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A NEW LOOK AT THE THIN-PLATE SCATTERING PROBLEM

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ABSTRACT

In this paper the H-integral equation is investigated for the problem of plane wave diffraction by a thin plate. It is found that by itself the limiting form of the H-equation is not complete unless the condition $\hat{n} \cdot \overline{H} = 0$ is simultaneously enforced. By suitably combining the above mentioned equations, a new set of equations is derived for the two components of the surface current distribution on the plate. Several advantageous features of the new equations are pointed out and numerical results based upon the use of these equations are presented for the thin strip and the square plate problems.

NEW LOOK AT THE THIN-PLATE SCATTERING PROBLEM

When deriving a numerical solution to scattering problems, it is well-known that the H-integral equation is preferable to the E-equation when considering solid surface scatterers. It is also known that for thin scatterers one is forced to use the E-equation, since the H-equation is fraught with numerical instability problems in the zero thickness limit.

In this paper we study the H-equation for the limiting case of zero plate thickness and find that by itself the resulting H-equation is not complete. We also find an auxiliary condition, viz $\hat{n} \cdot \bar{H} = 0$ must be simultaneously satisfied in order to derive the correct solution. By suitably combining all of these equations we derive two new equations which are different from either the E or the H-equation and has some important and advnatageous features.

Although the thin plate problem has been studied by a number of authors [Mentzer, 1955; Ross, 1966] using the physical optics or GTD approach there still exists a need for deriving a self-existent numerical formulation of the problem which gives accurate results for arbitrary angles of incidence of the illuminating plane wave. The purpose of this paper is to discuss some aspects of numerical formulation and solution of this problem.

For a perfect electric conductor the H-equation takes the form

$$\bar{J} = \hat{n} \times \bar{H}_i + \hat{n} \times \int_S \bar{J} \times \nabla'\Phi \, dS', \qquad (1)$$

where $\bar{J}$ is the surface current density, $\bar{H}_i$ is the incident magnetic field, $\Phi = (4\pi r)^{-1} \exp(-jkr)$, $\hat{n}$ is the outward normal to the surface and r is the distance between the source and observation points. Throughout this work the source points will be denoted in prime, and the observation points in the unprimed coordinate systems.

Let us now consider a thin scatterer, e.g., a thin plate, as shown in Figure 1. Applying the H-equation (1) to the two surfaces of the plate we obtain

$$\frac{\bar{J}^+}{2} = \hat{z} \times \bar{H}_i(z_1^+) + \int_{S(-)} \bar{J}^-(\hat{z}\cdot\nabla'\Phi)\Big|_{S(+)} dS' \tag{2a}$$

$$\frac{\bar{J}^-}{2} = -\hat{z} \times \bar{H}_i(z_2^-) + \int_{S(+)} \bar{J}^+ (-\hat{z}\cdot\nabla'\Phi)\Big|_{S(-)} dS' \tag{2b}$$

where the superscripts (+) and (-) serve to indicate the top and bottom surfaces respectively, and the integrals over S' in (2) are interpreted in the Cauchy principal value sense. The kernel function in the integrand of (2) may be written

$$\hat{z}\cdot\nabla'\Phi = \hat{z}\cdot\nabla' \left(\frac{e^{-jkr}}{4\pi r}\right) = \frac{(1 + jkr)}{4\pi r^3}(z - z')\, e^{-jkr} \tag{3}$$

where $r = [(x - x')^2 + (y - y')^2 + (z - z')^2]^{1/2}$.

For a thin scatterer the solution of the H-equation would require the simultaneous solution of (2a) and (2b) for the two surface currents $\bar{J}^+$ and $\bar{J}^-$. However, for a small Δ these two equations are almost parallel, and hence their numerical inversion is unstable. By subtracting and adding the above two equations one can generate two new equations. The first equation which is for $(\bar{J}^+ - \bar{J}^-)$, is $O(1)$, while the second for $(J^+ + J^-)$ is $O(\Delta)$, Δ being the thickness of the plate.

Subtracting (2b) from (2a) we obtain

$$\bar{J}^+ - \bar{J}^- = 2\hat{z} \times \bar{H}_i \tag{4}$$

which is independent of Δ. More importantly, it does <u>not</u> contain the desired unknown for our problem, viz., $\bar{J}^+ + \bar{J}^- = \bar{J}_S$, and thus is of little use. To get an equation for $\bar{J}_S$ we add (2a) and (2b) and derive

$$\frac{\bar{J}^+ + \bar{J}^-}{2} = \hat{z} \times [\bar{H}_i(z^+) - \bar{H}_i(z^-)] + \Delta \int_S (\bar{J}^+ + \bar{J}^-)\, G\, dS' \tag{5}$$

where

$$\Delta = z^+ - z^-, \quad G = \frac{(1 + jkr_o)}{4\pi r_o^3} e^{-jkr},$$

and r_o is given by $r_o = [(x - x')^2 + (y - y')^2 + \Delta^2]^{1/2}$.

