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MULTICONDUCTOR TRANSMISSION LINES

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Abstract

A scattering-matrix formalism is presented for analysis of transmission-line sections consisting of an arbitrary number of conductors.

multiconductors, transmission lines

ABSTRACT

Following a review of the properties of TEM field distributions on multiconductor transmission lines, the scattering matrices for sections consisting of an arbitrary number of conductors are derived in a close analogy with the well-known formalism for single transmission lines. It is shown that the so-called one-by-one modes are of more general use than the odd-and-even modes. The formulas for scattering matrices of two-sided and one-sided sections are expressed directly in terms of the induction coefficient matrix. These formulas are well suited for numerical evaluation on a digital computer. The analytical treatment is also possible and practical for two conductors above the ground. Examples of a directional coupler and an all-pass network are carried out in detail. A simple design procedure for the maximally-flat phase shifter having arbitrary phase concludes the report.

PREFACE

This report is revised and extended edition of the report entitled "Scattering Matrix Approach to the Multiconductor Transmission Lines", originally published by the Department of Electrical Engineering in December, 1970. It treats the steady-state response of single-velocity waves on the multiconductor transmission lines (abbreviated as MTL). The presentation is of a tutional nature although also some original material is presented, which has not been published elsewhere. The time and space limitations prevented this report from including the treatment of multivelocity waves.

MULTICONDUCTOR TRANSMISSION LINES

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Section 1.

ELECTROMAGNETIC FIELD ON MULTICONDUCTOR TRANSMISSION LINES

The final goal of this report is establishing useful circuit relationships for microwave circuits consisting of multiconductor transmission lines (abbreviated MTL). The best way to derive the circuit relationships is to start from the knowledge of the electromagnetic field distribution between conductors. The field distribution is, in turn, determined by the Maxwell's equations and the particular boundary conditions. In this first Section, we will give a brief review of the properties of the electromagnetic field distribution on the MTL. However, the properties will primarily be merely stated, without an attempt to derive them exactly.

A simple example of the MTL is shown in Figure 1.1. This particular line consists of a rectangular shielding cylinder and two conductors of a circular cross section. The space between the conductors is filled with a homogeneous and isotropic dielectric, e.g. air.

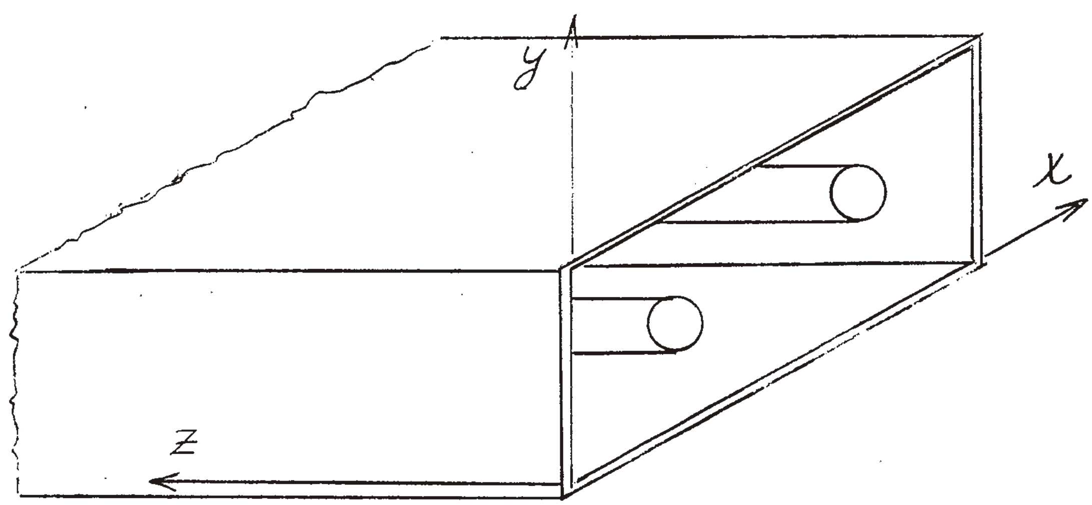


Figure 1.1

The system of coordinates will be oriented so that x and y describe transverse variations, and z is pointing in the direction of energy propagation.

The electromagnetic field inside this structure for guiding electromagnetic waves must satisfy the Maxwell's equations and the boundary conditions. As we are considering time-harmonic fields, the electric and magnetic fields will be governed by Helmholtz's vector wave equation (see e.g. [18], p. 373):

$$\nabla^2 \vec{E} + k^2 \vec{E} = 0 \quad ; \quad \nabla^2 \vec{H} + k^2 \vec{H} = 0 \tag{1.1}$$

where k is the intrinsic propagation constant of the medium:

$$k = \omega\sqrt{\mu\varepsilon} \ . \tag{1.2}$$

Assuming that the wave propagates along the z-direction with a propagation constant γ :

$$\vec{E}(x,y,z) = \vec{E}(x,y)\ e^{\pm\gamma z} \tag{1.3}$$

we may partition the operator ∇^2 into its transverse and longitudinal parts:

$$\nabla^2 = \nabla^2_{xy} + \frac{\partial^2}{\partial z^2} = \nabla^2_{xy} + \gamma^2 \ . \tag{1.4}$$

(1.1) thus becomes

$$\nabla^2_{xy} \vec{E} + (\gamma^2 + k^2) \vec{E} = 0 \ ; \ \nabla^2_{xy} \vec{H} + (\gamma^2 + k^2) \vec{H} = 0 \tag{1.5}$$

For given boundaries (e.g. the boundaries chosen in Figure 1.1) there is a large number of solutions satisfying (1.5). As it is well known, one can classify these solutions into TEM, TE, TM and HEM types (see [5]). The last three types may have infinitely many distinct solutions for a given structure of conductors. Each solution which alone satisfies given boundary conditions is called a mode.

The fields with which this report is concerned are TEM waves. These waves have a common property in that their propagation constant γ is equal to the intrinsic propagation constant k of the medium between the conductors;

$$\gamma^2 + k^2 = 0 \quad \text{or} \quad \gamma = \pm jk \quad . \tag{1.6}$$

Strictly speaking, the pure TEM field may exist only on a system with perfect conductors. Practical transmission lines have conductors with only finite conductivity. Any current density J_z flowing along the conductors in z direction will produce the z component of the electric field E_z according to Ohm's law

$$J_z = \sigma E_z \tag{1.7}$$

where σ is the conductivity of the material. Therefore, if $E_z \neq 0$, the field is not a TEM. Usually the z-component is so small in comparison with transverse components that the actual field configuration, although not a pure TEM, is a good approximation of the TEM type of field.

When (1.6) is substituted in (1.5), one finds that the electric and magnetic fields of a TEM wave satisfy Laplace's equation in the transverse plane:

$$\nabla^2_{xy} \vec{E} = 0 \quad \text{and} \quad \nabla^2_{xy} \vec{H} = 0 \quad . \tag{1.8}$$

From the above we conclude that the transverse distribution of the TEM field even at microwave frequencies, is the same as that of the static field, which has the same boundaries.

On a MTL consisting of N + 1 conductors (the " + 1" conductor is the shielding cylinder in the case from Figure 1.1), there are N distinct TEM modes possible. This property may be derived from the principle of superposition as follows.

The differential operator ∇^2_{xy} from (1.8) is a linear operator, i.e.

$$\nabla^2_{xy}\left(\alpha f_1(x,y) + \beta f_2(x,y)\right) = \alpha \nabla^2_{xy} f_1(x,y) + \beta \nabla^2_{xy} f_2(x,y) \quad . \qquad (1.9)$$

Suppose that one has found two functions (= modes) f_1 and f_2 which independently satisfy the differential equation of the type (1.8):

$$\nabla^2_{xy} f_1(x,y) = 0 \quad \text{and} \quad \nabla^2_{xy} f_2(x,y) = 0 \, . \qquad (1.10)$$

Due to the linearity of the operator ∇^2_{xy}, any linear combination of f_1 and f_2 will also satisfy the differential equation. The physical meaning of this statement can be illustrated with the example shown in Figure 1.2.

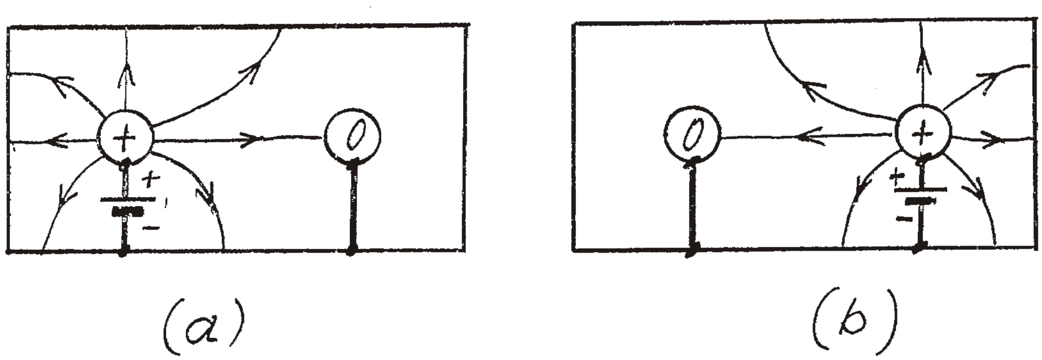


Figure 1.2

In Figure 1.2 one can see two different excitations of a two-conductor-plus-ground system. In Figure 1.2(a) conductor 1 is held at the potential +1 Volt above the ground, while conductor 2 is held at the ground potential. Likewise, in Figure 1.2(b) conductor 2 is held at the potential +1 Volt, while conductor 1 is grounded. The two different problems shown in Figure 1.2 have to be solved separately, starting from Laplace's differential equation and from the given potential distribution on the conductors. Let us assume that we have found the solutions to these two problems, and denoted them by ϕ_1 (x,y) and ϕ_2 (x,y):

$\phi_1(x,y)$---potential distribution on the MTL when conductor 1 is at +1V and conductor 2 is at zero potential,

$\phi_2(x,y)$---potential distribution on the MTL when conductor 2 is at +1V and conductor 1 is at zero potential.

Once we know these two solutions, ϕ_1 and ϕ_2, for a given (2 + 1) conductor system, we can use them for expressing the most general distribution of the potential in this system. For the sake of example, Figure 1.3 shows the same system of two conductors driven in such a manner that conductor 1 is held at the potential -0.5V and conductor 2 is at +2.5V.

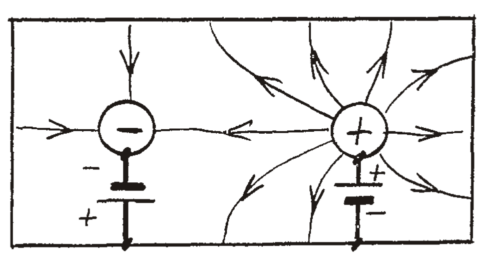


Figure 1.3

In order to find the potential distribution ϕ_3 for the situation shown in Figure 1.3 we do not have to solve again the differential equation subject to the new boundary conditions. We will simply use the superposition of the two basic modes ϕ_1 and ϕ_2 as follows:

$$\phi_3 = -0.5\ \phi_1 + 2.5\ \phi_2$$

The same reasoning may be extended to systems with any number of conductors. In order to find the general solution to the field distribution on the system of N + 1 conductors, it is necessary to solve N separate problems; each problem consists of one of the conductors being held at the potential 1 Volt, while all the other conductors are at zero potential. Each of the solutions is a mode for a MTL at hand. The best name for these modes seems to be "the one-by-one modes." These one-by-one modes will be compared with the odd-and-even modes in Section 6.

A convenient property of the TEM field is that it is possible to define the voltages between conductors as the line integrals of the electric field. For the voltage of conductor 1 with respect to ground (= shield) in Figure 1.1 we have

$$V_{10} = -\int_0^1 \vec{E} \cdot d\vec{l} \quad . \tag{1.11}$$

As long as the integration path lies in the (x,y) plane, there will be no magnetic flux crossing this plane (since $H_z = 0$ for a TEM field), so that the result of integration in (1.11) is independent of the chosen path (compare e.g. [18], p. 267). Therefore, V_{10} is a meaningful, unique number, which is called the voltage of conductor 1 with respect to the ground.

As far as the currents on conductors are concerned, the TEM field has a nice property that the currents flow only in z direction. That is, the transverse component of the surface current on a conductor would require the existence of a z-component of the magnetic field, which is zero by the definition of the TEM type of the field. As is well known, the boundary condition which relates the surface current and the tangential magnetic field in the vicinity of a perfect conductor is the following (see e.g. [18], p. 257):

$$\vec{n} \times \vec{H} = \vec{J} \quad . \tag{1.12}$$

As has already been mentioned, the resistivity of conductors is one reason why the actual field on MTL cannot be of the pure TEM type. Another important physical situation which prevents the existence of the pure TEM field is the <u>inhomogeneity of the dielectric material</u> between the conductors. When one portion of the cross section of the MTL is filled with a dielectric material different from that of the rest of the cross section, a pure TEM field cannot satisfy the boundary conditions of such a system, as the following example shows.

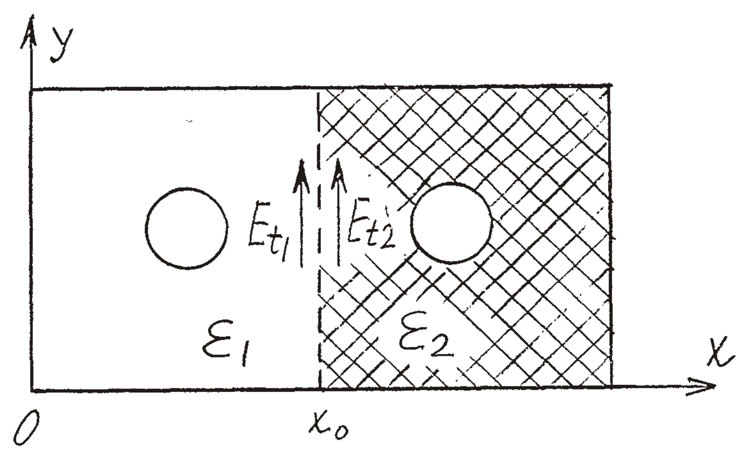


Figure 1.4

The same configuration of conductors as in Figure 1.1 is again shown in Figure 1.4 with the addition of filling the right-hand portion of the line with a dielectric ε_2 which is different from ε_1. It is well known that

Maxwell's equations require that the tangential component of the electric field at the interface of two mediums must be continuous:

$$E_{t1} = E_{t2} \tag{1.13}$$

This condition must be satisfied for every point on the interface, which means for every z. Now, if the propagation constants in mediums ε_1 and ε_2 follow from (1.6) we must have

$$\beta_1 = \omega\sqrt{\mu_o\varepsilon_1} \quad \text{and} \quad \beta_2 = \omega\sqrt{\mu_o\varepsilon_2} \quad . \tag{1.14}$$

If $\varepsilon_1 \neq \varepsilon_2$ then also $\beta_1 \neq \beta_2$ so that the propagation constants of pure TEM waves in two mediums are different. The continuity condition (1.13) would require

$$E_{t1}\ (x_o,y)\ e^{-j\beta_1 z} = E_{t2}\ (x_o,y)\ e^{-j\beta_2 z} \quad . \tag{1.15}$$

Obviously, when E_{t1} and E_{t2} have any non-zero values, (1.15) cannot be satisfied for every z. Therefore, pure TEM waves cannot exist on a system with inhomogeneous dielectrics. The field which will be actually established on such a MTL will have the same propagation constant β in both mediums. For the case that $\varepsilon_1 < \varepsilon_2$, the propagation constant will be situated somewhere between the following limits

$$\omega\sqrt{\mu\varepsilon_1} < \beta < \omega\sqrt{\mu\varepsilon_2} \tag{1.16}$$

The field configuration will contain a non-zero component E_z, so that the field will be not a pure TEM. However, as long as the transverse dimensions of the line are small in comparison to the wavelength, E_z is very small in comparison with transverse components of E. This is the reason why such a case is often referred to as the "quasi-TEM" field distribution. An important case belonging to this class of field is the field on a microstrip transmission line.

Inequality (1.16) states only that the propagation constant β will have a value somewhere between the values corresponding to two dielectric constants ε_1 and ε_2. When the system with inhomogenous dielectrics from Figure 1.4 is driven in a way indicated in Figure 1.2(a), most of the energy will be stored in the left part of the line, so that the propagation constant of this mode will be close to the value $\omega\sqrt{\mu\varepsilon_1}$. On the other hand, the mode from Figure 1.2(b) would have most of the energy located in the dielectric ε_2, so that the corresponding propagation constant will be close to the value $\omega\sqrt{\mu\varepsilon_2}$. In general, it comes out that each quasi-TEM mode on the MTL with inhomogeneous dielectrics has a slightly different propagation constant. This situation is often referred to as multivelocity waves, and it will be mentioned again in Section 3.

We have established the fact that the electromagnetic field on the microstrip transmission lines is not strictly TEM. Therefore, in order to find the exact solution to the field distribution one has to use Helmholtz's equation (1.1), and not Laplace's equation (1.8). However, most of the investigators use Laplace's equation because it is simpler, and it is still a good approximation for practical use. The best known solution of Laplace's equation for the single-conductor microstrip transmission line is by Wheeler [21], which reference also contains a number of useful diagrams. Another solution of the Laplace's equation for the microstrip transmission line is by Yamashita and Mittra [23]. They use a variational formulation which is very simple for practical computations. For the two-conductor microstrip system, Laplace's equation has been solved successfully by Bryant and Weiss (the method of solution is explained in [2]; many additional diagrams are published in [16]).

The exact solution to Helmoltz's equation on a microstrip transmission line was first published by Zysman and Varon [22]. They showed that in order to satisfy boundary conditions for inhomogeneous dielectrics, the field must be of the HEM type (hybrid electromagnetic i.e. a combination of TE and TM fields). Their numerical results indicate that the propagation constant exhibits a dispersion. However, this effect is negligible below the frequency 2 GHz (for the alumina substrate of thickness 0.05"). More recent solution of the Helm holtz's equation on microstrip has been presented by Mittra and Itoh [24].

Section 2.

COEFFICIENTS OF POTENTIAL, INDUCTION, AND CAPACITANCE

The electric field of the TEM wave on a multiconductor transmission line (MTL) satisfies Laplace's equation in the transverse plane. Therefore, the computation of capacitance coefficients may be borrowed from the books dealing with electrostatic problems. The fact that the capacitances were computed for a static situation does not make the microwave application of the MTL any less accurate or less exact. As long as we are having TEM waves, the use of static capacitances is legitimate and exact, because the transverse distribution of field is governed by Laplace's equation. It is well known that for the given boundary conditions, the solution of Laplace's equation is unique; therefore, the microwave TEM field and the static field look the same as long as they both have the same boundary conditions.

