a) - \kappa k^2 H_0'(\kappa_0 a) J_0(\kappa a)]} . \tag{4.25}
$$

We again simplify this expression by means of the Wronskian (3.14) and obtain the induced current at $z = 0$,

$$
I = \frac{4iE_0}{\omega\mu_0 \sin\theta (\varepsilon_0 \omega/\sigma + i) H_0(\kappa_0 a)} \left[1 - \frac{\kappa k^2}{\kappa_0 K^2} \frac{H_1(\kappa_0 a) \; J_0(\kappa a)}{H_0(\kappa_0 a) \; J_1(\kappa a)} \right]^{-1} \tag{4.26}
$$

This result reduces to that in equation (3.15) for normal incidence.

When the conductivity tends to infinity, the moduli of K and κ tend to infinity and the analysis in Section 3 shows that the second term in the bracket tends to zero. This result then agrees with that of reference (1).

On the other hand, when the conductivity is finite, equations (2.12) and (4.1) show that $|K|$ and $|\kappa|$ become small for sufficiently small ω and we can use equations (3.39) through (3.42) to show that the modulus of the second term in the bracket becomes large compared to 1, which can be neglected to obtain the approximate value of the induced current

$$I(\omega) \approx \pi a^2 \sin\theta \, \sigma E_0(\omega) . \tag{4.27}$$

This again is the current induced directly by the incident wave. As discussed in Section 3, the late-time behavior of the current induced by an incident pulse is determined by the form of the transfer function for small ω, and we find the same qualitative changes in the induced current between perfect and imperfect conductors.

Another limiting case of interest is the one in which θ tends to zero, that is, when the incident wave travels almost parallel to the cylinder. The case of the perfect conductor gives an induced current that is approximately

$$I \approx \frac{2\pi E_0}{\omega\mu_0\theta[\log(\theta ka/2) + \gamma]} , \tag{4.28}$$

which diverges when θ tends to zero. We now take the complete expression (4.26). As, for small θ ,

$$\kappa^2 \approx i\,\sigma\mu_0\omega \tag{4.29}$$

and κ remains finite, the second term in the bracket is proportional to $1/(\theta^2 \log\theta)$ and becomes large, so that

$$I \approx \frac{2\pi\sigma E_0 a J_1(\kappa a)}{\kappa\, J_0(\kappa a)} \; \theta, \tag{4.30}$$

which tends to zero with θ. There is then an angle for which the current is a maximum for a good conductor. This angle can be obtained from equation (4.26).

5. CONCLUSIONS

The expressions we have obtained for the current induced in a conducting cylinder by a plane monochromatic wave--essentially the transfer function in this problem--allow us to study the general properties of the response to an incident pulse. In particular, we found some qualitative changes due to the presence of a resistivity both in the late-time behavior of the current and in the small angle limit.

The early-time behavior of the current, related to the high-frequency region of the Fourier transform of this function, is not particularly affected by the terms containing a finite σ when compared to the limit of a perfect conductor, and the results obtained by Barnes are valid. We should even be able to observe the behavior for large t in some cases, before the changes brought about by a finite conductivity set in.

The late-time behavior of the current is determined by the transfer function for low frequencies, and in this limit there is a marked difference between the expressions obtained for finite conductivity and for a perfect conductor. For finite conductivity, the scattered field becomes negligible for sufficiently low frequencies and the field inside the cylinder is that of the incident wave. Consequently, the induced current is proportional to this field, and its late-time behavior is not that of $1/\log t$ but the one of the incident pulse.

The analysis of the case where the wave comes in at an arbitrary angle with respect to the cylinder introduces no qualitative changes with respect to the simpler case of normal incidence. The special limit of a small angle between the direction of propagation of the incident wave and the axis of the cylinder also shows a marked difference due to a small resistivity; for a perfect conductor, the induced current tends to infinity when the angle tends to zero, whereas for finite conductivity the limit is zero.

The qualitative differences are due to the changing relative importance of the two terms in the bracket in equation (4.15) or equation (4.26), from which we can obtain an idea of the order of magnitude of the quantities involved in a particular case. A more precise calculation of the late-time behavior of the current induced in a cylinder that shows that transition between the two regimes, or a determination of the angle of incidence that gives a maximum induced current, would require numerical calculations of the actual responses.

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13. ABSTRACT

We find an expression for the axial conduction current induced in an infintely long circular cylinder in free space with a small resistivity by an incident electromagnetic plane wave. We study in particular the qualitative changes in the late-time behavior of the response to a pulse and the limiting case for small angles between the axis of the cylinder and the direction of propagation of the wave when compared to the case of a perfectly conducting cylinder.

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