Although $\bar{H}_i(z^+) - \bar{H}_i(z^-)$ is of $O(\Delta)$, $(\bar{J}_+ + \bar{J}^-)/2$ given by (5) is nevertheless $O(1)$ since the integral in the r.h.s. of (5) is $O(1/\Delta)$. To show this we rewrite the above integral as

$$\int_S (\bar{J}^+ + \bar{J}^-)\, G\, dS' = \frac{\bar{J}^+ + \bar{J}^-}{2\Delta} + (\nabla_t^2 + k^2) \int_S (\bar{J}^+ + \bar{J}^-)\phi\, dS', \tag{6}$$

as $\Delta \to 0$.

$$\nabla_t^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{7}$$

The integral in the r.h.s. of (6) is bounded, and in fact turns out to be $O(1)$ in the limit $\Delta \to 0$. Substituting (6) in (5), letting $\Delta \to 0$, and equating the coefficients of the terms that are of $O(\Delta)$, one obtains the desired equation* for $\bar{J}_s$, viz.,

$$(\nabla_t^2 + k^2) \int_S \bar{J}_s\, \phi\, dS' = \frac{\partial}{\partial z} (\hat{z} \times \bar{H}_i)\Big|_S \tag{8}$$

Alternatively, this equation may also be derived by starting with the E-equation and making use of the condition $\hat{n} \cdot \bar{H} = 0$, although the latter condition is implied in the E-equation. Note also that (8), which is the limiting form of the H-equation for a thin scatterer, is simply the normal derivative of the original H-integral equation given in (1). Note also that this equation is not complete since the equations for the two scalar components of the surface current are uncoupled. If (8) were complete, it would be possible to solve for the two current components independently, irrespective of the geometrical shape of the plate. It is evident that the equations for the two components of $\bar{J}_s$ should be coupled except in the limiting case of an infinite plate.

An investigation of (8) reveals the cause of this difficulty, viz., that the solution of this equation does not necessarily satisfy the required boundary condition $\hat{n} \times \bar{E} = 0$ on the surface of the thin scatterer. Although for a solid scatterer the condition $\hat{n} \times \bar{H} = \bar{J}_s$ implies the satisfaction of $\hat{n} \times \bar{E}$, and hence, $\hat{n} \cdot \bar{H} = 0$ on the surface of the scatterer, neither of these is automatically satisfied when the scatterer is thin. However, it is possible to satisfy $\hat{n} \times \bar{E} = 0$ and derive a unique solution by solving the equation $\hat{n} \times \bar{H} = \bar{J}_s$ in conjunction with the auxiliary condition $\hat{n} \cdot \bar{H} = 0$.

The investigation also reveals that Equation (8) admits non-trival, homogeneous solutions. That is, non-zero solutions of this equation can be constructed even when its r.h.s. is identically zero. Furthermore, homogeneous solutions of (8) exist even though the matrix version of this equation is not singular and its determinant is not zero. As shown later these homogeneous solutions play a very important role.

In spite of the above features of (8), the equation is consistent with the E-equation as shown below. The E-equation for the thin plate may be written in the form

$$\hat{z} \times \{k^2 \int_S \bar{J}_S \,\phi\, dS' + \int_S (\bar{J}_S \cdot \nabla'_S)\nabla' \phi\, dS'\} + j\omega\varepsilon\, \hat{z} \times \bar{E}_i = 0 \quad (9)$$

Subtracting (8) from (9) we get, after simplification,

$$\nabla_t \{\hat{z} \cdot (\bar{H}_i + \nabla \times \int_S \bar{J}_S \,\phi\, ds') = 0 \text{ on S, or } \nabla_t(\hat{z} \cdot \bar{H}_{tot}) = 0 \text{ on S} \quad (10)$$

a result which is evidently true since $\hat{z} \cdot \bar{H}_{tot}$ = normal component of total $\bar{H}$ is zero on the surface S, implying that its tangential derivative is also zero. The conclusion is that (8) and (10) are required to be satisfied simultaneously in order to satisfy (9), the latter being a statement of the boundary condition $\bar{E}_{tan} = 0$ on S.

In the next section we develop a procedure for constructing a solution to (8) that simultaneously satisfies the auxiliary condition $\hat{z} \cdot \bar{H}_{tot} = 0$.

DERIVATION OF NEW EQUATIONS

Numerically, a straightforward method for combining (8) and (10) is to simultaneously solve a set of algebraic equations representing the discretized versions of these equations. Numerical experiments based on this approach have shown that it leads to erroneous results; however, a correct and consistent method for combining these equations has been developed and is outlined below. For the present, we restrict the derivation to the case of plane wave incidence, which is the case of most common interest. However, generalization of the method to an arbitrary incident wave is also possible by a slight extension of the method.

The strategy for generating the desired equations for $\bar{J}_S$ is as follows. First, the two scalar components of (8) are integrated and both the homogeneous and inhomogeneous solutions are included in the resulting integrated form. The inclusion of a homogeneous solution is necessary for completeness. This step also assures that J_x and J_y are properly coupled and that they are nonzero when either H_{yi} or H_{xi} or both (but not H_{zi}) are zero. Second, the unknown coefficients of the homogeneous terms in the two equations are related by invoking the $\hat{z} \cdot \bar{H}_{tot} = 0$ condition. Finally, these coefficients are determined by invoking the condition $\hat{\nu} \cdot \bar{J} = 0$, where $\hat{\nu}$ is a vector lying in the x,y plane and is normal to the contour of the plate.