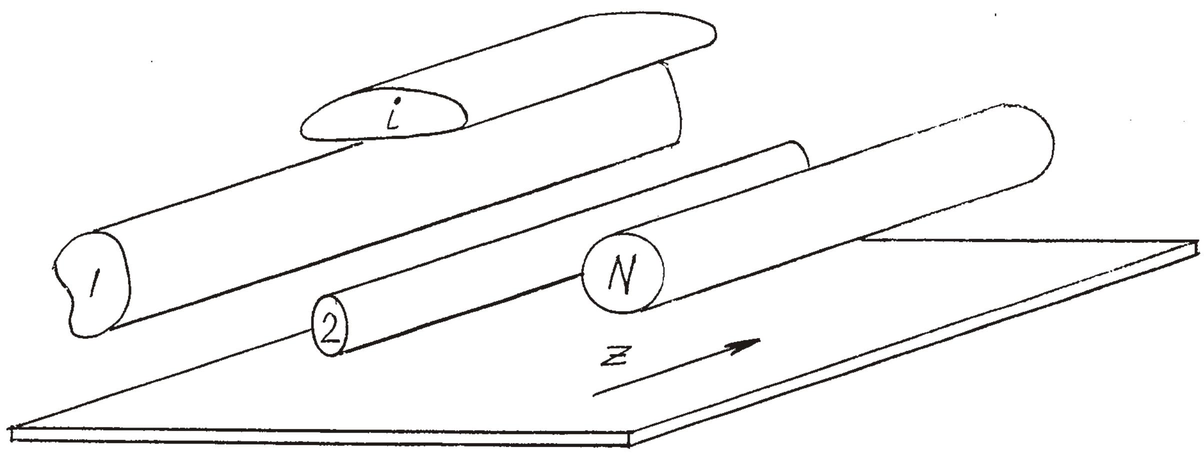


Figure 2.1

Figure 2.1 represents the system of N straight conductors of arbitrary shape above the ground plane. We want to investigate the distribution of potentials and charges on such a system. Denote:

q_i ... line charge of i-th conductor (in Coulombs/meter)

V_i ... voltage of i-th conductor w.r. to ground.

If one takes the potential of the ground plane to be zero, then the potential of the i-th conductor is also equal to V_i. This potential is a linear function of all the charges:

$$V_i = P_{i1}q_1 + P_{i2}q_2 + \cdots + P_{ii}q_i + \cdots + P_{iN}q_N \text{ for } i = 1, 2, \cdots, N. \quad (2.1)$$

The constants of proportionality P_{ij} are called coefficients of potential (see [18], p. 315). They are determined solely by the geometrical configuration of conductors (their shapes and distances). Except for a few simple cases, the exact computation of coefficients P_{ij} is a difficult problem for which there is usually no closed form solution, so that numerical methods must be used. But if the analytic solution is too difficult one can always build the physical model and determine the coefficients P_{ij} experimentally.

For a system of linear equations such as (2.1) the matrix notation is of great advantage. We define the N-dimensional voltage vector:

$$|V\rangle = \begin{pmatrix} V_1 \\ V_2 \\ \vdots \\ V_N \end{pmatrix}, \quad (2.2)$$

and the charge vector:

$$|q\rangle = \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_N \end{pmatrix}. \quad (2.3)$$

Their relationship is determined by the N X N square matrix $\underline{P}$ of induction coefficients

$$\underline{P} = \begin{pmatrix} P_{11} & P_{12} \ldots P_{1N} \\ \cdot & \cdot \; \cdot \; \cdot \; \cdot \; \cdot \\ P_{N1} & P_{N2} \ldots P_{NN} \end{pmatrix} \tag{2.4}$$

The equation (2.1) can be cast in a more compact notation

$$|V\rangle = \underline{P} \; |q\rangle \; . \tag{2.5}$$

This kind of notation originates from Dirac and is widely used in physics to describe linear systems (see [15], p. 245). The advantage of this notation in comparison with the common matrix notation is in the fact that one can immediately tell whether a given algebraic form represents a column vector, row vector, matrix (=operator), or a scalar; e.g. the column vector $|q\rangle$ has its dual row vector which is denoted by $\langle q|$ (transpose conjugate of $|q\rangle$)

$$\langle q| = (q_1^*, q_2^*, \ldots q_N^*) \; .$$

One can obtain interesting results with this notation when he attempts to compute the stored energy in the system of conductors from Figure 2.1. The stored electrostatic energy per unit length of such a system is (see e.g. [18], p. 103)

$$U_E = \frac{1}{2} \sum_{i=1}^{N} q_i V_i \; . \tag{2.6}$$

According to the rules of matrix multiplication (row vector times column vector), (2.6) can be written as

$$U_E = \frac{1}{2} \langle q|V\rangle \; , \tag{2.7}$$

since q_i's are real numbers. When the value of vector $|V\rangle$ from (2.5) is substituted in the above scalar product, one obtains

$$U_E = \frac{1}{2} \; \langle q|\underline{P}|q\rangle \; . \tag{2.8}$$

The stored energy of the electromagnetic system from Figure 2.1 is equal to the work required to move the charges from the ground plane (= zero potential) to their present positions above the ground plane. Every charge is resisting this movement, due to its image which is attracting it toward the ground plane. The total work done is positive, no matter what combination of charges q_i is there. A matrix which always gives a positive result to the expression (2.8)

$$0 < \langle q|\underline{P}|q\rangle \quad \text{for any } |q\rangle \tag{2.9}$$

is called positive definite. Such a matrix $\underline{P}$ has a property that all its eigenvalues are real and positive. A positive definite matrix with real elements may be divided into a symmetric and an antisymmetric part. By using the property of reciprocity of the electrostatic field it can be shown (see e.g. [4], p. 205) that matrix $\underline{P}$ has only a symmetric part, i.e.

$$P_{ij} = P_{ji} \quad . \tag{2.10}$$

In matrix notation

$$\underline{P}^T = \underline{P} \quad , \tag{2.11}$$

where the superscript ^{T} denotes the transpose of a matrix.

A positive definite matrix always has an inverse. Therefore, one can find the inverse relationship to (2.5)

$$|q\rangle = \underline{P}^{-1}|V\rangle = \underline{K}\,|V\rangle \quad . \tag{2.12}$$

The inverse of $\underline{P}$ was denoted by $\underline{K}$. In the expanded form (2.12) can be written as

$$q_i = K_{i1}V_1 + K_{i2}V_2 + \ldots K_{ii}V_i + \ldots K_{iN}V_N \text{ for } i = 1, 2, \ldots N \quad . \tag{2.13}$$

As we can see, the coefficients K_{ij} represent the constants of proportionality between voltage and charge. The units in which K_{ij}'s are expressed are Coulombs/Volt·Meter = Farads/Meter. However, it would be erroneous to call K_{ij}'s capacitance coefficients. The proper name for them is <u>coefficients of (electrostatic) induction</u> (see again [18], p. 315). In order to define the <u>capacitance coefficients</u> C_{ij}, one must use a linear relationship of the following form

$$q_i = C_{i1}(V_i - V_1) + C_{i2}(V_i - V_2) \ldots + C_{ii}V_i + \ldots C_{iN}(V_i - V_N). \quad (2.14)$$

It is important to observe that C_{ij}'s are coefficients of linear relationship between charges and <u>mutual</u> voltages between the i-th and j-th conductors. If we want to find the relationship between K's and C's we have to group together the terms containing V_i in (2.14);

$$q_i = -C_{i1}V_1 - C_{i2}V_2 \ldots + (C_{i1} + C_{i2} + \ldots + C_{ii} \ldots + C_{iN})V_i \ldots -C_{iN}V_N. \quad (2.15)$$

By comparing (2.15) and (2.13) we find the following relationship

diagonal elements: $K_{ii} = \sum_{j=1}^{N} C_{ij}$, (2.16)

Off-diagonal elements: $K_{ij} = -C_{ij}$ for $i \neq j$. (2.17)

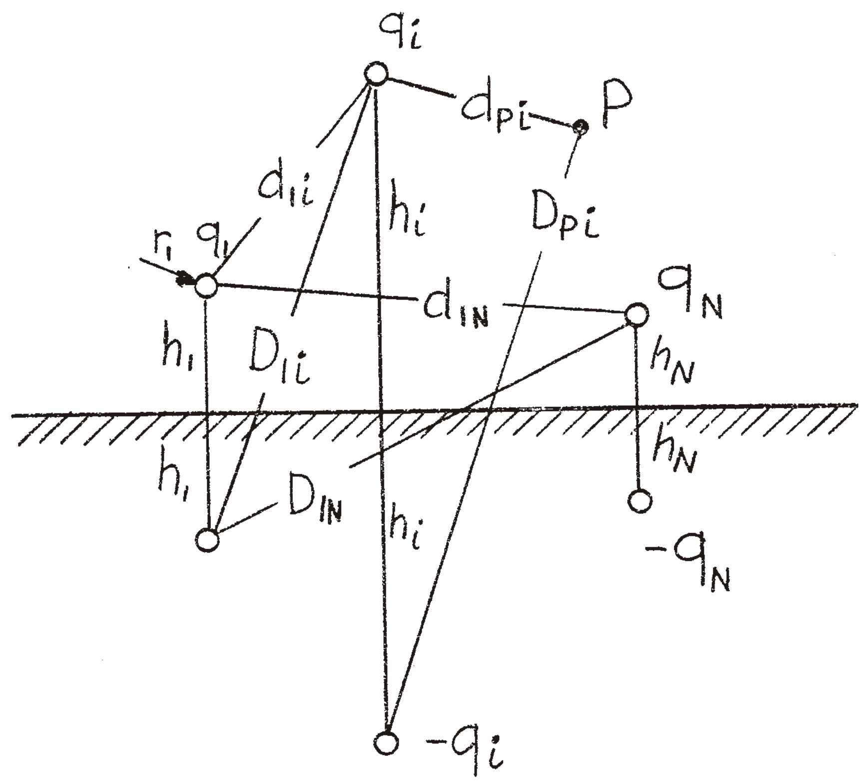


Figure 2.2

For a system of straight conductors of a circular cross section above the ground plane as shown in Figure 2.2, there is a very simple approximate method of computing the elements of the matrix P (see e.g. [20], p. 124). In books dealing with the power transmission, this method is widely used, but it is frequently explained in a lengthy, and sometimes confusing, way. The confusion is usually caused by the attempt to compute matrix C directly, without first computing matrix P and inverting it afterwards. Such a direct computation of C is possible only in some special cases, when the conductors are located in certain symmetric positions. We are going to show that the step-by-step computation which starts with the matrix P is very straightforward and general, so that it includes also all the special cases of symmetrically placed conductors.

According to the principle of images, the distribution of the electric field above the ground plane in Figure 2.2 can be computed by removing the ground plane and introducing the image line charge $-q_i$ for every conductor. For the observer located at point P, the potential produced by a pair of charges q_i and $-q_i$ is given by (see [20], p. 116)

$$V_{pi} = \frac{q_i}{2\pi\epsilon} \ln \left(\frac{D_{pi}}{d_{pi}}\right) \tag{2.18}$$

There is no added constant in this expression, since we choose that the ground plane has zero potential. When the point P from Figure 2.2 is on the ground plane, d_{pi} becomes equal to D_{pi} and the resulting potential is zero, as required.

Equation (2.18) is exact for the ideal line charges of zero thickness. When the observer's location P moves closer and closer to the charge q_i, distance d_{pi} goes to zero and the potential V_{pi} from (2.18) approaches infinity. This unpleasant situation is corrected by introducing something called quasi line charges. That is, one of the equipotential lines around the ideal line charge q_i may be chosen to coincide with the actual metal boundaries of the i-th wire. As long as other conductors are not very close to the i-th conductor, the equipotential line is very closely approximated by a circle centered at q_i. Therefore, a wire of a circular cross section with radius r_i and charged with the amount of q_i Coulombs/meter will produce the same potential distribution as an ideal, infinitely thin, line charge q_i.

This is the reason for the name quasi line charge. When the observer P is placed on the surface of the i-th quasi line charge, the distance d_{pi} becomes equal to the wire radius r_i, and the distance D_{pi} from the observer to the image charge becomes equal to twice the height h_i above the ground. (2.18) becomes

$$V_{ii} = \frac{q_i}{2\Pi\varepsilon} \ln \left(\frac{2h_i}{r_i}\right) \tag{2.19}$$

The total potential V_i of the i-th quasi line charge is found by the superposition of all N potentials caused by pairs of line charges and their images. The expression obtained for the total potential V_i is a sum of the form (2.1), where the individual coefficients P_{ij} are given by

$$P_{ij} = \frac{1}{2\Pi\varepsilon} \ln \left(\frac{D_{ji}}{d_{ji}}\right) \quad \text{for } j \neq i \tag{2.20}$$

and

$$P_{ii} = \frac{1}{2\Pi\varepsilon} \ln \left(\frac{2h_i}{r_i}\right) \quad \text{for } j = i \tag{2.21}$$

(2.20) gives the off-diagonal elements of matrix $\underline{P}$, and (2.21) gives the elements on the main diagonal. These two formulas are all the formulas that we need. When $\underline{P}$ has been computed from these two formulas for a particular geometry at hand, matrix $\underline{K}$ is obtained on any computer by simple inversion of matrix $\underline{P}$ in a matter of seconds (or even milliseconds). Elements of the capacitance matrix $\underline{C}$ can be computed afterwards by using (2.16) and (2.17), if necessary. We will see later that the study of wave propagation on MTL's requires only the knowledge of $\underline{P}$ or $\underline{K}$, so that $\underline{C}$ is not needed at all.

For a certain system of conductors of irregular shape, the computation of elements of matrix $\underline{P}$ may be too difficult, and we may decide to measure these elements. This is performed by taking the appropriate length d of the MTL and measuring the capacitance of the i-th conductor to the ground, as shown in Figure 2.3. During this measurement we usually connect all the other conductors to the ground, as shown in the same figure.

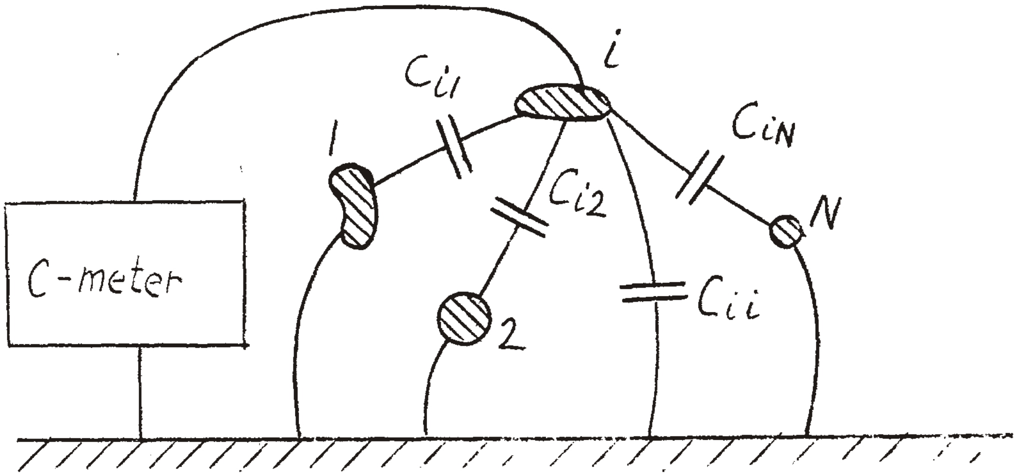


Figure 2.3

If the measured value of capacitance is C_{meas}, we have

$$\frac{C_{meas}}{d} = C_{i1} + C_{12} + .. + C_{ii} .... + C_{iN} = K_{ii} \; .$$

The measured value of capacitance per unit length is equal to the induction coefficient K_{ii}, and not to capacitance coefficient C_{ii} as one is often misled to think. By repeating the same measurement on any other conductor while all the remaining conductors are grounded, we can get all the elements on the main diagonal of matrix $\underline{K}$. One will recall from (2.17) that the off-diagonal elements of induction coefficients K_{ij} are the same numbers (with negative sign) as the mutual capacitances C_{ij}. These elements should be measured with the special capacitance meter which allows the ungrounded operation of the two conductors between which the mutual capacitance is to be measured. However, the need for the special instrument may be avoided by measuring capacitances of several conductors joined together in various combinations and solving the resulting system of linear equations afterwards.

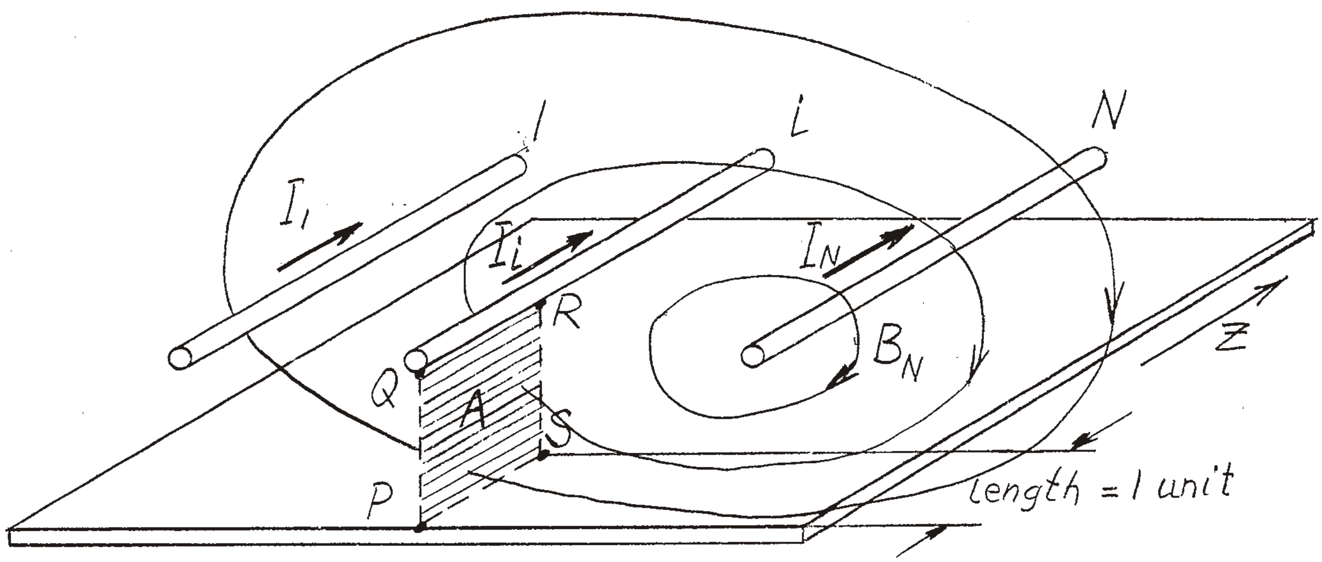


Figure 2.4

At this place it is appropriate to introduce the magnetic induction coefficients. Figure 2.4 represents a system of N parallel conductors above the ground plane. Each conductor is supposed to carry a constant D. C. current (e.g. current I_i on the i-th conductor). All these currents are returned back through the ground plane.

The magnetic field produced by the current I_N in the N-th conductor is also indicated in Figure 2.4. The closed lines around the N-th conductor represent the field of the magnetic flux density B_N. The magnetic flux over the area A under the i-th conductor, and produced by the current I_N on the N-th conductor, may be found by computing the surface integral of the flux of field B_N across the area A. The length of this area in z direction is taken to be unity, so that we are talking about the magnetic flux per unit length. The total magnetic flux Ψ_i' (per unit length) belonging to the i-th conductor (i.e. the total flux across area A) can be computed by summing up all the individual fluxes caused by currents I_1 through I_N.

$$\Psi_i' = \sum_{j=1}^{N} L_{ij} I_j \tag{2.22}$$

The coefficients of proportionality between the currents and corresponding fluxes are denoted by L_{ij}. They are called magnetic induction coefficients.

In the matrix notation, (2.22) becomes

$$|\Psi'\rangle = \underline{L}\,|I\rangle \quad . \tag{2.23}$$

The coefficients of matrix $\underline{L}$ are determined solely by the geometrical configurations of the MTL. In the case of a TEM field configuration, L_{ij} coefficients do not have to be computed separately, if we have determined the coefficients K_{ij} previously. Namely, matrix $\underline{L}$ is related to matrix $\underline{K}$ as follows:

$$\underline{L} = \frac{1}{v^2}\,\underline{K}^{-1} \quad . \tag{2.24}$$

Therefore, we can save the labor of computing the elements of $\underline{L}$ from the given geometry.

Section 3.

TRANSMISSION-LINE EQUATIONS

The discussion in the previous section was limited to the static electric and magnetic fields. In Figure 2.4 we have seen the static magnetic flux density B_N produced by the D. C. current I_N. The total flux over the area A under the i-th conductor produced by all currents from I_1 to I_N was given by (2.22).

Let us next consider the time-harmonic case. All the quantities like current, voltage, flux, etc. are assumed to include the factor exp (jωt). The well-known Maxwell's equation

$$\oint \vec{E} \cdot d\vec{l} = -j\omega \int \vec{B} \cdot d\vec{S}$$

tells us that the total magnetic flux Ψ_i across area A multiplied with (-jω) is equal to the induced voltage V_i in the contour PQRS (Figure 2.4)

$$V_i = -j\omega \, \Psi_i \tag{3.1}$$

Ψ_i denotes here the total magnetic flux across the area A, and not the flux per unit length as Ψ_i' in Section 2.

In Figure 3.1 we see the differential length Δz of the i-th conductor of a MTL. The corresponding area across which the magnetic flux should be evaluated is denoted by ΔA. The corresponding induced voltage in the contour of this area is denoted by ΔV_i. The total magnetic flux across ΔA is denoted by $\Delta\Psi_i$. It follows from (3.1) that

$$\Delta V_i = -j\omega \, \Delta\Psi_i \tag{3.2}$$

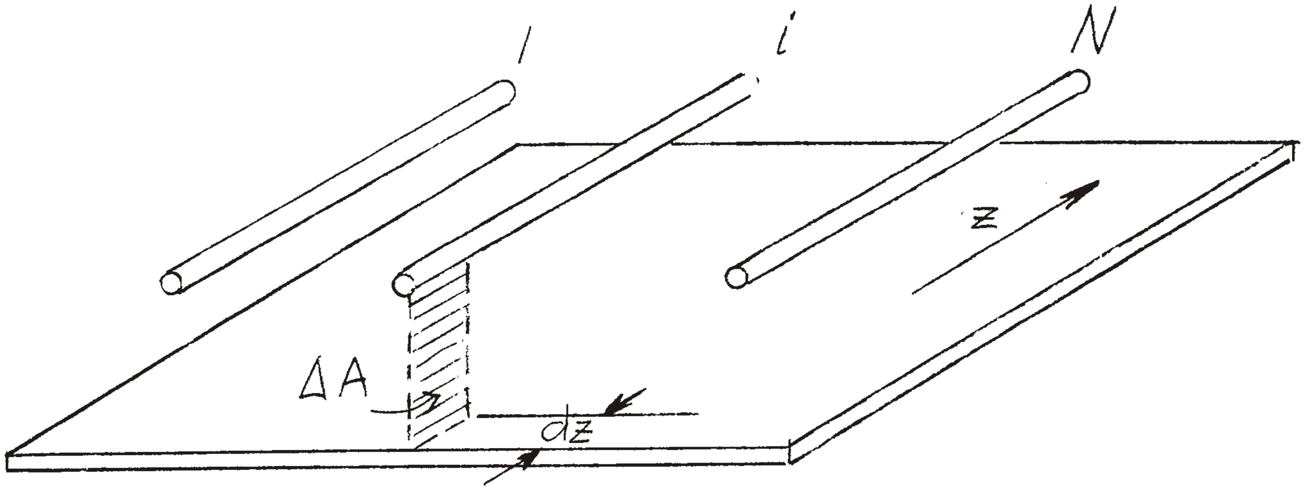


Figure 3.1

Flux $\Delta\Psi_i$ across the area of length Δz is equal to the flux Ψ_i' per unit length multiplied by the corresponding length Δz

$$\Delta\Psi_i = \Psi_i' \Delta z \tag{3.3}$$

Ψ_i' can be expressed in terms of individual currents I_j by using (2.22)

$$\Delta V_i = -j\omega \Delta z \sum_{j=1}^{N} L_{ij} I_j \tag{3.4}$$

Dividing by Δz and letting Δz become an infinitesimally small quantity, we obtain

$$\frac{\Delta V_i}{\Delta z} \simeq \frac{dV_i}{dz} = -j\omega \sum_{j=1}^{N} L_{ij} I_j \tag{3.5}$$

In matrix notation

$$\frac{d}{dz} |V> = -j\omega \underline{L} |I> \tag{3.6}$$

This is the first of the transmission-line equations for the MTL. It is formally very similar to the equation for the single transmission line (see e.g. [3], p. 83)

$$\frac{dV}{dz} = -j\omega\, L\, I$$

The second transmission-line equation will be found by using the other Maxwell's equation

$$\oint \vec{H}\cdot d\vec{l} = \int \vec{i}\cdot d\vec{S} + j\omega\int \vec{D}\cdot d\vec{S} \quad .$$

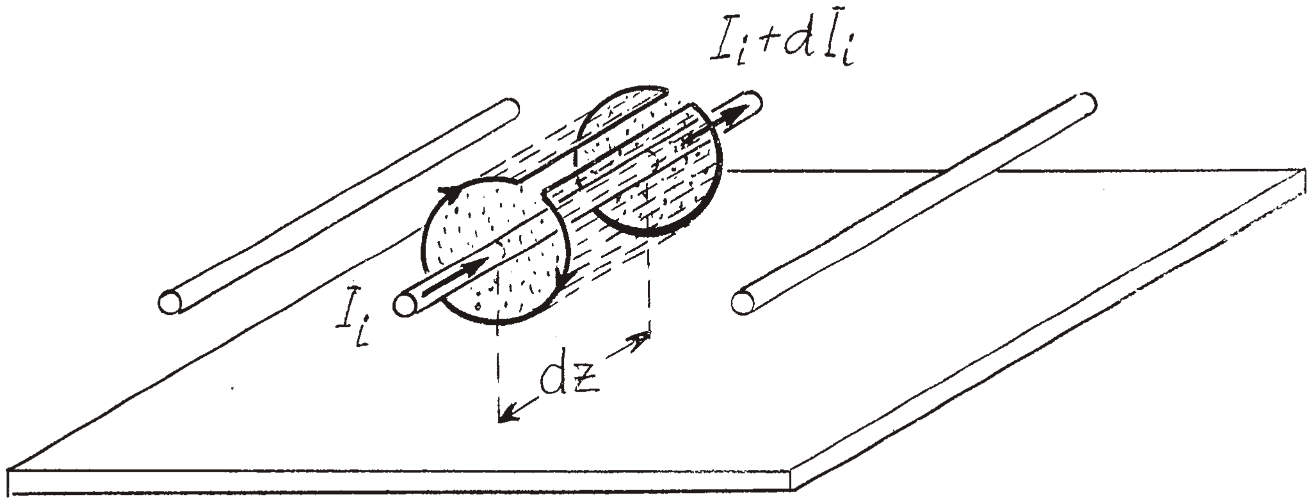


Figure 3.2

This equation will be applied on the closed path which is indicated by the solid line in Figure 3.2. The surface enclosed by this path may be interpreted in two ways:

1) The dotted surface consisting of two circular areas, perpendicular to the z axis. As we are considering TEM waves, the electric flux density $\vec{D}$ is everywhere tangential to the dotted surface, so that the total surface integral of $\vec{D}$ is zero. Maxwell's equation gives

$$\oint \vec{H}\cdot d\vec{l} = I_i - (I_i + \Delta I_i) = -\Delta I_i \quad . \qquad (3.7)$$

2) The dashed cylindrical surface, wrapped around the wire. There is no conduction current flow across this surface, but there is an appreciable electric flux since $\vec{D}$ is essentially perpendicular to the surface. The same Maxwell's equation applied to this surface gives

$$\oint \vec{H} \cdot d\vec{l} = j\omega \int \vec{D} \cdot d\vec{S} = j\omega q \Delta z \quad . \qquad (3.8)$$

In the second part of the above equation we have used Gauss' law which states that the electric flux is equal to the total charge inside the volume. From (3.7) and (3.8) it follows that

$$\frac{\Delta I_i}{\Delta z} \simeq \frac{dI_i}{dz} = -j\omega q_i \qquad (3.9)$$

which is just another form of the statement of charge continuity (see e.g. [6], p. 101). Expressing q_i by the use of (2.13) we get

$$\frac{dI_i}{dz} = -j\omega \sum_{j=1}^{N} K_{ij} V_j \quad . \qquad (3.10)$$

This is the second transmission-line equation for the MTL, which in matrix form becomes

$$\frac{d}{dz} |I\rangle = -j\omega \underline{K} |V\rangle \quad . \qquad (3.11)$$

Again we can recognize the close resemblance to the corresponding equation for the simple transmission line

$$\frac{di}{dz} = -j\omega C V \quad .$$

However, the equation for the MTL has the matrix $\underline{K}$ in place of the capacitance C. As discussed in Section 2, K's and C's have the same physical dimension (Farads/meter), but the matrices $\underline{K}$ and $\underline{C}$ differ considerably from each other. However, for a single conductor above the ground, $\underline{K}$ and $\underline{C}$ are (1 x 1) matrices, identical to each other.

Equations (3.6) and (3.11) represent the coupled system of two differential equations in two dependent variables:

$$\frac{d}{dz} |V\rangle = -j\omega \, \underline{L} \, |I\rangle \tag{3.6}$$

$$\frac{d}{dz} |I\rangle = -j\omega \, \underline{K} \, |V\rangle \; . \tag{3.7}$$

This is the so-called Hamiltonian form of the coupled wave equations (see e.g. [12], p. 4). One can eliminate one of the dependent variables from each of the above two equations and obtain the so-called Lagrangean form as follows

$$\frac{d^2}{dz^2} |V\rangle = -\omega^2 \, \underline{L} \, \underline{K} \, |V\rangle \tag{3.12}$$

$$\frac{d}{dz^2} |I\rangle = -\omega^2 \, \underline{K} \, \underline{L} \, |I\rangle \; . \tag{3.13}$$

The straightforward way of solving an equation of the type (3.12) is to rewrite it in a new basis in which the matrix $\underline{L}\,\underline{K}$ becomes diagonal. To do this we have to find the eigenvalues k_i^2 and eigenvectors V_i' of the matrix $\omega^2 \underline{L}\,\underline{K}$. The matrix equation (3.12) is then reduced to N independent scalar equations of the following form

$$\frac{d^2 V_i'}{dz^2} = -k_i^2 V_i' \quad \text{for } i=1,2, \ldots N \; . \tag{3.14}$$

The solution of each of these equations is an exponential function of the type

$$V_i' = V_i^+ \; e^{-jk_i z} + V_i^- \; e^{jk_i z} \quad . \qquad (3.15)$$

In the most general case, the matrix $\omega^2 \underline{L} \; \underline{K}$ for the system of N conductors may have N distinct eigenvalues. Therefore, we can have N different waves with different velocities to propagate in either positive or negative z direction. Such multivelocity waves can take place for different reasons. One reason is imperfect conductivity, which causes the electromagnetic field to penetrate inside the metal conductors. The examples of multivelocity waves caused by finite conductivity have been discussed in detail by many of the power-transmission workers (see e.g. [1]). However, even if conductors are assumed to have a perfect conductivity, multivelocity waves will appear when the dielectric material in inhomogeneous. Lines whose cross-section consists of two or more dielectric materials cannot propagate pure TEM waves. Therefore, multiconductor lines made in microstrip techniques will propagate multivelocity waves, while MTL's made in strip-line will propagate pure TEM waves (see Figure 3.3). Another situation which seems to produce the multivelocity waves is a system of

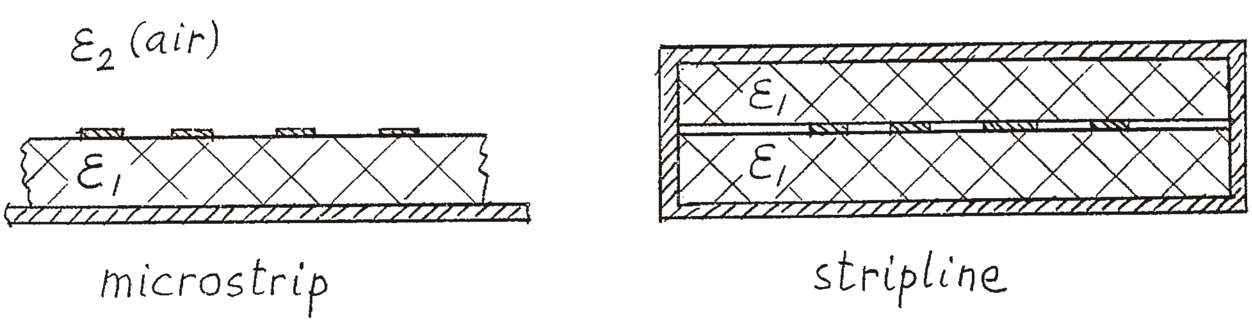


Figure 3.3

perfect conductors which are curved in a form of circular arc (see [7]).

However, a straight section of a MTL with perfect conductors supports TEM waves, which have the property to propagate with one single velocity. This velocity is equal to the velocity of light in the dielectric medium of which the MTL is made. This means that all the eigenvalues of $\omega^2 \underline{L}\,\underline{K}$ must be degenerate (equal to each other) and equal to the free-space propagation constant:

$$k_i^2 = k^2 = \omega^2 \mu\varepsilon \qquad \text{for } i=1,2,\ldots N \quad . \tag{3.16}$$

When the eigenvalues are degenerate, one has a freedom in choosing the eigenvectors. In our case we can take eigenvectors V_i' to be equal to the original voltages V_i. The equation (3.14) becomes

$$\frac{d^2V_i}{dz^2} = -k^2\, V_i \qquad \text{for } i=1,2,\ldots N \quad . \tag{3.17}$$

Comparing this with (3.12) we conclude that the single-velocity TEM wave must have the matrix $\omega^2 \underline{L}\,\underline{K}$ equal to the identity matrix $\underline{I}$ multiplied by a constant:

$$\omega^2\, \underline{L}\,\underline{K} = k^2\, \underline{I} \quad . \tag{3.18}$$

Therefore, whenever matrix $\underline{K}$ is known, one can compute matrix $\underline{L}$ from it

$$\underline{L} = \frac{k^2}{\omega^2}\underline{K}^{-1} = \frac{1}{v_{TEM}^2}\,\underline{K}^{-1} \tag{3.19}$$

where v_{TEM} denotes the velocity of the TEM wave.

The solution to the system of equations (3.17) is in the form of incident and reflected wave:

$$V_i = V_i^+\, e^{-jkz} + V_i^-\, e^{jkz} \qquad \text{for } i=1,2,\ldots N \quad . \tag{3.20}$$

The constants V_i^+ and V_i^- are determined by the boundary conditions at the generator side and load side of the MTL. The manner of determining these constants will be discussed in Section 4.

In matrix notation, (3.20) becomes

$$|V> = e^{-jkz}|V^{+}> + e^{jkz}|V^{-}> \tag{3.21}$$

The corresponding solution for currents can be obtained by substituting (3.21) into (3.6). We obtain

$$-jke^{-jkz}|V^{+}> +jke^{jkz}|V^{-}> = -j\omega \underline{L}|I> \tag{3.22}$$

In analogy with the simple transmission lines, we define the matrices for characteristic admittance and characteristic impedance

$$\underline{Y}_o = \underline{Z}_o^{-1} = v_{TEM}\underline{K} \tag{3.23}$$

By using this notation, (3.22) can be rewritten as

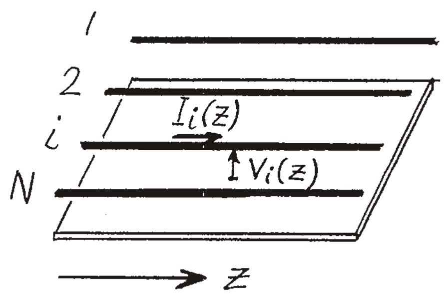
$$|I> = \underline{Z}_o^{-1}(e^{-jkz}|V^{+}> - e^{jkz}|V^{-}>) , \tag{3.24}$$

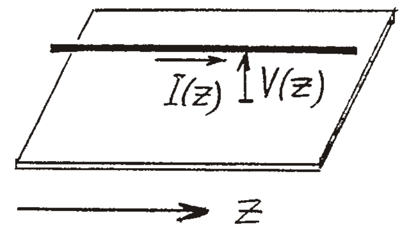
which expression is a direct analogy of the corresponding formula for simple transmission lines

$$I = \frac{1}{Z_o}(e^{-jkz}V^{+} - e^{-jkz}V^{-})$$

The analogy between matrix formulas for MTL's and scalar formulas for single transmission lines is summarized in Figure 3.3.