Expressing $(\nabla_t^2 + k^2)$ explicitly as $\left(\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + k^2 \Phi\right)$, and solving the differential equation, one can derive the following equations for the scalar components J_x and J_y. Assuming plane wave incidence with $k_x^2 + k_y^2 = k^2$,

$$jk_{zi} \int J_x \Phi ds' = H_{yi} + \int_C A(k_x) k_x e^{j[k_x x + k_y y]} dk_x \qquad (11)$$

$$jk_{zi} \int J_y \Phi ds' = -H_{xi} + \int_C A(k_x) \sqrt{k^2 - k_x^2}\, e^{j[k_x x + k_y y]} dk_x \qquad (12)$$

where the integrals in (11) and (12) are spectral representations of the homogeneous solutions of the operator $(\nabla_t^2 + k^2)$, and $A(k_x)$ is as yet unknown. That (11) and (12) indeed simultaneously satisfy (8) and $\hat{z} \cdot \bar{H}_{tot} = 0$ may be verified by direct substitution.

At this point we note some rather important and useful features of (11) and (12). In contrast to the conventional E-equation given in (9), the new equations are uncoupled for the two components J_x and J_y and the kernel functions of (11) and (12) are identical. The above features allow one to use matrix sizes that are half that required by the E-equation. In addition, the fill-time for the matrix elements is smaller since the kernel function is simpler and its singularity less severe in comparison to the E-equation kernel. In light of the above, the new equations (11) and (12) would be expected to be more efficient numerically than the conventional E-equation.

NUMERICAL VERIFICATION

The primary purpose of this paper was to introduce the underlying concepts leading to the derivation of the new equations for the thin plate problem, deferring the presentation of extensive numerical results to a later publication. However, for the sake of completeness it seems appropriate to include some representative examples to illustrate the use of the uncoupled integral equations (11) and (12). With this in mind we briefly discuss in this section the problems of diffraction by an infinitely long strip and by a square plate, both of which are electrically thin.

THIN STRIP (H-POLARIZATION)

Consider the geometry of the strip problem shown in Figure 2. Let us first consider the case of an incident plane wave with its $\bar{H}$ vector parallel to the edge of the strip, i.e., a H-polarized incident wave. It is well known that this problem is reducible to a scalar one via the introduction of a scalar potential proportional to H_y. The only non-zero component of the surface current is J_x, which may be shown to satisfy the integral equation

$$\int J_x \left. \frac{\partial^2 g}{\partial z \partial z'} \right|_{\substack{z' = 0 \\ z = 0}} dx' = \left. \frac{-\partial}{\partial z} (\hat{z} \times \bar{H}_i) \right|_{z = 0} \tag{13}$$

with $g = \frac{1}{4j} H_o^{(2)} (k|\bar{\rho} - \bar{\rho}'|)$, the two-dimensional Green's function, $\rho = \sqrt{x^2 + z^2}$, $H_i = \hat{y}\, e^{-jkx \cos\theta + jkz \sin\theta}$ and θ is the incident angle. Equation (13) is equivalent to the E-equation for the thin strip problem. and can be discretized using the moment method [Harrington, 1968] and solved via the usual matrix inversion procedures. The results of this computation afford a convenient comparison with those derived from the alternate form of the equation, i.e., the integrated version, which will now be discussed. For this two-dimensional case, the integrated equation degenerates to

$$jk_{zi} \int_o^a J_x \, g \, dx' = H_{yi} + A \cos k(x - a/2) + B \sin k\,(x - a/2) \tag{14}$$

where a is the half-width of strip and A, B are unknowns. Equation (14) is a slightly rearranged version of (11) with g the same as in (13). This equation is again transformed into a matrix form and solved for J_x with A and B as parameters. Finally, these two unknowns A and B are determined by imposing the two end conditions, viz., $J_x = 0$ at $|x| = a$.

It should be pointed out that slight differences in the numerical solutions of (13) and (14) are unavoidable because of the way they are discretized. Specifically, in the numerical solution of (14) the end conditions, viz., $J_x = 0$, are imposed not exactly at $|x| = a$ but at $|x| = (a - \Delta x/2)$, where Δx is the width of the segment used in the discretization of the interval $|x| < a$. Thus the width of the strip effectively becomes $2a - \Delta x$ rather than $2a$. On the other hand, while solving (13) the boundary conditions on J_x at $|x| = a$ are not imposed directly; instead, it is tacitly assumed that the above boundary conditions are automatically satisfied. Once again, because of discretization, it appears that the J_x computed from (13) appears to go to zero at $|x| = (a + \Delta x/2)$ and hence the effective width of the strip appears to be $2a + \Delta x$. Keeping this in mind we note that the two curves shown in Figure 3 appear to agree reasonably well. It should be mentioned that similar differences arise in the solution of a thin-wire antenna problem using Pocklington's equation [Poggio and Miller, 1973] which corresponds to (13), and Hallen's equation [Collin and Zucker, 1969] which is the counterpart of (14). However, this problem can be alleviated by using triangular rather than rectangular pulses for basis functions for the representation of J_x.