Multiconductor Transmission Line (MTL)	Single Transmission Line





Differential Equations:

$$\frac{d}{dz}|V\rangle = -j\omega \underline{L}\,|I\rangle \qquad\qquad \frac{dV}{dz} = -j\omega\, L\, I$$

$$\frac{d}{dz}|I\rangle = -j\omega \underline{K}\,|V\rangle \qquad\qquad \frac{dI}{dz} = -j\omega\, C\, V$$

Solutions:

$$|V\rangle = e^{-jkz}\,|V^+\rangle + e^{jkz}\,|V^-\rangle \qquad\qquad V = e^{-jkz}\,V^+ + e^{jkz}\,V^-$$

$$|I\rangle = \underline{Z}_o^{-1}\left(e^{-jkz}\,|V^+\rangle - e^{jkz}\,|V^-\rangle\right) \qquad\qquad I = \frac{1}{Z_o}\left(e^{-jkz}\,V^+ - e^{jkz}\,V^-\right)$$

Velocity:

$$\underline{L}\,\underline{K} = \frac{1}{v_{TEM}^2}\,\underline{I} \qquad\qquad L\,C = \frac{1}{v_{TEM}^2}$$

Characteristic Impedance:

$$\underline{Z}_o = \frac{1}{v_{TEM}}\,\underline{K}^{-1} \qquad\qquad Z_o = \frac{1}{v_{TEM}C}$$

Figure 3.3

SCATTERING MATRIX OF A TWO-SIDED SECTION OF A MTL

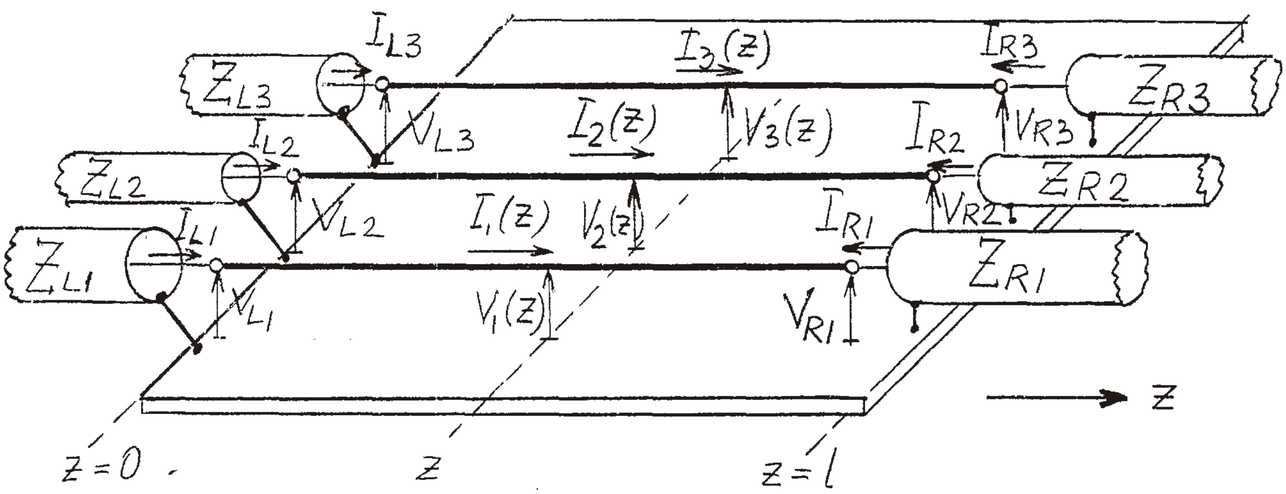


Figure 4.1

When a MTL is used in some microwave network, the circuit will look like the one shown in Figure 4.1. In that figure we have represented a three-conductor line which is at both sides connected to outer transmission lines. The characteristic impedances of the outer transmission lines connected to left-hand MTL terminals are denoted by Z_{L1}, Z_{L2}, Z_{L3}; those at the right-hand terminals are denoted by Z_{R1}, Z_{R2}, Z_{R3}. The outer lines are completely shielded from each other whereas the section of the MTL between z=0 and z=1 contains mutual couplings between different lines. The MTL in Figure 4.1 has only 3 conductors above the ground plane, but in a general case we will have N conductors. It is obvious that a two-sided section of an N-conductor transmission line represents a network with 2N ports. Our desire is to find the properties of this linear, reciprocal, 2N-port. The most useful representation of a microwave multiport is through its scattering matrix. In this section we will derive the expression for a scattering matrix of a section of a MTL of a length l, assuming that the induction coefficient matrix $\underline{K}$ and the velocity v_{TEM} are known. As discussed in Section 2, knowing $\underline{K}$ is the same thing as knowing the capacitance matrix $\underline{C}$ since they are simply related. v_{TEM} is the velocity of light in the dielectric medium surrounding the conductors, and it can be easily evaluated from the relative dielectric constant

ε_r of the material

$$v_{TEM} = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{1}{\sqrt{\mu_o\varepsilon_o}\sqrt{\varepsilon_r}} = \frac{c}{\sqrt{\varepsilon_r}} \tag{4.1}$$

where c is the velocity of light in free space, and $\mu = \mu_o$ is the permeability of free space. v_{TEM} is related to the angular frequency ω and to the propagation constant k as follows

$$v_{TEM} = \frac{\omega}{k} . \tag{4.2}$$

The voltage of the i-th conductor with respect to the ground at the point z is denoted by $V_i(z)$. Similarly the current of i-th conductor at the position z is denoted by $I_i(z)$. Note that the direction of $I_i(z)$ is positive when pointing in (+z) direction. For an N-conductor transmission line, we will define the N-dimensional voltage and current vectors

$$|V(z)> = \begin{pmatrix} V_1(z) \\ V_2(z) \\ . \\ . \\ . \\ V_N(z) \end{pmatrix} ; \quad |I(z)> = \begin{pmatrix} I_1(z) \\ I_2(z) \\ . \\ . \\ . \\ I_N(z) \end{pmatrix} . \tag{4.3}$$

It was found in Section 3 that voltage and current are expressed as functions of z in the following matrix form [see (3.21) and (3.24)]:

$$|V(z)> = e^{-jkz} |V^+> + e^{jkz} |V^-> \tag{4.4}$$

$$|I(z)> = \underline{Z}_o^{-1} (e^{-jkz} |V^+> - e^{jkz} |V^->) \tag{4.5}$$

$|V^+>$ and $|V^->$ are N-dimensional constant vectors representing the amplitudes of waves traveling in positive and negative directions of z, respectively. Their form is the following:

$$|V^+> = \begin{pmatrix} V_1^+ \\ V_2^+ \\ \cdot \\ \cdot \\ \cdot \\ V_N^+ \end{pmatrix} \quad ; \quad |V^-> = \begin{pmatrix} V_1^- \\ V_2^- \\ \cdot \\ \cdot \\ \cdot \\ V_N^- \end{pmatrix} . \qquad (4.6)$$

$\underline{Z_o}$ is a symmetric, positive definite, matrix with real coefficients. It can be easily computed from the electric induction matrix $\underline{K}$ as follows [see (3.23)]:

$$\underline{Z_o}^{-1} = v_{TEM} \underline{K} .$$

When we define the parameters of a multiport, we think of a black box. Only voltages and currents at its terminals are of interest, the details of voltage and current distribution inside the box are not of interest for the user. The user of the black box wants only to know how the terminal voltages and currents influence each other. This relationship is described by the impedance matrix, admittance matrix, or- preferably for the microwave engineer-the scatering matrix. The voltages and currents at the left-hand side terminals will be denoted by column vectors $|V_L>$ and $|I_L>$. Similarly the right-hand side voltage vector will be denoted by $|V_R>$ and the current vector by $|I_L>$. Note the current directions in Figure 4.1: in accordance with network analysis, all the terminal currents are pointed into the multiport. In this way, current I_{R1} is oriented in the opposite direction from I_1 (z = 1). This fact is important in choosing the proper sign in (4.17) below.

At z = 0, the voltage and current of a MTL are:

$$|V_L> = |V^+> + |V^-> \qquad (4.7)$$

$$|I_L> = \underline{Z_o}^{-1} (|V^+> - |V^->). \qquad (4.8)$$

In general case, each of the outer transmission lines has a different characteristic impedance; e.g. i-th outer line on the left-hand side has the characteristic impedance Z_{Li}. Each such line can support two waves: one propagating toward the MTL, its amplitude being V_{Li}^{+}, and other propagating away from the MTL, its amplitude being V_{Li}^{-}. At the left-hand side terminal, voltage and current on all the outer lines can be represented by the following matrix equations:

$$|V_L> = |V_L^{+}> + |V_L^{-}> \tag{4.9}$$

$$|I_L> = \underline{Z}_L^{-1} (|V_L^{+}> - |V_L^{-}>) \quad . \tag{4.10}$$

$\underline{Z}_L$ represents a diagonal matrix containing all the characteristic impedances of outer lines connected to left-hand side terminals:

$$\underline{Z}_L = \text{diag.} (Z_{L1}, Z_{L2}, \ldots, Z_{LN}) \quad . \tag{4.11}$$

We will assume that all Z_{Li}'s are real numbers.

The rest of our derivation will be devoted to the elimination of parameters which describe the situation <u>inside</u> the MTL : $|V^{+}>$ and $|V^{-}>$. We desire to find the relationship between parameters describing the <u>outer</u> network: $|V_L^{+}>$, $|V_L^{-}>$, $|V_R^{+}>$ and $|V_R^{-}>$. From (4.7) and (4.9) we have

$$|V_L^{+}> + |V_L^{-}> = |V^{+}> + |V^{-}> \quad . \tag{4.12}$$

From (4.8) and (4.10) we have

$$\underline{Z}_o^{-1} |V^{+}> - \underline{Z}_o^{-1} |V^{-}> = \underline{Z}_L^{-1} |V_L^{+}> - \underline{Z}_L^{-1} |V_L^{-}> \quad . \tag{4.13}$$

From these two equations we can express $|V^{+}>$ and $|V^{-}>$ as follows:

$$2|V^{+}> = (\underline{I} + \underline{Z}_o \underline{Z}_L^{-1}) |V_L^{+}> + (\underline{I} - \underline{Z}_o \underline{Z}_L^{-1}) |V_L^{-}> \tag{4.14}$$

$$2|V^{-}> = (\underline{I} - \underline{Z}_o \underline{Z}_L^{-1}) |V_L^{+}> + (\underline{I} + \underline{Z}_o \underline{Z}_L^{-1}) |V_L^{-}> \quad . \tag{4.15}$$

At the right-hand terminals z = l and we find from (4.4) and (4.5) the following expressions for the voltages and currents of the MTL

$$|V_R> = e^{-jkl}\ |V^+> + e^{jkl}\ |V^-> \qquad (4.16)$$

$$-|I_R> = \underline{Z}_o^{-1}\ (e^{-jkl}\ |V^+> - e^{jkl}\ |V^->). \qquad (4.17)$$

It is important to note the negative sign of the current in (4.17).

We define the diagonal matrix $\underline{Z}_R$ of characteristic impedances of outer lines on the right-hand side

$$\underline{Z}_R = \text{diag.}\ (Z_{R1}, Z_{R2}, \ldots Z_{RN}) \qquad (4.18)$$

The amplitude of the wave traveling toward the MTL on the i-th outer line on the right-hand side is denoted by V_{Ri}^+. Similarly, the amplitude of the wave traveling away from the MTL on the i-th outer line is V_{Ri}^-. In matrix notation, the voltages and currents on the outer lines at z = l are

$$|V_R> = |V_R^+> + |V_R^-> \qquad (4.19)$$

$$|I_R> = \underline{Z}_R^{-1}\ (|V_R^+> - |V_R^->)\ . \qquad (4.20)$$

Just as we did for the left-hand terminals, we can eliminate $|V_R>$ and $|I_R>$ from (4.16) through (4.20) and obtain

$$2e^{-jkl}\ |V^+> = (\underline{I} - \underline{Z}_o\ \underline{Z}_R^{-1})\ |V_R^+> + (\underline{I} + \underline{Z}_o\ \underline{Z}_R^{-1})\ |V_R^-> \qquad (4.21)$$

$$2e^{jkl}\ |V^-> = (\underline{I} + \underline{Z}_o\ \underline{Z}_R^{-1})\ |V_R^+> + (\underline{I} - \underline{Z}_o\ \underline{Z}_R^{-1})\ |V_R^->\ . \qquad (4.22)$$

Eliminate $|V^+>$ from (4.14) and (4.21)

$$(\underline{I} + \underline{Z}_o\,\underline{Z}_L^{-1})\,|V_L^+> + (\underline{I} - \underline{Z}_o\,\underline{Z}_L^{-1})\,|V_L^-> =$$

$$e^{jkl}\,(\underline{I} - \underline{Z}_o\,\underline{Z}_R^{-1})\,|V_R^+> + e^{jkl}\,(\underline{I} + \underline{Z}_o\,\underline{Z}_R^{-1})\,|V_R^-> \qquad (4.23)$$

Eliminate $|V^->$ from (4.15) and (4.22):

$$(\underline{I} - \underline{Z}_o\,\underline{Z}_L^{-1})\,|V_L^+> + (\underline{I} + \underline{Z}_o\,\underline{Z}_L^{-1})\,|V_L^-> =$$

$$e^{-jkl}\,(\underline{I} + \underline{Z}_o\,\underline{Z}_R^{-1})\,|V_R^+> + e^{-jkl}\,(\underline{I} - \underline{Z}_o\,\underline{Z}_R^{-1})\,|V_R^-> \qquad (4.24)$$

The last two equations contain only the amplitudes of the waves on outer transmission lines, as desired. It remains to arrange and normalize them properly.

First, we introduce normalized amplitudes of <u>incident waves</u> (traveling toward the MTL):

$$|a_L> = \underline{Z}_L^{-1/2}\,|V_L^+> \quad ; \quad |a_R> = \underline{Z}_R^{-1/2}\,|V_R^+> \qquad (4.25)$$

and of <u>scattered waves</u> (traveling away from the MTL):

$$|b_L> = \underline{Z}_L^{-1/2}\,|V_L^-> \quad ; \quad |b_R> = \underline{Z}_R^{-1/2}\,|V_R^-> \; . \qquad (4.26)$$

The amplitudes a_i and b_i have been chosen so that $1/2|a_i|^2$ represents the power of the incident wave on the i-th line, while $1/2|b_i|^2$ represents the power of the scattered wave on the i-th line. (see e.g. [18], p. 603). It is important to mention that some outstanding authors define the normalized wave amplitudes in slightly different ways (e.g. [10] is using r.m.s. values of voltages and currents instead of peak values, while [9] normalizes a's and b's in another way).

Further simplification in our formulas is possible by introducing new symbols $\underline{P}$ and $\underline{Q}$ as follows:

$$\underline{P}_L = (\underline{I} + \underline{Z}_o\,\underline{Z}_L^{-1})\,\underline{Z}_L^{1/2} \quad ; \quad \underline{P}_R = (\underline{I} + \underline{Z}_o\,\underline{Z}_R^{-1})\,\underline{Z}_R^{1/2} \qquad (4.27)$$

$$\underline{Q}_L = (\underline{I} - \underline{Z}_o\,\underline{Z}_L^{-1})\,\underline{Z}_L^{1/2} \quad ; \quad \underline{Q}_R = (\underline{I} - \underline{Z}_o\,\underline{Z}_R^{-1})\,Z_R^{1/2} \; . \qquad (4.28)$$

In the new notation, (4.23) and (4.24) become

$$\underline{P}_L \, |a_L\rangle + \underline{Q}_L \, |b_L\rangle = e^{jkl} \, (\underline{Q}_R \, |a_R\rangle + \underline{P}_R \, |b_R\rangle) \tag{4.29}$$

$$\underline{Q}_L \, |a_L\rangle + \underline{P}_L \, |b_L\rangle = e^{-jkl} \, (\underline{P}_R \, |a_R\rangle + \underline{Q}_R \, |b_R\rangle) \tag{4.30}$$

One can eliminate $|b_R\rangle$ from both equations by multiplying (4.29) by $e^{-jkl} \, \underline{P}_R^{-1}$ and (4.30) by $e^{jkl} \, \underline{Q}_R^{-1}$. Taking the difference of these two equations we obtain

$$(e^{jkl} \, \underline{Q}_R^{-1} \, \underline{P}_L - e^{-jkl} \, \underline{P}_R^{-1} \, \underline{Q}_L) \, |b_L\rangle =$$

$$= (e^{-jkl} \, \underline{P}_R^{-1} \, \underline{P}_L - e^{jkl} \, \underline{Q}_R^{-1} \, \underline{Q}_L) \, |a_L\rangle + (\underline{Q}_R^{-1} \, \underline{P}_R - \underline{P}_R^{-1} \, \underline{Q}_R) \, |a_R\rangle. \tag{4.31}$$

Similarly, elimination of $|b_L\rangle$ results in

$$(e^{jkl} \, \underline{Q}_L^{-1} \, \underline{P}_R - e^{-jkl} \, \underline{P}_L^{-1} \, \underline{Q}_R) \, |b_R\rangle =$$

$$= (\underline{Q}_L^{-1} \, \underline{P}_L - \underline{P}_L^{-1} \, \underline{Q}_L) \, |a_L\rangle + (e^{-jkl} \, \underline{P}_L^{-1} \, \underline{P}_R - e^{jkl} \, \underline{Q}_L^{-1} \, \underline{Q}_R) \, |a_R\rangle \, . \tag{4.32}$$

The system of equations (4.31) and (4.32) may be represented in a new matrix notation extended to 2N dimensions:

$$|b\rangle = \underline{S} \, |a\rangle \, . \tag{4.33}$$

The new vectors $|b\rangle$ and $|a\rangle$ are 2N-dimensional column vectors

$$|b\rangle = \begin{pmatrix} |b_L\rangle \\ \\ |b_R\rangle \end{pmatrix} \quad ; \quad |a\rangle = \begin{pmatrix} |a_L\rangle \\ \\ |a_R\rangle \end{pmatrix} . \tag{4.34}$$

$\underline{S}$ is a 2N x 2N matrix, partitioned in four N x N matrices as follows

$$\underline{S} = \begin{pmatrix} \underline{S}_{LL} & \underline{S}_{LR} \\ \underline{S}_{RL} & \underline{S}_{RR} \end{pmatrix}. \qquad (4.35)$$

Equation (4.33) is the desired scattering matrix formulation for a 2N-port consisting of a two-sided section of an N-conductor transmission line. Subscripts L and R refer to the left-hand and right-hand side of the network. The partitioned parts of the matrix $\underline{S}$ can be written by inspection from (4.31) and (4.32);

$$\underline{S}_{LL} = (e^{jkl}\, \underline{Q}_R^{-1}\, \underline{P}_L - e^{-jkl}\, \underline{P}_R^{-1}\, \underline{Q}_L)^{-1}\; (e^{-jkl}\, \underline{P}_R^{-1}\, \underline{P}_L - e^{jkl}\, \underline{Q}_R^{-1}\, \underline{Q}_L) \qquad (4.36)$$

$$\underline{S}_{LR} = (e^{jkl}\, \underline{Q}_R^{-1}\, \underline{P}_L - e^{-jkl}\, \underline{P}_R^{-1}\, \underline{Q}_L)^{-1}\; (\underline{Q}_R^{-1}\, \underline{P}_R - \underline{P}_R^{-1}\, \underline{Q}_R) \qquad (4.37)$$

$$\underline{S}_{RL} = (e^{jkl}\, \underline{Q}_L^{-1}\, \underline{P}_R - e^{-jkl}\, \underline{P}_L^{-1}\, \underline{Q}_R)^{-1}\; (\underline{Q}_L^{-1}\, \underline{P}_L - \underline{P}_L^{-1}\, \underline{Q}_L) \qquad (4.38)$$

$$\underline{S}_{RR} = (e^{jkl}\, \underline{Q}_L^{-1}\, \underline{P}_R - e^{-jkl}\, \underline{P}_L^{-1}\, \underline{Q}_R)^{-1}\; (e^{-jkl}\, \underline{P}_L^{-1}\, \underline{P}_R - e^{jkl}\, \underline{Q}_L^{-1}\, \underline{Q}_R)\;. \qquad (4.39)$$

The above four equations have a very broad use in dealing with MTL. These equations give all the necessary information for a black-box description of a section of a MTL which is incorporated in any kind of a microwave network. The quantities that are needed in the computation of scattering matrix S are: the electric inductance matrix $\underline{K}$, the diagonal matrices $\underline{Z}_L$ and $\underline{Z}_R$ of characteristic impedances of left-hand and right-hand ports, the length of the section l, and the TEM propagation constant k. The computation of scattering matrix coefficients from the above formulas is also well suited for computer programming. Finally, one should note that the above formulas require the inversion of N x N matrices only, although the resulting scattering matrix is of the size 2N x 2N.