THE THIN-PLATE PROBLEM

The numerical advantages of the new form of Equations (11) and (12) become more evident when one deals with a two-dimensional structure, e.g., a thin plate (or the complementary problem of an aperture in a thin plate). We consider the problem of scattering of a plane wave incident at an arbitrary angle on a perfectly conducting square plate whose linear dimension is a. As shown in Figure 4, the z direction coincides with the normal to the surface of the plate. Although we report only some illustrative results of the surface current computation for an incident E-vector parallel to the plate, it should be mentioned that extensive results have been obtained for RCS computations and will appear in a later publication.

Using the moment method in connection with (11-12) obtain the following two equations for J_x and J_y

$$jk_{zi} \sum_{m=1}^{M} \sum_{n=1}^{M} J_x(n,m)\ \phi(k,\ell;n,m)\ \Delta x \Delta y = H_{yi}(k,\ell) + \sum_{p=1}^{4M} A_p\ k_{xp}\ e^{+jk_{xp}x+jk_{yp}y}\Big|_{k,\ell} \tag{15}$$

$$jk_{zi} \sum_{m=1}^{M} \sum_{n=1}^{M} J_y(n,m)\ \phi(k,\ell;n,m)\ \Delta x \Delta y = -H_{xi}(k,\ell) + \sum_{p=1}^{4M} A_p\ k_{yp}\ e^{+jk_{xp}x+jk_{yp}y}\Big|_{k,\ell} \tag{16}$$

In the above equations the surface current distributions have been represented by two-dimensional pulse functions and point matching has been used in conjunction with this representation to derive the matrix elements $\phi(k,\ell,n,m)$. As usual, special care is required in the evaluation of the self-patch integral which is carried out analytically to yield

$$\phi(k,\ell;k,\ell)\Delta x \Delta y = \int^{\Delta x} \int^{\Delta y} \phi(k,\ell;x,y)\ dxdy = \frac{\lambda s}{\pi M}\ \ell n\ (1 + \sqrt{2}) \tag{17}$$

where $s = a/\lambda$ and $\Delta x = \Delta y = a/M$. The only other comment regarding the numerical transformation of the integral equations (15) and (16) into a matrix form pertains to the choice of k_{xp}, the x-components of the wave numbers of the plane wave forcing functions appearing in the r.h.s. of (11) and (12). In this computation the k_{xp}'s were assumed to be uniformly distributed, i.e.,

$$k_{xp} = \frac{2\pi}{\lambda} \cos\left(\frac{2\pi}{p} + \delta\right)\ ;\quad p = 1, \ldots 4M$$

where δ, which can be an arbitrary angle that was set equal to 5°. Note that 4M of these forcing functions are required in order to satisfy the boundary conditions to be imposed on J_x and J_y along the edges parallel to the y and x axis, respectively. Numerically the above boundary conditions imply that

i) $J_x = 0$ at extreme top and bottom rows of patches;

ii) $J_y = 0$ at extreme left and right columns of patches.

The matrix equation for J_x and J_y, each of which is $M^2 \times M^2$ in size, can now be solved numerically in terms of the constants A_n's. These constants are in turn determined, and the solution for J_x and J_y completed by applying the boundary conditions on J_x and J_y as indicated above.

We present some numerical results for the two components of the current distribution on a square plate for a normally incident plane wave with E polarized parallel to the plate along the y axis. The plate is one wavelength square, i.e., $a = \lambda$, and was sectionalized into 81 subsections. The results however do not substantially deteriorate when the number of subsections is reduced to only 36.

A general program, which was written for handling arbitrary incident angles, was used to derive the solution and no advantage was taken of the symmetry of the problem, although the matrix size could be reduced considerably for this special case. Figure 5 shows that the currents exhibit the proper behavior at the edges, viz., the normal component of the current goes smoothly to zero at the edge, and the parallel component exhibits a singular-type behavior. It was also verified that J_x was zero in the middle patches and was asymmetric about $y = a/2$, while J_y was symmetric. This behavior of the current distribution should be expected for incident E parallel to the y axis. Though not shown here, we found by increasing the size of the plate, the current distribution near the center of the plate slowly tends to the physical optics current $2\hat{n} \times \bar{H}_i$ as the plate size is increased. The estimated size of the plate is approximately 5λ square before this becomes an accurate approximation.

The general program was also used to solve the oblique incidence case and typical results are shown in Figure 6, for an angle of incidence of 85° from the normal.

Finally, it was found that the equations are quite stable and numerically well-behaved for all of the plate sizes and angles of incidence studied here.

CONCLUSIONS

In this paper we have examined the limiting case of the H-equation for the thin-plate problem and have found an additional condition, viz., $\hat{n} \cdot \overline{H}_{tot} = 0$, must be imposed on this equation (where $\hat{n}$ is the normal to the surface) in order to derive a unique solution to the problem. It is shown how these equations can be combined to derive a new set of equations for the surface current components. These new equations have several advantages that are pointed out in the text. Some numerical results illustrating the application of the new equations to the thin-strip and the square-plate diffraction problems are presented.

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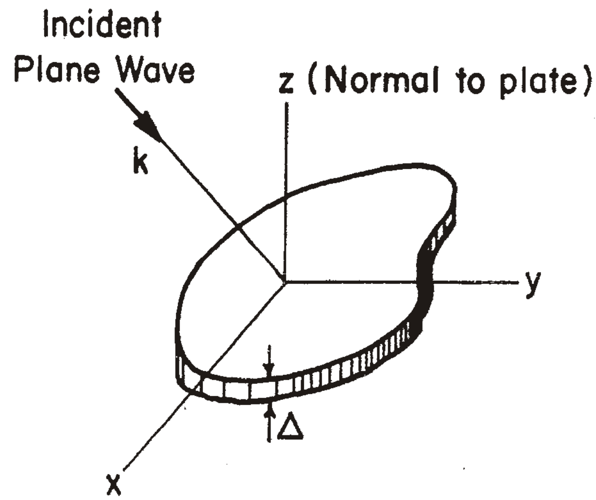


Figure 1. Thin plate diffraction problem. The plate lies in the x-y plane. Its top surface is designated by $S^{(+)}$ and the bottom surface by $S^{(-)}$.

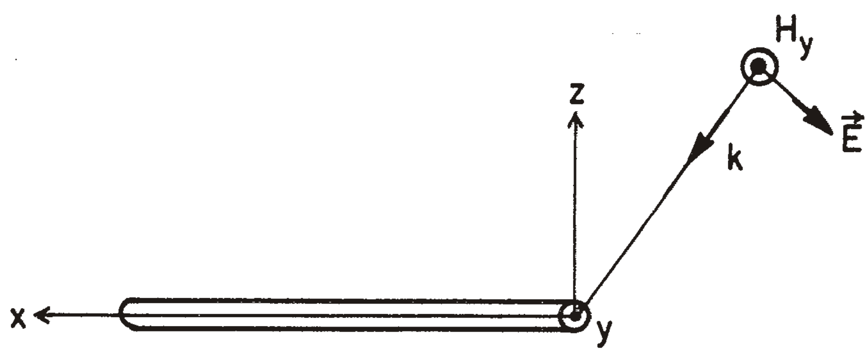
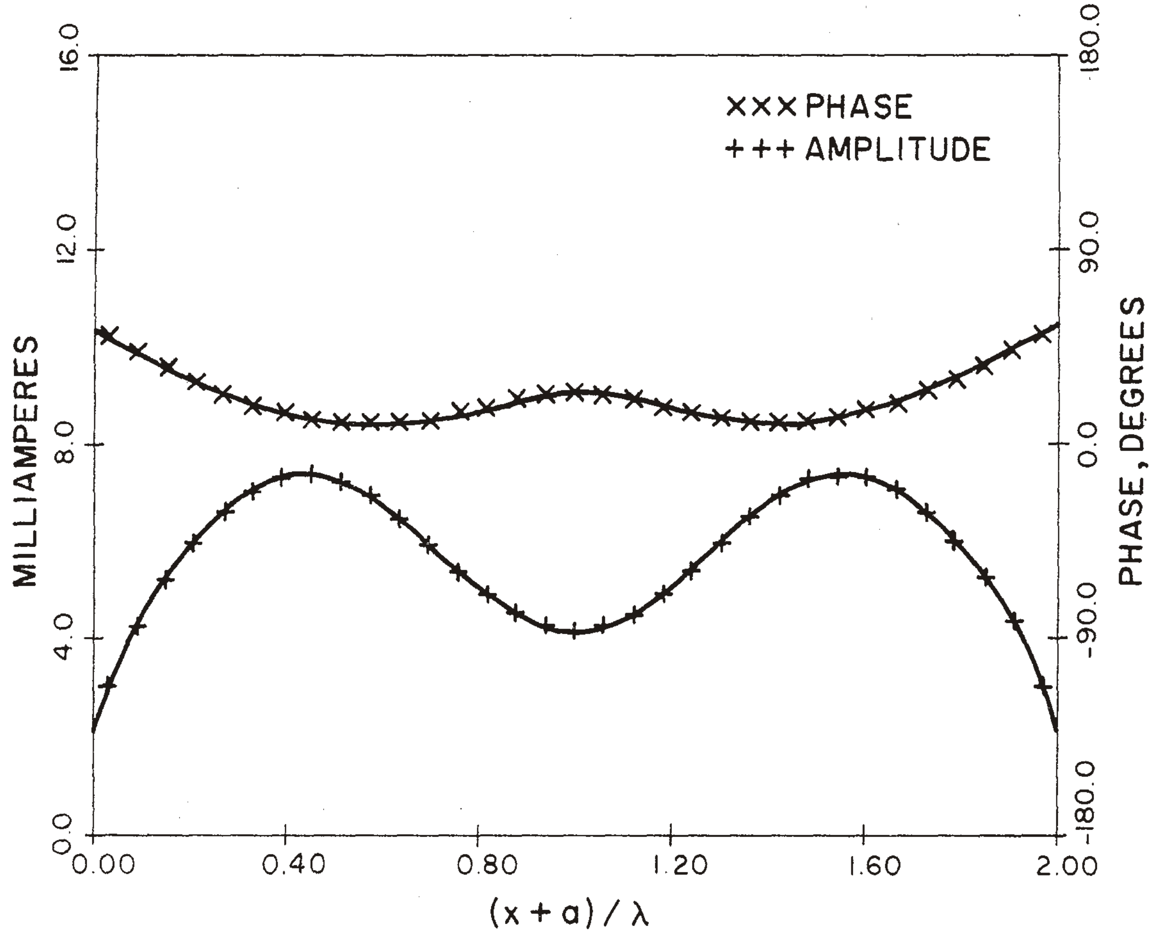


Figure 2. The thin strip diffraction problem (H-polarization).