Starting with (4.29) and (4.30) we can arrive at the scattering matrix $\underline{\underline{S}}$ in a somewhat different way (see [13], Section 2.2). Namely, by grouping b's on the left-hand side and a's on the right-hand side, (4.29) and (4.30) become:

$$\underline{\underline{Q}}_L|b_L> - e^{jkl}\ \underline{\underline{P}}_R|b_R> = -\underline{\underline{P}}_L|a_L> + e^{jkl}\ \underline{\underline{Q}}_R|a_R> \qquad (4.40)$$

$$\underline{\underline{P}}_L|b_L> - e^{-jkl}\ \underline{\underline{Q}}_R|b_R> = -\underline{\underline{Q}}_L|a_L> + e^{-jkl}\ \underline{\underline{P}}_R|a_R> . \qquad (4.41)$$

In a 2N-dimensional matrix notation this can be written as

$$\begin{pmatrix} \underline{\underline{Q}}_L & -e^{jkl}\ \underline{\underline{P}}_R \\ \underline{\underline{P}}_L & -e^{-jkl}\ \underline{\underline{Q}}_R \end{pmatrix} \begin{pmatrix} |b_L> \\ |b_R> \end{pmatrix} = \begin{pmatrix} -\underline{\underline{P}}_L & e^{jkl}\ \underline{\underline{Q}}_R \\ -\underline{\underline{Q}}_L & e^{-jkl}\ \underline{\underline{P}}_R \end{pmatrix} \begin{pmatrix} |a_L> \\ |a_R> \end{pmatrix} , \qquad (4.42)$$

Comparing (4.42) with (4.33) we conclude that the scattering matrix $\underline{\underline{S}}$ can be written in the following form:

$$\underline{\underline{S}} = \begin{pmatrix} \underline{\underline{Q}}_L & -e^{jkl}\ \underline{\underline{P}}_R \\ \underline{\underline{P}}_L & -e^{-jkl}\ \underline{\underline{Q}}_R \end{pmatrix}^{-1} \begin{pmatrix} -\underline{\underline{P}}_L & e^{jkl}\ \underline{\underline{Q}}_R \\ -\underline{\underline{Q}}_L & e^{-jkl}\ \underline{\underline{P}}_R \end{pmatrix} . \qquad (4.43)$$

This form appears to be more compact than (4.36) to (4.39), but it requires the inversion of a larger matrix, of the size 2N x 2N.

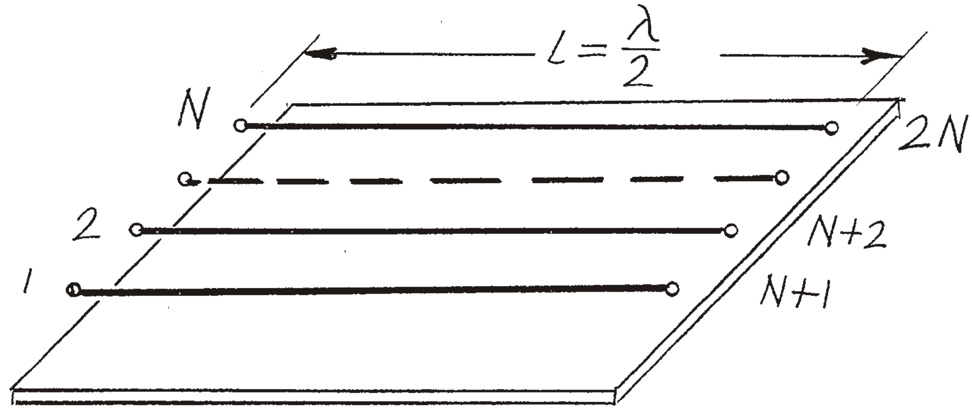


Figure 4.2

An interesting case to which our formulas can be easily applied is the half-wavelength section of the N-conductor transmission line shown in Figure 4.2. Let us choose each characteristic impedance on the right-hand side of the i-th line to be the same as on the left-hand side, namely $Z_{Li} = Z_{Ri}$. In matrix notation:

$$\underline{Z}_L = \underline{Z}_R \quad . \tag{4.44}$$

Since $kl = \pi$, we obtain from (4.36) to (4.39):

$$\underline{S}_{LL} = \underline{0} \;;\; \underline{S}_{LR} = -\underline{I} \;;\; \underline{S}_{RL} = -\underline{I} \;;\; \underline{S}_{RR} = \underline{0} \;. \tag{4.45}$$

$\underline{0}$ denotes the zero matrix and $\underline{I}$ denotes the identity matrix. The 2N x 2N scattering matrix of the half-wavelength section is, therefore,

$$\underline{S} = \begin{pmatrix} \underline{0} & -\underline{I} \\ -\underline{I} & \underline{0} \end{pmatrix}. \tag{4.46}$$

This represents an interesting situation. Since the main diagonal of $\underset{\sim}{S}$ is zero, all the ports are internally matched. Furthermore, the wave entering port 1 comes out of port N + 1 only (with 180° phase shift), the wave entering port 2 comes out of port N + 2 only, etc. The situation is the same as if all the N conductors of the MTL were completely shielded from each other.

Section 5.

SCATTERING MATRIX OF A ONE-SIDED SECTION OF A MTL

In the previous Section we derived the expression for scattering matrix of a 2N-port which results from a two-sided straight section of a MTL. In the present Section we will investigate the properties of the N-port which is produced by terminating one side of a MTL section in a known network T, as shown in Figure 5.1.

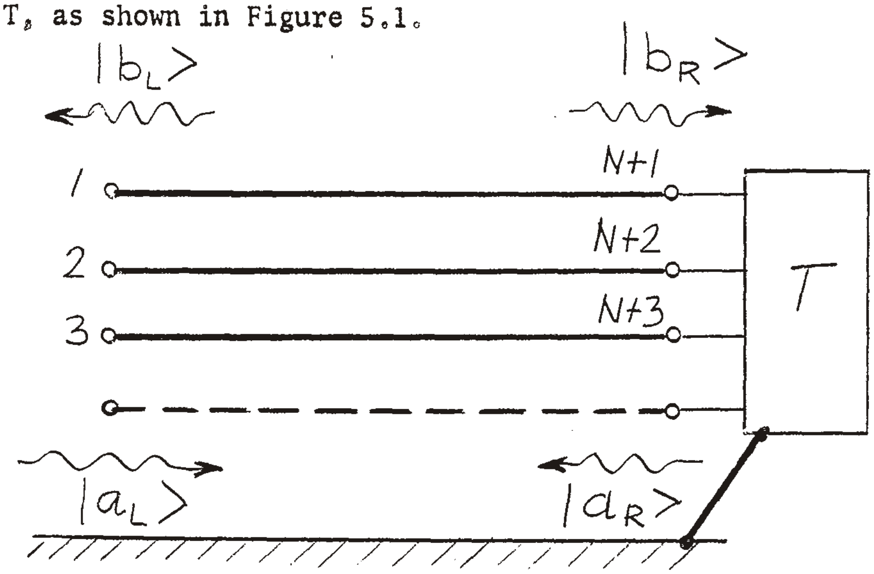


Figure 5.1

We choose to terminate the right-hand side, which is denoted by a subscript R, and leave the left-hand side available for use as an N-port. The incident wave vector on the right-hand side of the MTL is denoted by $|a_R>$. For the network T, the same quantity $|a_R>$ represents the scattered wave (going away from T). The network T has a known scattering matrix $\underline{T}$:

$$|a_R> = \underline{T} \; |b_R> \; . \tag{5.1}$$

Our intention is to find the scattering matrix of the N-port of the left-hand side terminals designated 1 to N. We will start with equations

(4.40) and (4.41), which are here repeated for convenience

$$\underline{Q}_L|b_L> - e^{jkl}\ \underline{P}_R|b_R> = -\underline{P}_L|a_L> + e^{jkl}\ \underline{Q}_R|a_R> \tag{5.2}$$

$$\underline{P}_L|b_L> - e^{-jkl}\ \underline{Q}_R|b_R> = -\underline{Q}_L|a_L> + e^{-jkl}\ \underline{P}_R|a_R> . \tag{5.3}$$

We can eliminate $|a_R>$ from these two equations by the use of (5.1):

$$\underline{Q}_L|b_L> + \underline{P}_L|a_L> = e^{jkl}\ (\underline{P}_R + \underline{Q}_R\underline{T})\ |b_R> \tag{5.4}$$

$$\underline{P}_L|b_L> + \underline{Q}_L|a_L> = e^{-jkl}\ (\underline{Q}_R + \underline{P}_R\underline{T})\ |b_R> . \tag{5.5}$$

For shorter notation, the terms in parentheses may be denoted by new symbols $\underline{G}$ and $\underline{H}$:

$$\underline{G} = \underline{P}_R + \underline{Q}_R\underline{T}\ \ ;\ \ \underline{H} = \underline{Q}_R + \underline{P}_R\underline{T}\ . \tag{5.6}$$

Assuming that $\underline{G}$ and $\underline{H}$ have their inverses, we can multiply (5.4) from the left by $e^{-jkl}\ \underline{G}^{-1}$ and (5.5) by $e^{jkl}\ \underline{H}^{-1}$ and compare the results:

$$e^{-jkl}\ (\underline{G}^{-1}\ \underline{Q}_L|b_L> + \underline{G}^{-1}\ \underline{P}_L|a_L>) = e^{jkl}\ (\underline{H}^{-1}\ \underline{P}_L|b_L> + \underline{H}^{-1}\ \underline{Q}_L|a_L>) . \tag{5.7}$$

Grouping $|b_L>$ and $|a_L>$ terms we arrive at the following

$$(e^{-jkl}\ \underline{G}^{-1}\ \underline{Q}_L - e^{jkl}\ \underline{H}^{-1}\ \underline{P}_L)\ |b_L> = (e^{jkl}\ \underline{H}^{-1}\ \underline{Q}_L - e^{-jkl}\ \underline{G}^{-1}\ \underline{P}_L)\ |a_L> \tag{5.8}$$

We have arrived at the desired expression for the scattering matrix relating $|b_L>$ and $|a_L>$, when the right-hand side of the MTL is terminated in a known network T. Assuming that the matrix on the left-hand side of equation (5.8) has an inverse, we can write

$$|b_L\rangle = \underline{L}\,|a_L\rangle \quad , \tag{5.9}$$

where $\underline{L}$ is the wanted scattering matrix

$$\underline{L} = (e^{-jkl}\,\underline{G}^{-1}\,\underline{Q}_L - e^{jkl}\,\underline{H}^{-1}\,\underline{P}_L)^{-1}\,(e^{jkl}\,\underline{H}^{-1}\,\underline{Q}_L - e^{-jkl}\,\underline{G}^{-1}\,\underline{P}_L). \tag{5.10}$$

$\underline{G}$ and $\underline{H}$ are defined by (5.6), while $\underline{P}$ and $\underline{Q}$ are defined in the previous Section by (4.27) and (4.28). The expression obtained for $\underline{L}$ solves completely the "Black box" problem of the one-sided MTL. The expression is very well suited for computer programming and is valid for any N.

As an example, let us use (5.10) in order to find the scattering matrix of a short-circuited section of a MTL, as shown in Figure 5.2.

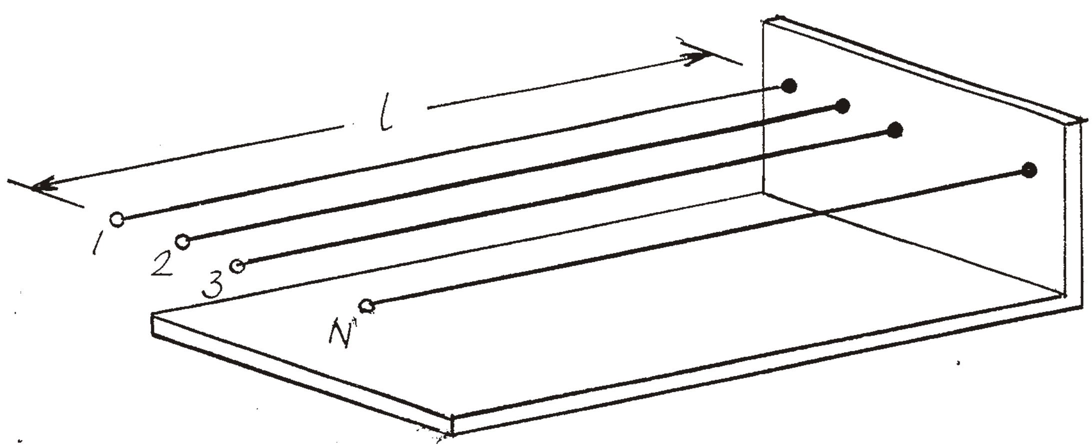


Figure 5.2

The scattering matrix of a short circuit at the right-hand side is the negative identity matrix (reflected wave voltages at the short circuit are equal and opposite to the incident wave voltages)

$$\underline{T} = -\underline{I} \ . \qquad (5.11)$$

Then, from (5.6)

$$\underline{G} = \underline{P}_R - \underline{Q}_R \ ; \quad \underline{H} = \underline{Q}_R - \underline{P}_R \ ,$$

or

$$\underline{G} = -\underline{H} \ . \qquad (5.12)$$

(5.10) therefore gives

$$\underline{L} = (e^{-jkl}\, G^{-1}\, \underline{Q}_L + e^{jkl}\, \underline{G}^{-1} \underline{P}_L)^{-1}\, (-e^{jkl}\, \underline{G}^{-1}\, \underline{Q}_L - e^{-jkl}\, \underline{G}^{-1}\, \underline{P}_L) . \qquad (5.13)$$

$\underline{G}^{-1}$ can be cancelled out:

$$\underline{L} = (e^{-jkl}\, \underline{Q}_L + e^{jkl}\, \underline{P}_L)^{-1}\, (-e^{jkl}\, \underline{Q}_L - e^{-jkl}\, \underline{P}_L) \qquad (5.14)$$

When the length l becomes equal to quarter-wavelength, $kl = \pi/2$ and (5.14) reduces to

$$\underline{L} = \underline{I} \qquad (5.15)$$

Therefore, when the quarter-wavelength section is short-circuited at the end, the input side behaves as an open circuit. Since $\underline{L}$ in (5.15) is a diagonal matrix, every incident wave at port i is reflected back at the same port, and none of its energy emerges at other ports. Therefore, the quarter-wavelength section of a MTL behaves exactly as a group of separate lines, shielded from each other, and short-circuited at one end. In a similar way, the open-circuited quarter-wavelength section

$$\underline{T} = \underline{I}$$

will result in a short circuit at the input:

$$\underline{L} = -\underline{I} \ .$$

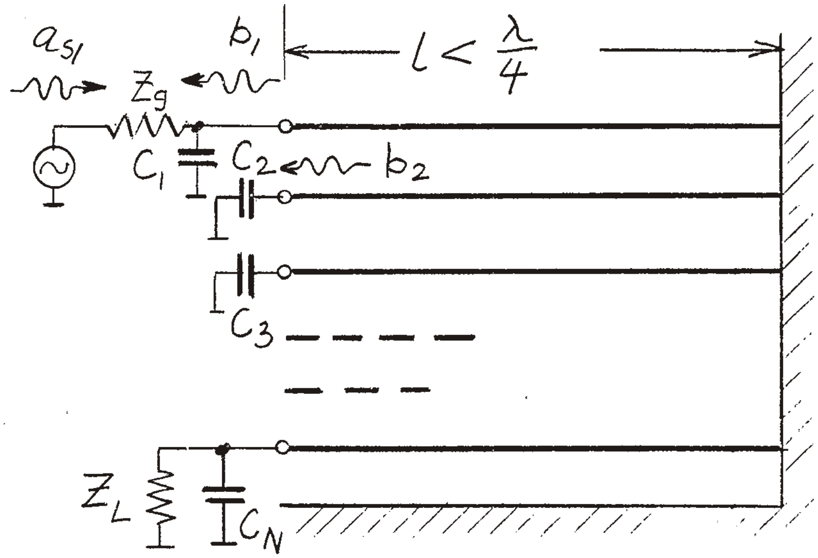


Figure 5.3

An important application of the short-circuited MTL is the comb-line filter shown in Figure 5.3. In order to make possible the propagation of energy from port 1 to port N, we have to add the capacitances C_1 through C_N at each port, and make $l < \lambda/4$ (see [14], Section 8.13), because we have just found that the quarter-wavelength section of MTL causes complete decoupling between the ports.

In the design of comb-line filters it is customary to neglect the mutual coupling between all conductors except the first neighbors. Such procedure is quite justified for the purposes of synthesis, in order to keep the algebra in manageable form. However, for computational checking of the frequency response of the filter which has been synthesized by this method it is possible to apply the more accurate matrix formalism developed here. Thus, we can check the actual behavior of the filter by taking into account all the mutual couplings, and not only between the first neighbors.

In order to use the scattering matrix $\underline{L}$ from (5.14) in the computation of the response of a comb-line filter, we will follow the procedure described in Ref. [17], Section C-1.2.

First we compute the reflection coefficients Γ_i looking from each port outwards. For the ports 2 to N-1 the only circuit element connected to the port is capacitance C_i. Its reflection coefficient is

$$\Gamma_i = \frac{1 - j\omega C_i Zo_i}{1 + j\omega C_i Zo_i} \tag{5.16}$$

Zo_i is the nominal characteristic impedance of port i.

For the first and last ports, the termination consists of a parallel combination of resistance and capacitance. Therefore

$$\Gamma_1 = \frac{1 - j\omega C_1 Z_{01} - Z_{01}/Z_g}{1 + j\omega C_1 Z_{01} + Z_{01}/Z_g} \tag{5.17}$$

The simplest choice is to take $C_1 = 0$ and $Zg = Zo_1$, so that $\Gamma_1 = 0$. A similar expression is obtained for Γ_N when Z_g is replaced by Z_L, etc. Next, we form the diagonal matrix of all the reflection coefficients

$$\underline{\Gamma} = \text{diag.}\ (\Gamma_1, \Gamma_2, \ldots \Gamma_N) \quad . \tag{5.18}$$

From matrix $\underline{\Gamma}$ which describes the outer network, and scattering matrix $\underline{L}$ which describes our one-sided section of a MTL, we compute the new matrix $\underline{\Sigma}$ as follows

$$\underline{\Sigma} = (\underline{I} - \underline{L}\,\underline{\Gamma})^{-1}\,\underline{L} \tag{5.19}$$

Furthermore, we define the vector of all generators of incident waves a_{s1} to a_{sn}. Since in our case only one generator exists at port 1, the vector of generated incident waves is

$$|a_s\rangle = \begin{pmatrix} a_{s1} \\ 0 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{pmatrix} \text{(N-dimensional)} \tag{5.20}$$

Then, the vector $|b\rangle$ of outgoing waves at ports 1 to N is

$$|b\rangle = \underline{\Sigma}\, |a_s\rangle \tag{5.21}$$

Since we are interested only in the output at port N, we have to compute only the element Σ_{N1} of the matrix $\underline{\Sigma}$. Element Σ_{N1} is the desired response of a filter, since it gives the amplitude and the phase of the wave coming out of port N when a generator of the incident wave is applied to port 1. The procedure described can easily be programmed to a computer, the use of which is unavoidable in filter design, anyway.

As only one element of matrix $\underline{\Sigma}$ gives the complete response of a two-port filter, it is obvious that the described procedure may be used to solve considerably more complex networks, like duplexers, multiplexers etc. In general, one can attach to every of the ports a generator, or a load. The matrix $\underline{\Sigma}$ contains all the information on transmission properties of the resulting multiport network.

Section 6.

TWO-CONDUCTOR TRANSMISSION LINES

By now, we have learned how to evaluate the scattering matrix of a MTL consisting of N conductors above the ground. The numerical results will be obtained on a digital computer in the following way:

Step 1. Define input data:

N ... number of conductors above the ground

kl ... electric length of the MTL

$\underline{K}$ ... matrix of coefficients of electrostatic induction, from (2.16)

$\underline{Z}_L, \underline{Z}_R$ ... diagonal matrices of left-side and right-side characteristic impedances, from (4.11) and (4.18)

Step 2. Compute the auxiliary matrices:

$\underline{P}_L$, $\underline{P}_R$, $\underline{Q}_L$, and $\underline{Q}_R$ from (4.27) and (4.28)

Step 3. Compute the submatrices:

$\underline{S}_{LL}$, $\underline{S}_{LR}$, $\underline{S}_{RL}$ and $\underline{S}_{RR}$ from (4.36) through (4.39). To perform this step one needs some computer subroutine for the inversion of a matrix with complex elements.

Step 4. Print out the values of the resulting scattering matrix $\underline{S}$ according to (4.35)

The above procedure remains unchanged, whatever the number of conductors and whatever their shape. Although $\underline{S}$ is not given in an explicit form but in the form of the step-by step procedure, the numerical results from the computer will be all the information that we will need for a practical design of a MTL device.

For some simple cases, it is possible to obtain not only the numerical results but also the explicit analytic expressions for S without going into a prohibitive number of algebraic manipulations. Such is the case of two conductors above the ground, which we will study in the present Section.

Odd-And-Even Modes vs. One-by-One Modes

A mode on a transmission system is an independent solution to the Maxwell's equations, one which alone satisfies the boundary conditions of the system in the transverse plane. The transmission system is assumed to be uniform in the direction of propagation (usually taken to coincide with the z axis of a Cartesian or cylindrical system of coordinates). The most general electromagnetic field which can exist on such a transmission system can be expressed as a linear combination of the fields of individual modes.

Just as we can talk about the modes constituting the total electric field on the system, we can also talk about the voltage modes, constituting the total voltage vector of the MTL. As long as we are dealing with the TEM types of the solutions to the Maxwell's equations, the voltage of the n-th conductor is defined uniquely as the line integral from the ground to the corresponding conductor. Reviewing what we have done in Section 1, we can easily visualize that our choice of voltage modes was a linear vector space which had the following basis

$$\begin{pmatrix} V_1 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ V_2 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ V_N \end{pmatrix}.$$

Each of these basis vectors represents one mode in which only one of conductors has a voltage different from zero (with respect to the ground), while the voltages of all other conductors are kept at zero. A few such modes are sketched in Figure 6.1. To give some name to these modes, we have called them one-by-one modes. Obviously, there are N distinct modes on a transmission system consisting of N conductors above the ground.

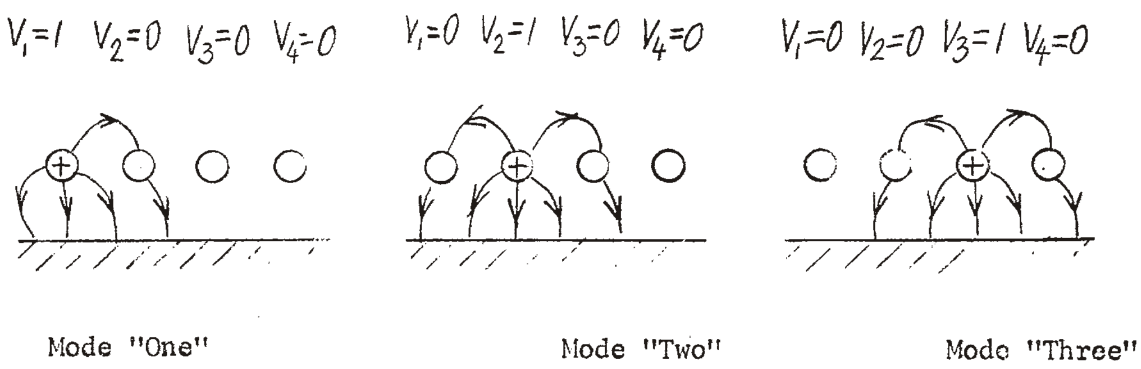


Figure 6.1

For N = 2, the two voltage modes of the one-by-one type are the following:

$$|V_A> = \begin{pmatrix}1\\0\end{pmatrix} \text{ and } |V_B> = \begin{pmatrix}0\\1\end{pmatrix} . \tag{6.1}$$

A linear combination of these two modes is sufficient to describe any actual voltage situation on this system of two conductors:

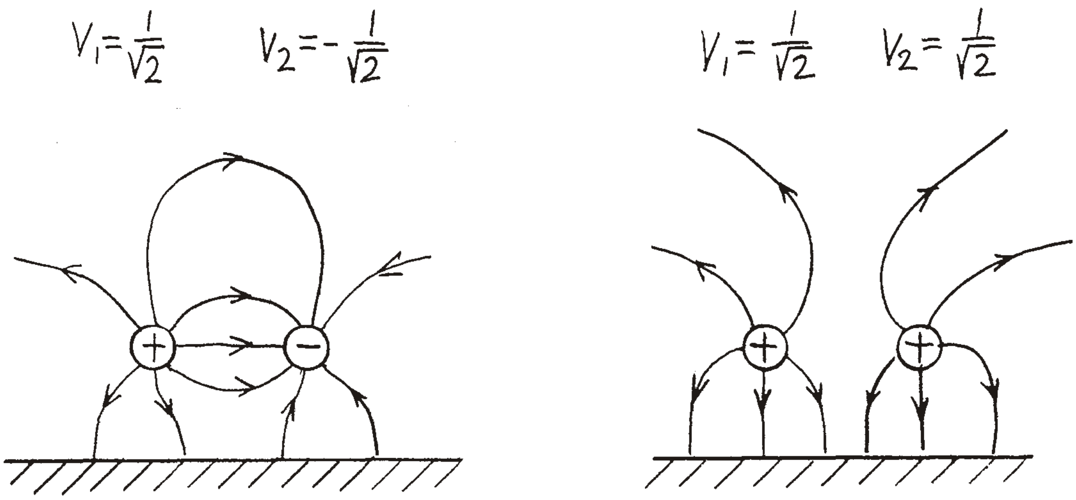
$$|V> = A|V_A> + B|V_B> , \tag{6.2}$$

where scalars A and B are the modal components of vector $|V>$.

As is well known from the theory of matrices and linear vector spaces, one can easily choose another basis in which to represent the desired quantities. In the case of two conductors above the ground, it is possible to define a new basis as follows:

$$|V_o> = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{and} \quad |V_e> = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} . \qquad (6.3)$$

These two voltage modes are called odd and even modes. The odd mode represents a physical situation in which the two conductors have equal and opposite voltages with respect to the ground, as in the left side of Figure 6.2, while in the even mode both conductors have the same voltage with respect to the ground, as is shown in the right side of Figure 6.2.



Odd Mode   Even Mode

Figure 6.2

The factor $\frac{1}{\sqrt{2}}$ in (6.3) is for normalization purposes. That is, the odd-and-even basis is also orthonormal, as one can see from the following

$$<V_o|V_o> = 1,$$

$$<V_e|V_e> = 1,$$

$$<V_o|V_e> = <V_e|V_o> = 0 .$$

By comparing (6.3) with (6.1) we find that the two representations are related as follows

$$|V_o\rangle = \frac{1}{\sqrt{2}} (|V_A\rangle - |V_B\rangle)$$

$$|V_e\rangle = \frac{1}{\sqrt{2}} (|V_A\rangle + |V_B\rangle)$$

Consequently, odd-and-even mode representation is just another linear combination of voltages on individual conductors. The solution of any circuit problem related to a two-conductor line may be found in either one-by-one or in odd-and-even modes. However, odd-and-even representation is useful only in systems consisting of two conductors above the ground, while the one-by-one representation is also useful for any other number of conductors. Since the one-by-one mode representation is more universally applicable, we can completely ignore the odd-and-even mode representation. We need be aware of odd-and-even modes only because they appear in the literature published by authors who prefer to use that notation when they discuss directional couplers, phase shifters, and similar microwave devices.

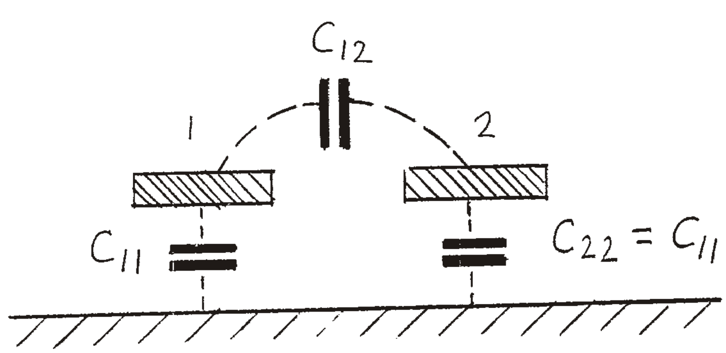


Figure 6.3

In Figure 6.3 we see two symmetric conductors (denoted by 1 and 2) above the ground and their relative capacitances. When we say symmetric conductors we mean that the cross sections of the two conductors are of the same shape and size and that they are placed at equal distances from the ground. This symmetry of the physical layout is not to be confused with the symmetry of the corresponding $\underline{K}$ matrix. That is, the electrostatic induction coefficient matrix $\underline{K}$ is always symmetric ($K_{ij} = K_{ji}$), regardless of whether the physical layout is symmetric or not. In other words, matrix $\underline{K}$ is symmetric even if $C_{11} \neq C_{22}$, i.e. even if the conductors are not symmetric.

According to (2.16), the matrix $\underline{K}$ of the two-conductor system from Figure 6.3 is

$$\underline{K} = \begin{pmatrix} C_{11} + C_{12} & -C_{12} \\ -C_{12} & C_{11} + C_{12} \end{pmatrix}. \tag{6.4}$$

Next, we will determine the characteristic impedance matrix $\underline{Z}_o$. For a symmetric pair of conductors, $\underline{Z}_o$ will consist of only two distinct elements, denoted by $\underline{Z}_s$ and $\underline{Z}_m$:

$$\underline{Z}_o = \begin{pmatrix} Z_s & Z_m \\ Z_m & Z_s \end{pmatrix}. \tag{6.5}$$

According to (3.23), $\underline{Z}_o$ is computed by inverting $\underline{K}$. The inverse of (6.4) is

$$\underline{K}^{-1} = \frac{1}{(C_{11} + C_{12})^2 - C_{12}^2} \begin{pmatrix} C_{11} + C_{12} & C_{12} \\ C_{12} & C_{11} + C_{12} \end{pmatrix} \tag{6.6}$$

Therefore, the elements of the matrix $\underline{Z}_o$ are

$$Z_s = \frac{C_{11} + C_{12}}{v\, C_{11}\, (C_{11} + 2C_{12})} \quad , \tag{6.7}$$

$$Z_m = \frac{C_{12}}{v\, C_{11}\, (C_{11} + 2C_{12})} \quad . \tag{6.8}$$

Let us compare these values with the characteristic impedance of the odd and-even modes. The usual definitions are (see e.g. [14], p. 193)

$$Z_{oe} = \frac{1}{vC_{11}} \tag{6.9}$$

for the even-mode impedance, and

$$Z_{oo} = \frac{1}{v(C_{11} + 2C_{12})} \tag{6.10}$$

for the odd-mode impedance. Comparing these two expressions with (6.7) and (6.8) reveals the following relationship:

$$Z_s = \frac{1}{2}\,(Z_{oe} + Z_{oo}) \quad , \tag{6.11}$$

$$Z_m = \frac{1}{2}\,(Z_{oe} - Z_{oo}) \quad . \tag{6.12}$$

Directional Coupler

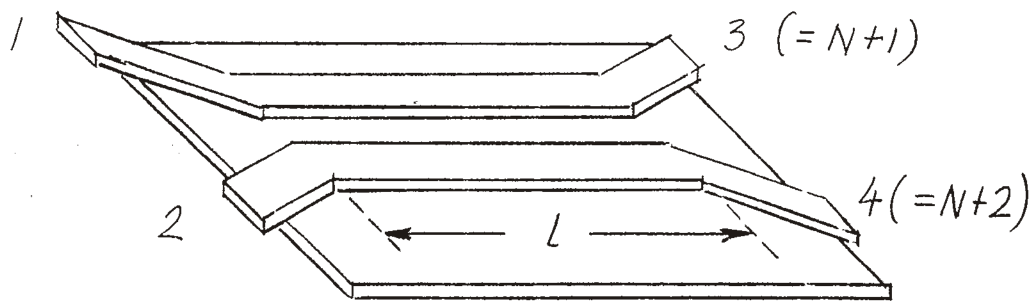


Figure 6.4

The most common configuration of the coupled-line TEM directional coupler is presented schematically in Figure 6.4. We will investigate its properties from the scattering-matrix point of view.

When a lossless 4-port is acting as a directional coupler, its scattering matrix should have the following form:

$$\underline{S} = \left(\begin{array}{cc|cc} 0 & S_{12} & S_{13} & 0 \\ S_{12} & 0 & 0 & S_{13} \\ \hline S_{13} & 0 & 0 & S_{12} \\ 0 & S_{13} & S_{12} & 0 \end{array}\right) = \begin{pmatrix} \underline{S}_{LL} & \underline{S}_{LR} \\ \underline{S}_{RL} & \underline{S}_{RR} \end{pmatrix} . \qquad (6.13)$$

When the elements on the main diagonal are zero, the device is internally matched (i.e. the device shows no reflections when terminated at every port in an outer resistance equal to the characteristic impedance of that port). The terms on the opposite diagonal are zero when the device has a property of isolation between the corresponding ports (e.g.

the energy of a wave entering port 1 splits into two parts which come out of ports 2 and 3, with nothing coming out of port 4, etc.).

We are interested in determining the conditions under which the network from Figure 6.4 has the properties of internal match and of isolation. According to (6.13), the internal match requires

$$\underline{S}_{LL} = S_{12}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{6.14}$$

For simplicity of computation, we choose that all ports are normalized to the same characteristic impedance Z_c. Then, the matrices (4.11) and (4.18) simplify to

$$\underline{Z}_L = \underline{Z}_R = Z_c\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = Z_c\underline{I} \;, \tag{6.15}$$

where $\underline{I}$ is the identity matrix. Furthermore, when $\underline{Z}_o$ is expressed as in (6.5), the auxiliary matrices $\underline{P}$ and $\underline{Q}$ from (4.27) and (4.28) become

$$\underline{P}_R = \underline{P}_L = \underline{P} = \frac{1}{\sqrt{Z_c}}\begin{pmatrix} Z_c + Z_s & Z_m \\ Z_m & Z_c + Z_s \end{pmatrix}, \tag{6.16}$$

$$\underline{Q}_R = \underline{Q}_L = \underline{Q} = \frac{1}{\sqrt{Z_c}}\begin{pmatrix} Z_c - Z_s & - Z_m \\ - Z_m & Z_c - Z_s \end{pmatrix}. \tag{6.17}$$

From (4.36) and (6.14) we obtain

$$(e^{jkl}\, \underline{Q}^{-1}\, \underline{P} - e^{-jkl}\, \underline{P}^{-1}\, \underline{Q})^{-1}\;\; (e^{-jkl} - e^{jkl}) = S_{12}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{6.18}$$

By noting that the inverse of the matrix on the right-hand side is equal to itself, i.e.