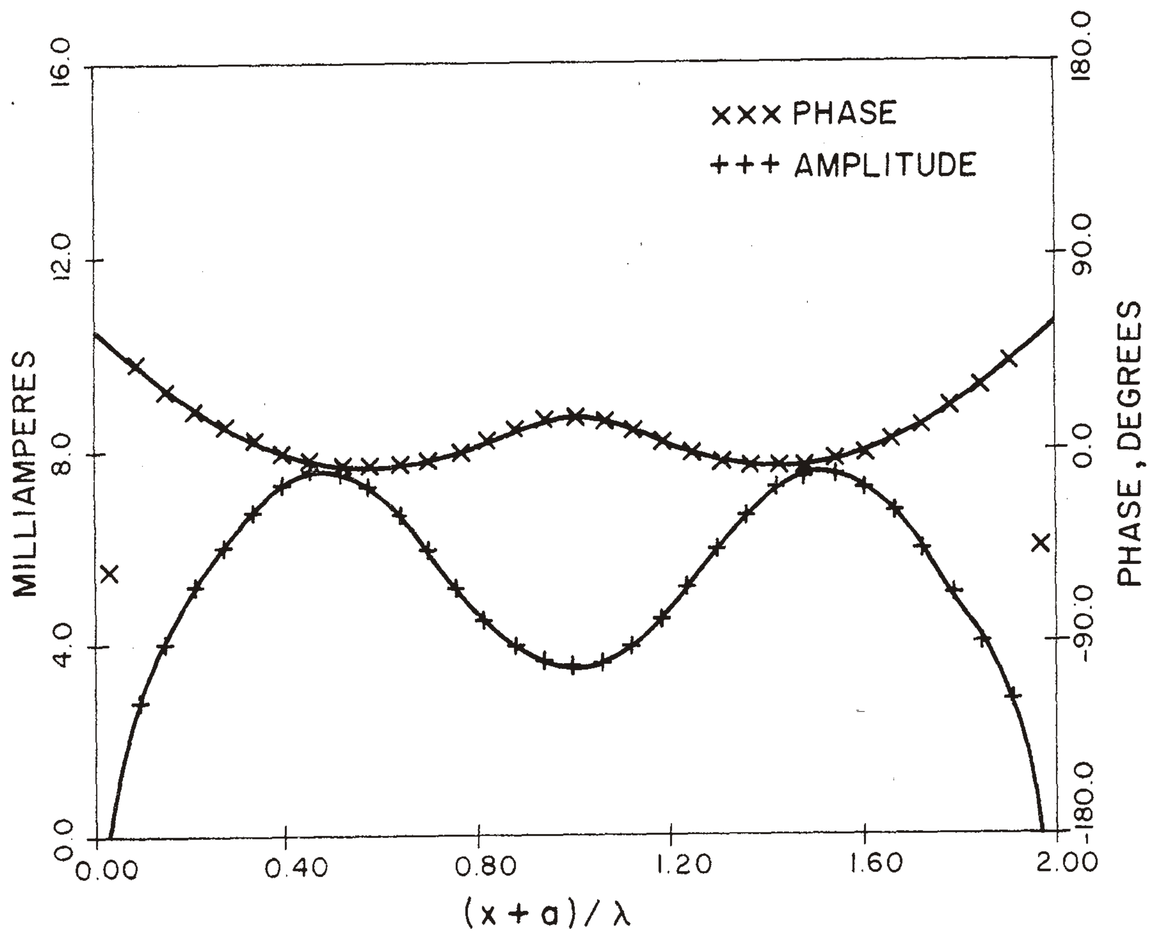


Fig. 3. Amplitude and phase of current distribution on a thin strip with $a/\lambda = 2$ and $\Lambda = 0.01$, due to a normally incident plane wave using pulse approximation and 33 sections. a) as obtained from (13), b) as obtained from (14)