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{6.19}$$

we can rewrite (6.18) in the following form

$$(e^{-jkl} - e^{jkl})\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = S_{12} \quad (e^{jkl} \underline{Q}^{-1} \underline{P} - e^{-jkl} \underline{P}^{-1} \underline{Q}) \ . \qquad (6.20)$$

Computation of (6.20) requires the inverse of matrices $\underline{P}$ and $\underline{Q}$. For shorter notation, the determinants of these two matrices will be denoted by D_1 and D_2:

$$D_1 = \det(\underline{P}) = (Z_c + Z_s)^2 - Z_m^2 \ , \qquad (6.21)$$

$$D_2 = \det(\underline{Q}) = (Z_c - Z_s)^2 - Z_m^2 \ . \qquad (6.22)$$

After performing the matrix operations required on the right-hand side of (6.20) and comparing the diagonal elements on both sides of this matrix equation, we obtain the following:

$$0 = (D_1 e^{jkl} - D_2 e^{-jkl}) \quad (Z_c^2 - Z_s^2 + Z_m^2) \ . \qquad (6.23)$$

When this condition is fulfilled, the device is internally matched. Since D_1 is different from D_2, the first factor in (6.23) cannot be made equal to zero. Therefore, the only possibility for internal matching is

$$Z_c = \sqrt{Z_s^2 - Z_m^2} \ . \qquad (6.24)$$

Next, we shall investigate the conditions for isolation between ports. The device from Figure (6.4) will have ports 1-4 and 2-3 isolated when the off-diagonal elements of matrix $\underline{S}_{LR}$ from (6.13) are zero. The expression for $\underline{S}_{LR}$ is given by (4.37), which for the case under consideration becomes

$$(e^{jkl} \underline{Q}^{-1} \underline{P} - e^{-jkl} \underline{P}^{-1} \underline{Q})^{-1} \quad (\underline{Q}^{-1} \underline{P} - \underline{P}^{-1} \underline{Q}) = S_{13} \underline{I} \qquad (6.25)$$

From the requirement that the off-diagonal elements on both sides of the equation are equal to each other we obtain

$$(e^{jkl} - e^{-jkl})\ (Z_c^2 - Z_s^2 + Z_m^2) = 0\ . \qquad (6.26)$$

When the above requirement is fulfilled, the ports 1-4 and 2-3 are isolated from each other. This time we see that, at least in principle, there are two ways to achieve the isolation, namely, by making either the first or the second factor in (6.26) equal to zero. Obviously, when the second factor is zero, the device will at the same time have the ports isolated and be internally matched. These two properties will be retained regardless of length the l of the device.

The other possibility for achieving the isolation is to choose such a length l that the first factor in (6.26) will become zero. This will happen when $l = (2n + 1)\ \lambda/2$. Such a case has been studied in Section 4, and it was found that not only ports 1 and 3 are isolated, but also 1 and 2 are isolated at the same time, so that all the energy entering port 1 comes out of port 3. Such a device is certainly not useful as a directional coupler.

Referring back to the case when condition (6.24) is fulfilled, we may ask ourselves the following question. If the directional coupler possesses the properties of isolation and of internal matching regardless of the length l, why is it that every practical microwave directional coupler is chosen to be one-quarter of wavelength long at the center frequency? We can find the answer to this question by investigating the frequency dependence of the coupling coefficient S_{12}.

In order to plot the <u>frequency dependence</u> of the element S_{12} of the scattering matrix $\underline{S}_{LL}$, we may use the computer printout of the formula (4.36), without even bothering to find the explicit analytic expression. However, the algebraic manipulations of 2 x 2 matrices may be performed in a straightforward way, and the analytical result of substituting (6.24) into (4.36) is found to be

$$S_{12} = \frac{D_1 D_2 (e^{-jkl} - e^{jkl})}{2Z_m Z_c (D_1 e^{jkl} + D_2 e^{-jkl})}$$

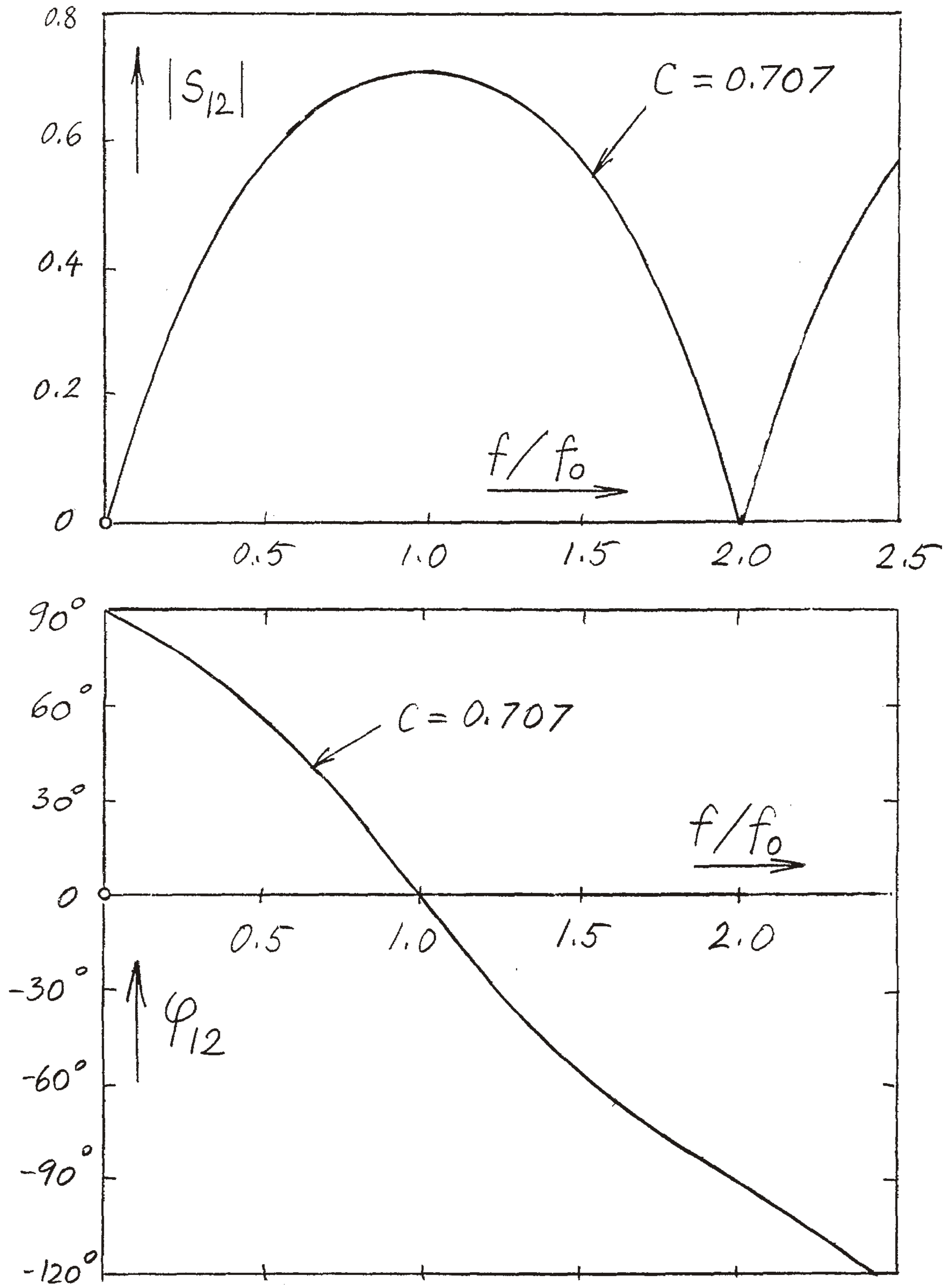
The above expression may be somewhat simplified by noting that at $kl = \Pi/2$ the value of S_{12} is

$$S_{12} \left(kl = \frac{\Pi}{2}\right) = C = \frac{Z_m}{Z_s} \quad . \tag{6.27}$$

With the use of C, our expression for S_{12} may be put in the form

$$S_{12} = \frac{jC \sin kl}{\sqrt{1 - C^2} \cos kl + j \sin kl} \quad , \tag{6.28}$$

which is the same as given e.g. in [14], p. 778. The amplitude and phase of S_{12} are plotted in Figure 6.5, as functions of the normalized frequency f/fo. It is clear that the coupling between ports 1 and 2 varies considerably with frequency, except in the vicinity of the frequency $f = f_o$, where the length l is equal to one-quarter wavelength. In the neighborhood of this point, the coupling becomes stationary with frequency. This explains why the design of every directional coupler is centered at $l = \lambda/4$.



0.8
0.6
0.4
0.2
0
|S12|
C = 0.707
f/fo
0.5
1.0
1.5
2.0
2.5
90°
60°
30°
0
-30°
-60°
-90°
-120°
C = 0.707
f/fo
φ12

Figure 6.5

All-pass Two-port

The all-pass network is an example of the one-sided section of a two-conductor transmission line. The right-hand side of the network consists of a simple junction of ports 3 and 4, as shown in Figure 6.6a.

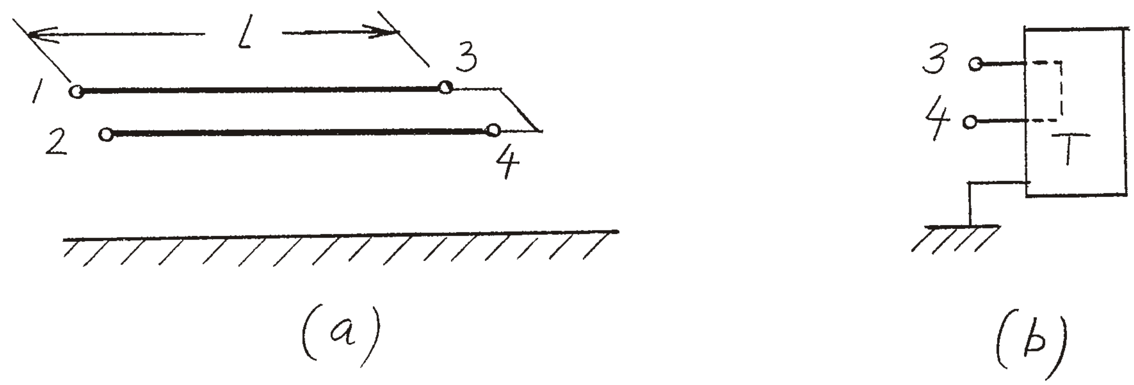


Figure 6.6

The network T, which is connecting ports 3 and 4, is indicated in Figure 6.6b. Assuming that ports 3 and 4 both have the same characteristic impedance to which they are normalized, the scattering matrix of the terminating network is [8]

$$\underline{T} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad (6.29)$$

The analytic expression for scattering matrix $\underline{L}$ of a two-port consisting of ports 1 and 2 will be found from (5.10). Let us mention again that for computation of numerical results we do not need to go through the derivation which follows, because (5.10) is already useful for evaluating the results on a computer. However, when we want to find the explicit analytic expressions for scattering matrix $\underline{L}$, we have to go through the following algebra.

For simplicity, we again choose that the characteristic impedances at all four ports be equal to each other, as we did in the case of the directional coupler:

$$\underline{Z}_L = \underline{Z}_R = Z_c \underline{I} \quad . \tag{6.15}$$

Auxiliary matrices $\underline{P}$ and $\underline{Q}$ are the same as in (6.16) and (6.17). The other pair of auxiliary matrices $\underline{G}$ and $\underline{H}$ follows from (5.6) and (6.29):

$$\underline{G} = \frac{1}{\sqrt{Z_c}} \begin{pmatrix} Z_c + Z_s - Z_m & Z_c - Z_s + Z_m \\ Z_c - Z_s + Z_m & Z_c + Z_s - Z_m \end{pmatrix} ,$$

$$\underline{H} = \frac{1}{\sqrt{Z_c}} \begin{pmatrix} Z_c - Z_s + Z_m & Z_c + Z_s - Z_m \\ Z_c + Z_s - Z_m & Z_c - Z_s + Z_m \end{pmatrix} .$$

Substituting these values in (5.10) we may compute all the elements of the matrix $\underline{L}$. In general, since the network is reciprocal, $L_{12} = L_{21}$. Furthermore, since we choose the same normalizing numbers at all ports, and since the two conductors are chosen to be symmetric, $L_{11} = L_{22}$. The computation reveals that L_{11} is proportional to the factor

$$(Z_c^2 - Z_s^2 + Z_m^2) \quad .$$

By choosing this factor to be zero, as in the case of the directional coupler (condition (6.24)), we can make $L_{11} = L_{22} = 0$. This means, that the device is internally matched. When (6.24) is fulfilled, we obtain

$$L_{12} = \frac{\sqrt{Z_s^2 - Z_m^2} \ \cos kl - j \ (Z_s - Z_m) \ \sin kl}{\sqrt{Z_s^2 - Z_m^2} \ \cos kl + j \ (Z_s - Z_m) \ \sin kl} \quad . \tag{6.30}$$

As the denominator is the complex conjugate of the numerator, the absolute value of L_{12} is unity. This should be so because the network is lossless, and $L_{11} = 0$. The phase angle θ of expression (6.30) is

$$\theta = \text{Arg } L_{12} = -2 \tan^{-1} \left(\sqrt{\frac{Z_s - Z_m}{Z_s + Z_m}} \tan kl \right) . \qquad (6.31)$$

The scattering matrix of the network is, therefore,

$$\underline{L} = e^{j\theta} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad (6.32)$$

where θ is given by (6.31). The network is internally matched, and it transmits every wave unattenuated from port 1 to port 2 or in the opposite direction. The phase of the wave passed through the network is shifted by the amount of θ radians. Therefore, this two-port is an all pass network.

Scattering matrix $\underline{L}$ from (6.32) implies that if the all-pass two-port is terminated in a matched load, the incident wave will be transmitted through the two-port with an unchanged amplitude and with a phase shift equal to θ. However, under normal operating conditions one may expect that the load impedance will slightly differ from the characteristic impedance. We want to investigate the error in the phase shift which is caused by such a mismatch. The network to be studied is shown in Fig. 6.7. The all-pass two-port is indicated by a dotted line, the generator has a reflection coefficient Γ_g, while the load has a reflection coefficient Γ_L. We define matrix $\underline{\Gamma}$ as

$$\underline{\Gamma} = \text{diag} \quad (\Gamma_g, \Gamma_L) \qquad (6.33)$$

To compute the phase shift between incident wave a_{s1} and outgoing wave b_2 we will use the $\underline{\Sigma}$ matrix procedure as defined in (5.19) and (5.21). The vector of generated incident wave is

$$|a_s\rangle = \begin{pmatrix} a_{s1} \\ 0 \end{pmatrix}$$

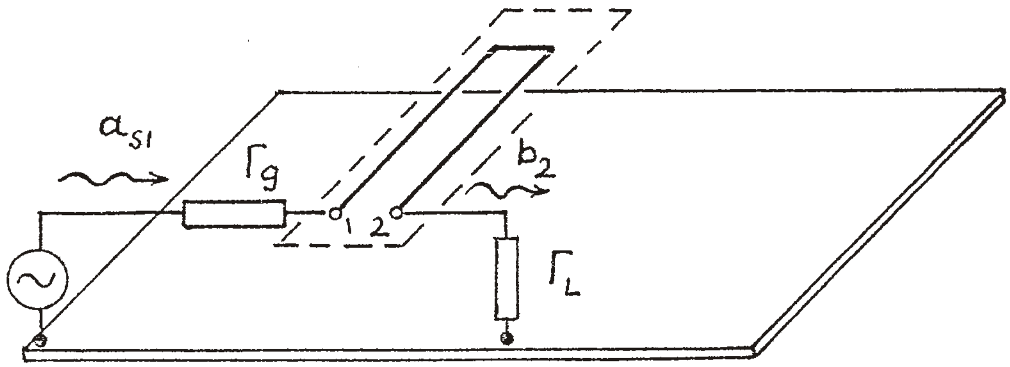


Figure 6.7

The vector of scattered waves is

$$|b> = \underline{\Sigma} \, | \, a_s >$$

If one knows the exact value of matrix $\underline{\Gamma}$, $\underline{\Sigma}$ may be computed from (5.19), and its element Σ_{12} will contain the desired information about the amplitude and phase of the ratio b_2/a_{s1} from Fig. 6.7. However, usually we know only the bounds on $|\Gamma_g|$ and $|\Gamma_L|$, and we want to investigate what is the resulting bound in phase and magnitude error. When

$$|\Gamma_g| \ll 1 \text{ and } |\Gamma_g| \ll 1$$

each term of the product $\underline{L}\,\underline{\Gamma}$ in (5.19) will be much smaller than unity. In this case, one can use the Neumann series for the inverse of identity plus small operator [26]

$$\underline{\Sigma} = (\underline{I} - \underline{L}\,\underline{\Gamma})^{-1}\underline{L} = (\underline{I} + \underline{L}\,\underline{\Gamma} + \underline{L}\,\underline{\Gamma}\,\underline{L}\,\underline{\Gamma} + \ldots)\underline{L}. \quad (6.34)$$

By using only the first three terms of this expression we obtain

$$\underline{\Sigma} \approx \underline{L} + \underline{L}\,\underline{\Gamma}\,\underline{L} + \underline{L}\,\underline{\Gamma}\,\underline{L}\,\underline{\Gamma}\,\underline{L}. \quad (6.35)$$

Substitute (6.32) and (6.33) into (6.35):

$$\underline{\Sigma} \approx e^{j\theta}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + e^{j2\theta}\begin{pmatrix} \Gamma_L & 0 \\ 0 & \Gamma_g \end{pmatrix} + e^{j3\theta}\begin{pmatrix} 0 & \Gamma_L\Gamma_g \\ \Gamma_L\Gamma_g & 0 \end{pmatrix}. \quad (6.36)$$

The term of interest is

$$\Sigma_{21} = \frac{b_2}{a_{s1}} \approx e^{j\theta} + e^{j3\theta}\Gamma_L\Gamma_g, \quad (6.37)$$

the graphical representation of which is shown in Fig. 6.8.

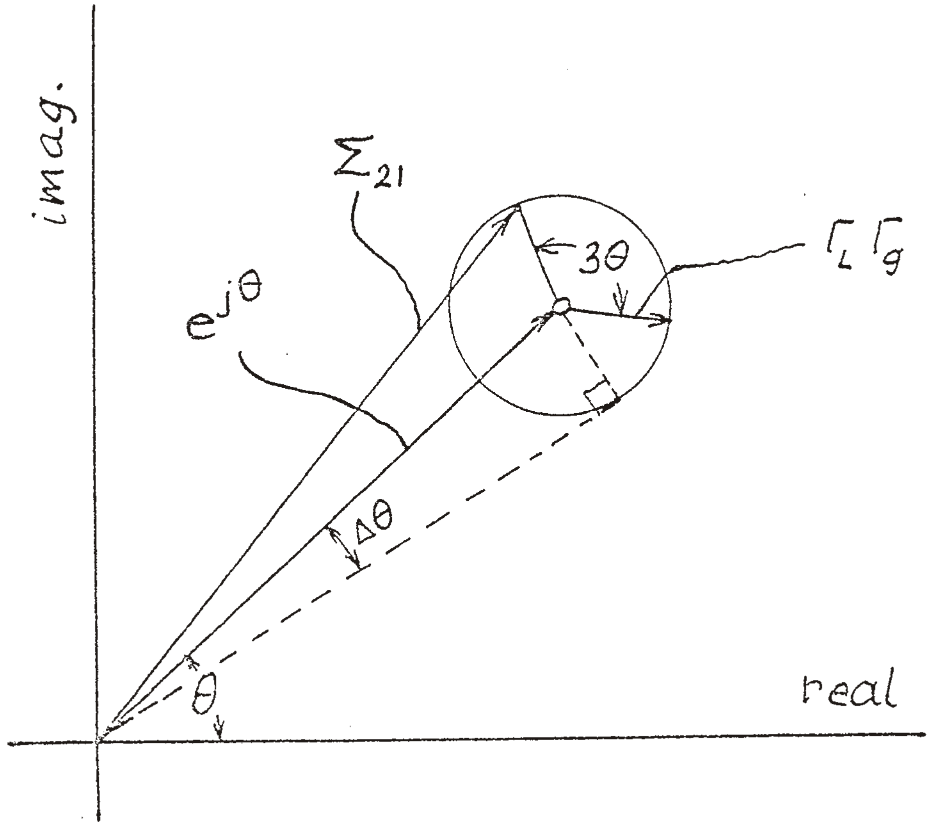


Figure 6.8

From Fig. 6.8, maximum phase error $\Delta\theta$ is

$$\Delta\theta = \sin^{-1}|\Gamma_L\Gamma_g| \approx |\Gamma_L\Gamma_g|. \tag{6.38}$$

One can see from (6.38) that the generator mismatch and the load mismatch have the same effect on the phase error. If each of Γ_L and Γ_g is having a magnitude 0.1 (corresponding to a standing wave ratio of 1.2), the resulting phase inaccuracy is about 0.6°. The same analysis can be applied also to a more complicated case when the all-pass scattering matrix is not ideal such as in (6.32), but it has some non-zero reflection coefficients on the main diagional. Furthermore, the analysis is applicable to any number of conductors.

Maximally-flat differential phase shifter

The arbitrary phase shift between ports 1 and 2 may be realized by inserting a simple transmission line between these two ports, as in Figure 6.9a. If the line is continuining on both sides into the lines of

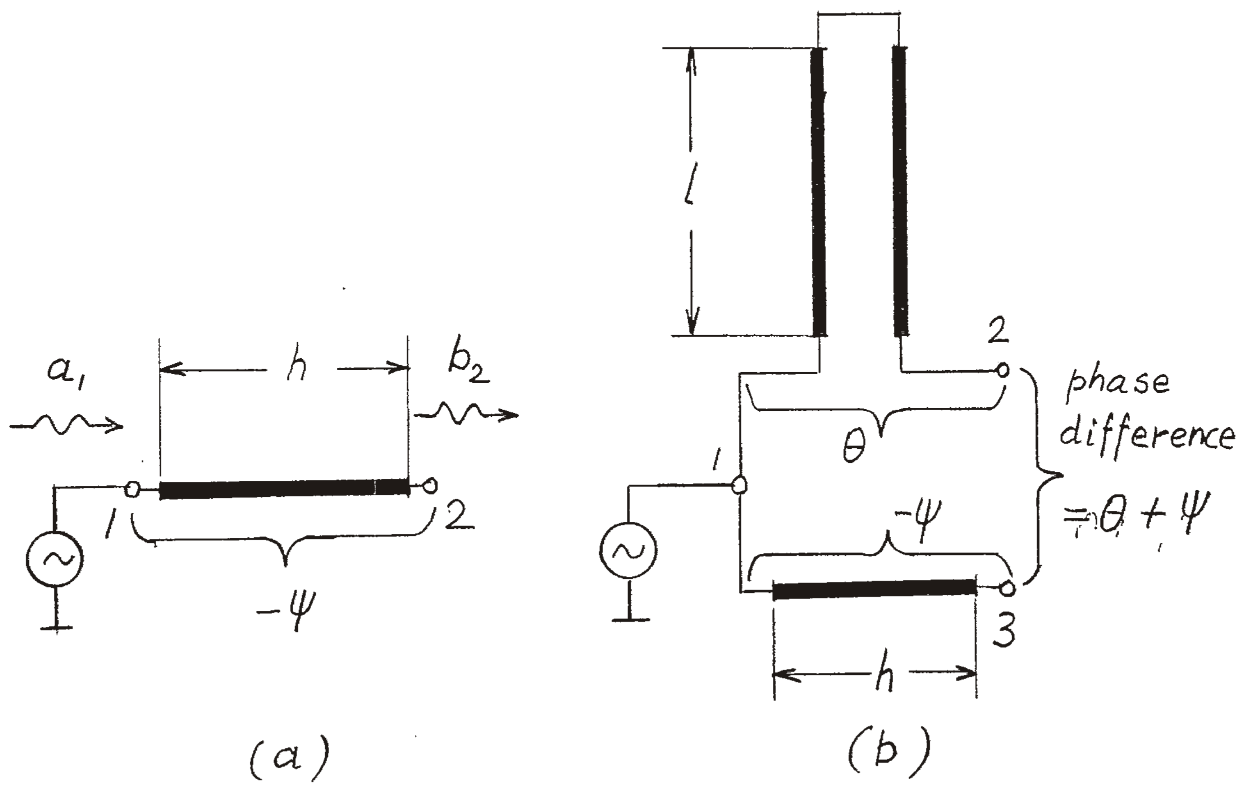


Figure 6.9

the same characteristic impedance, wave a_1 incident on port 1 will produce a pure traveling wave which will come out of port 2 as b_2:

$$b_2 = a_1\, e^{-j\psi}, \text{ where } \psi = kh\,. \tag{6.39}$$

Obviously, this crude phase shifter depends greatly upon frequency. As the propagation constant k is proportional to the frequency f, the phase shift ψ varies linearly with frequency:

$$\psi = k_o h + k_o h \frac{f-f_o}{f_o} = \psi_o + \Delta\psi \tag{6.40}$$

We have denoted with f_o the frequency at which the phase shift has value ψ_o. (6.40) indicates that the phase shift is decreasing with frequency.

If we want to synthesize the differential phase shifter which has essentially constant phase shift over a wide range of frequencies, we can use the structure from Figure 6.9b, as proposed originally by Schiffmann [19]. The idea is to combine the phase shift ψ of a section of a cable of length h with the phase shift θ of an all-pass network, as shown in Fig. 6.9b. The phase difference $\theta + \psi$ may be made virtually constant over the range of frequencies, if the slope of θ curve is made equal to the slope of ψ curve. Originally Schiffmann proposed to use such a device for 90° phase shifters only, and he did not give much information about how to choose its dimensions. We will see that by using (6.31) we can derive simple design formulas which are valid for any value of the phase shift, and which result in a maximally flat behavior of the total phase shift.

For shorter writing, we will use the following notation:

$\phi = k\ell$ ... the electric length of the all-pass network, (6.41)

$$m = \sqrt{\frac{Z_s - Z_m}{Z_s + Z_m}} = \sqrt{\frac{1 - C}{1 + C}} \quad \text{... coupling ratio.} \qquad (6.42)$$

Note that for very loose coupling of two conductors, $m \approx 1$. The tighter the coupling, the smaller the value of m. In the microstrip technique it is difficult to fabricate very strong coupling. Therefore, we desire that our m will be reasonably large, of the order 0.5 or so. In other words, if we have to choose between two possible designs, one requiring $m_1 = 0.1$ and the other $m_2 = 0.3$, we will usually prefer the larger one, m_2.

In the new notation, (6.31) becomes:

$$\theta = -2 \tan^{-1} (m \tan \phi). \qquad (6.43)$$

When the frequency varies around the value f_o, ϕ will be presented as

$$\phi = \phi_o + \Delta\phi = \phi_o + \phi_o \frac{f - f_o}{f_o}. \qquad (6.44)$$

For reasonably small values of $\Delta\phi$, θ can be expanded in the Taylor series around the value ϕ_o as follows:

$$\theta = \theta\Big|_{\phi_o} + \frac{d\theta}{\phi}\Big|_{\phi_o} \Delta\phi + \frac{1}{2}\,\frac{d^2\theta}{d\phi^2}\Big|_{\phi_o} \Delta\phi^2 + \ldots \qquad (6.45)$$

We will retain the first three terms of this expansion. The symbol $|\phi_o$ denotes that the quantity is to be evaluated at $\phi = \phi_o$. From (6.43) we find the coefficients of the Taylor series as follows:

$$\theta\Big|_{\phi_o} = -2 \tan^{-1} (m \tan \phi_o) \quad , \qquad (6.46)$$

$$\frac{d\theta}{d\phi}\Big|_{\phi_o} = \frac{-2m}{1 - (1 - m^2) \sin^2 \phi_o} \quad , \qquad (6.47)$$

$$\frac{d^2\theta}{d\phi^2}\Big|_{\phi_o} = \frac{-2m\,(1 - m^2) \sin 2\phi_o}{[1 - (1 - m^2) \sin^2 \phi_o]^2} \quad . \qquad (6.48)$$

Our design should give the following differential phase shift ϕ_{wan} between ports 3 and 1:

$$\phi_{wan} = \theta + \psi \; . \qquad (6.49)$$

This desired value of the phase shift should be independent of frequency, in the range of consideration. Substituting θ from (6.45), $\Delta\phi$ from (6.44), and ψ from (6.40) into the condition (6.49), we obtain an expression in the form of a polynomial of the second order. Equating both sides of the expression term by term results in three equations. From the constant term we obtain the first requirement:

$$\phi_{wan} = \theta_o + \psi_o \qquad (6.50)$$

From the linear term in frequency we obtain the second requirement:

$$\left.\frac{d\theta}{d\phi}\right|_{\phi_o} = -\frac{\psi_o}{\phi_o}. \tag{6.51}$$

Finally, the square term results in the third requirement:

$$\left.\frac{d^2\theta}{d\phi^2}\right|_{\phi_o} = 0. \tag{6.52}$$

Substituting (6.48) into the third requirement, we find that we must choose the length of the all-pass network according to

$$\sin 2\phi_o = 0, \text{ or } \phi_o = n\frac{\pi}{2} \text{ for } n = 1,2,3, \ldots \tag{6.53}$$

As one can see, there are many possibilities for maximally-flat phase behavior: n = 1,2, .. etc. However, we will consider seriously only the first two possibilities, namely n = 1 and n = 2.

From the first requirement, we obtain

$$\psi_o = \phi_{wan} + \pi \qquad \text{for } n = 1, \tag{6.54}$$

and

$$\psi_o = \phi_{wan} \qquad \text{for } n = 2. \tag{6.55}$$

Then, from the second requirement we obtain

$$m = \frac{\pi}{\phi_{wan} + \pi} \qquad \text{for } n = 1, \tag{6.56}$$

and

$$m = \frac{\phi_{wan}}{2\pi} \qquad \text{for } n = 2. \tag{6.57}$$

The last two equations determine the coupling ratio m. We see that for $\phi_{wan} = \pi$, both equations give the same result: $m = \frac{1}{2}$. However, for $\phi_{wan} < \pi$ the case n = 1 will give a larger m (looser coupling), while for $\phi_{wan} > \pi$, the case n = 2 will give a larger m.

From the **definition** of m (6.42), we see that the coupling coefficient C can be computed as

$$C = \frac{1 - m^2}{1 + m^2}. \tag{6.58}$$

Since much of the published data on coupled transmission lines is given in terms of odd-and-even characterisitic impendances, we will use (6.11) and

(6.12) together with (6.42) to obtain

$$\frac{Z_{oo}}{Z_{oe}} = m^2 \ . \qquad (6.59)$$

The other relationship between Z_{oe} and Z_{oo} is given by the requirement that the all-pass be internally matched. The condition is given by (6.24)

$$Z_c = \sqrt{Z_s^2 - Z_m^2} \ ,$$

or, in odd-and-even notation,

$$Z_c = Z_{oe} Z_{oo} \ . \qquad (6.60)$$

In the above, we have derived the formulas which enable us to design a maximally-flat phase shifter of the Schiffman type. The procedure is as follows

(1) Given values: ϕ_{wan} and Z_c.

(2) From (6.56) or (6.57) compute m and decide whether n = 1 or n = 2; n specifies the length of the phase shifter, according to (6.43).

(3) Choose the cross-sectional dimensions of the phase shifter so that (6.59) and (6.60) are satisfied; namely

$$Z_{oo} = mZ_c \ , \quad Z_{oe} = \frac{Zc}{m} \ . \qquad (6.61)$$

In order to check how good is the design, we may go back and compute the total phase shift $\psi + \theta$ as a function of frequency. This time we should not use the approximate Taylor's expansion for θ, but the exact function from (6.43).

As an example, one has computed a 50Ω, 120° phase shifter with the center frequency fo = 1.5GHZ. The length of the phase shifter is ℓ = 10 cm (for the air-filled transmission line), and the length of the auxiliary line is h = 6.67 cm. Furthermore, the coupling ratio is m = 0.333, and the odd and even mode characterisitic impedances are 16.67Ω and 150.0Ω. The computed phase shift as a function of frequency is shown in Fig. 6.10. As one can see from this figure, the phase shift deviates from the design value for less than 5° in a wide range of frequencies between 1.16 and 1.84 GHZ.

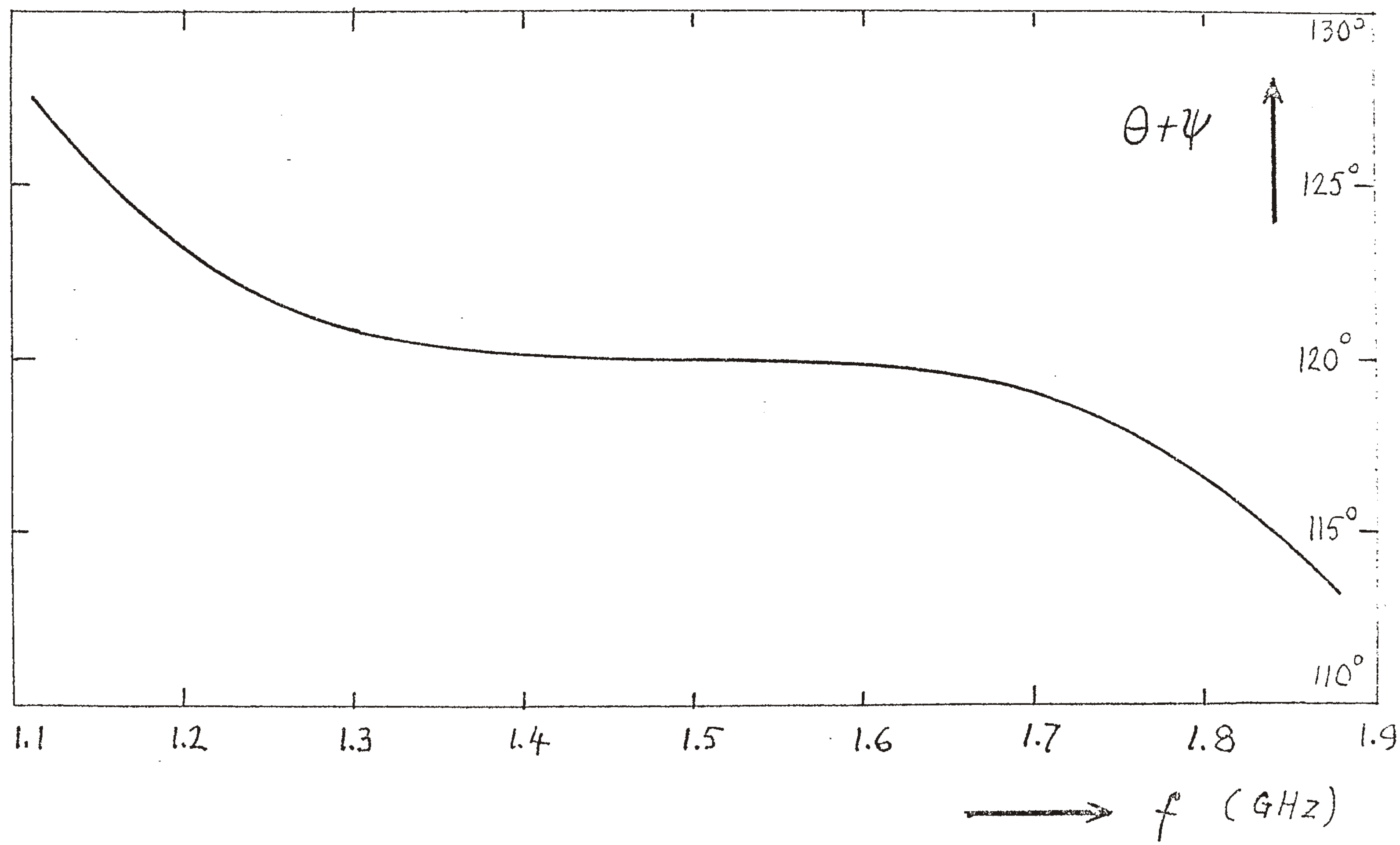


Figure 6.10

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