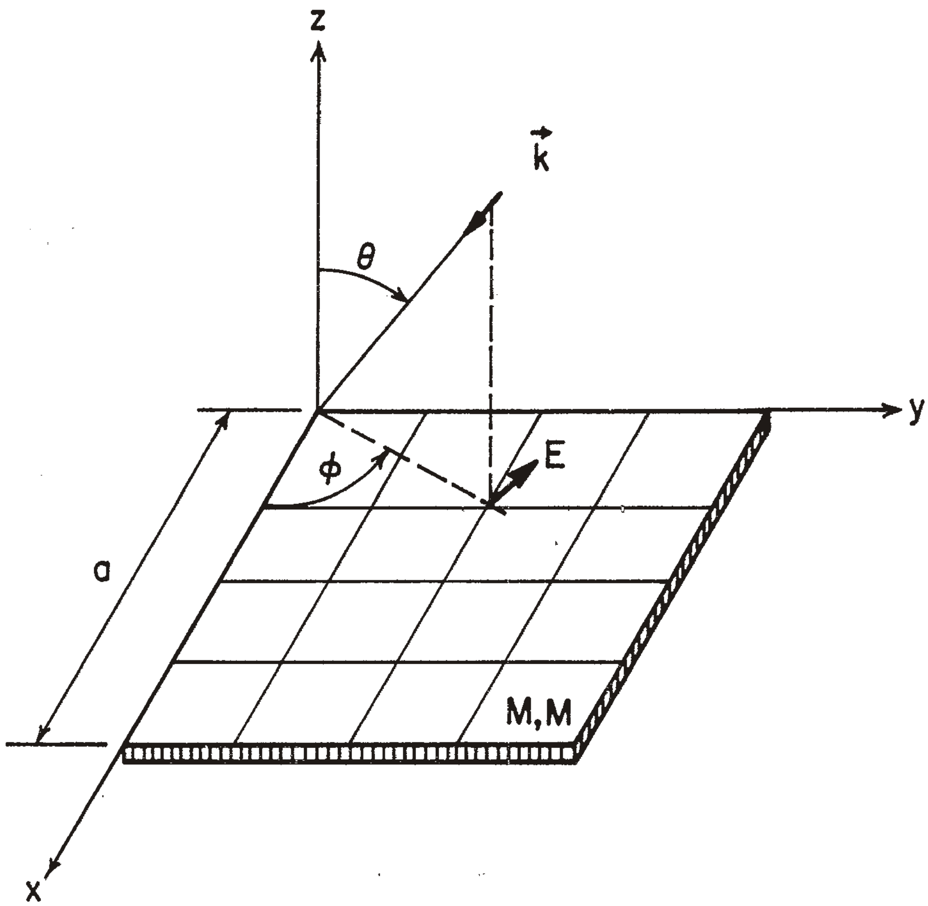


Figure 4. The thin square plate diffraction problem (E-polarization).

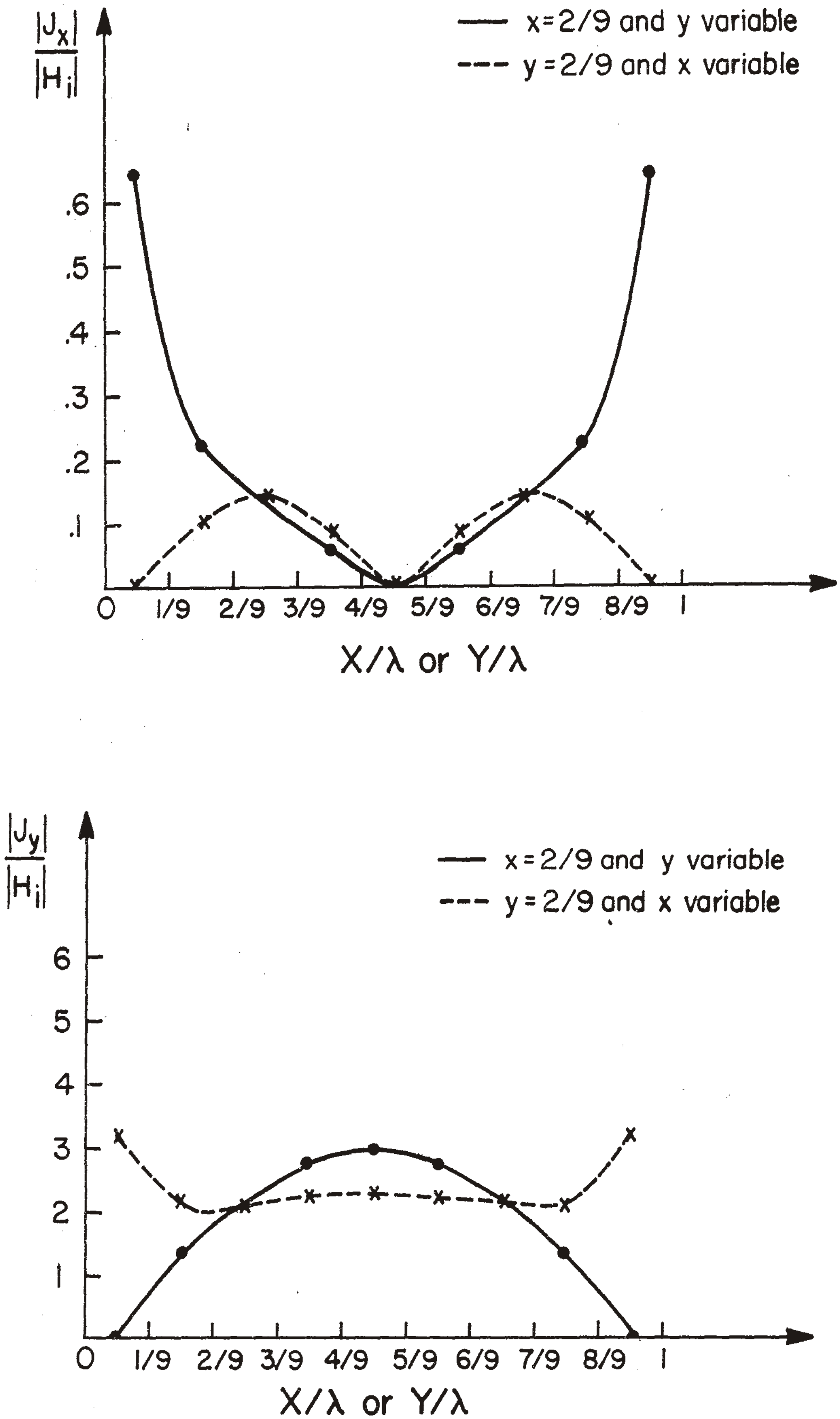
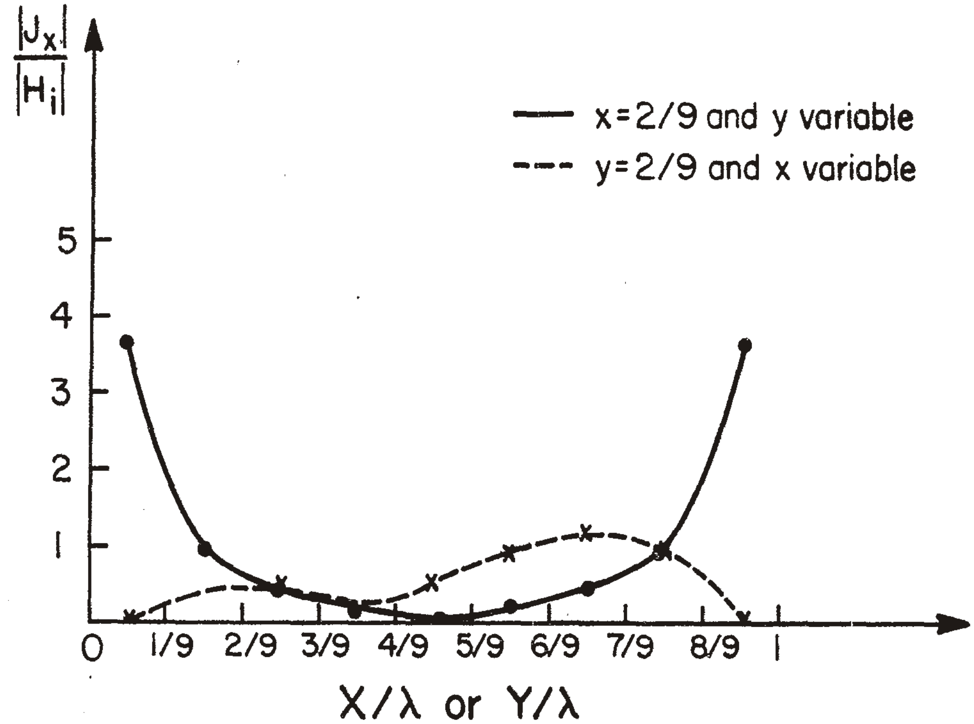


Figure 5. Amplitude of current distribution on a thin plate for normal incidence. $\theta = 0°$, $\phi = 0°$. (a) J_x component (b) J_y component.



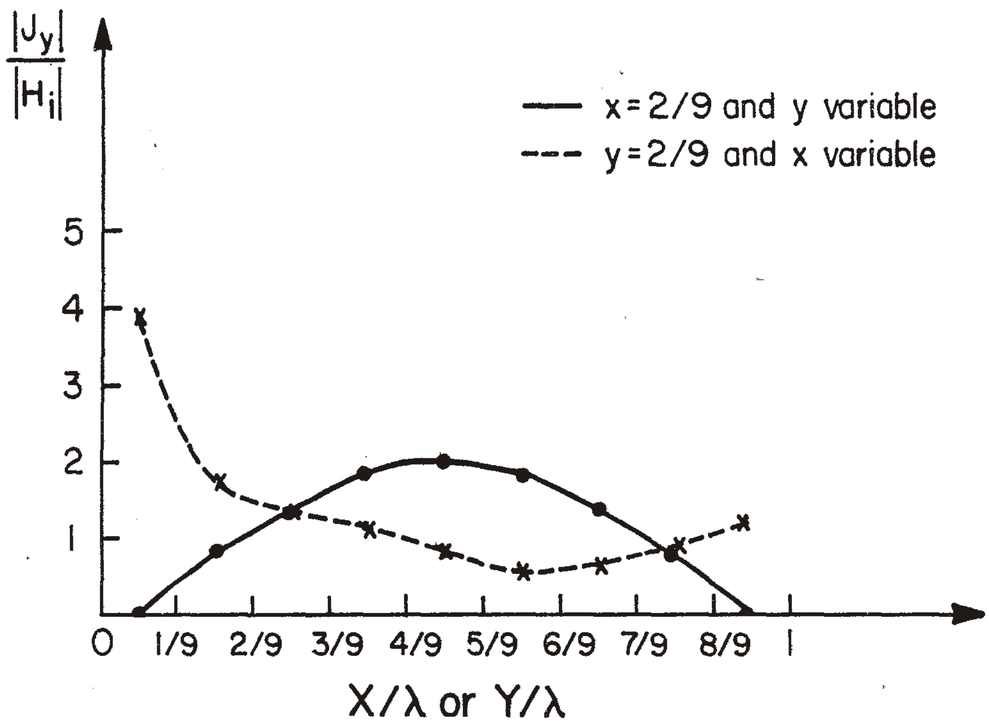


Figure 6. Amplitude of current distribution on thin plate for oblique incidence. $\theta = 85°$, $\phi = 0°$. (a) J_x component (b) J_y component.