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FORMULATION OF INTEGRAL EQUATIONS FOR AN ELECTRICALLY SMALL APERTURE IN A CONDUCTING SCREEN

BY

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Chalmers M. Butler
University of Mississippi
University, Ms. 38677

ABSTRACT

This note is the result of one phase of an effort to better understand the penetration of electromagnetic energy through holes and cracks in conducting enclosures. Integral equations of a form highly amenable to numerical solution procedures are formulated for electrically small apertures in conducting screens. The new equations are based on a Rayleigh series analysis and potential theory, and they characterize aperture fields valid to zeroth and first order in reciprocal wavelength.

I

INTRODUCTION

For EMP studies, it is desirous to characterize and quantitize electromagnetic penetration through apertures in conducting surfaces so that deleterious effects on electronic systems within aircraft and missiles, among other units, can be assessed. Even though the classic problem of penetration of time-harmonic electromagnetic fields through an aperture in a planar conducting screen, as depicted in Figure 1, has been the subject of intensive research [1] for many years, still there appears to be no truly satisfactory formulation of equations both applicable to general aperture shapes and amenable to recently developed numerical solution techniques.

In this note, a new set of equations, which are reasonably well suited to numerical analysis, is developed for electrically small apertures in planar screens of vanishing thickness, infinite extent, and perfect conductivity. The formulation is based upon a Rayleigh series expansion [1,2,3,4] and potential theory, and it leads to integral equations for aperture fields valid to zeroth and first order in reciprocal wavelength. Even though two dimensional, the integral equations remind one of Hallén's familiar equation of thin-wire theory and, indeed, they lend themselves readily to an efficient numerical

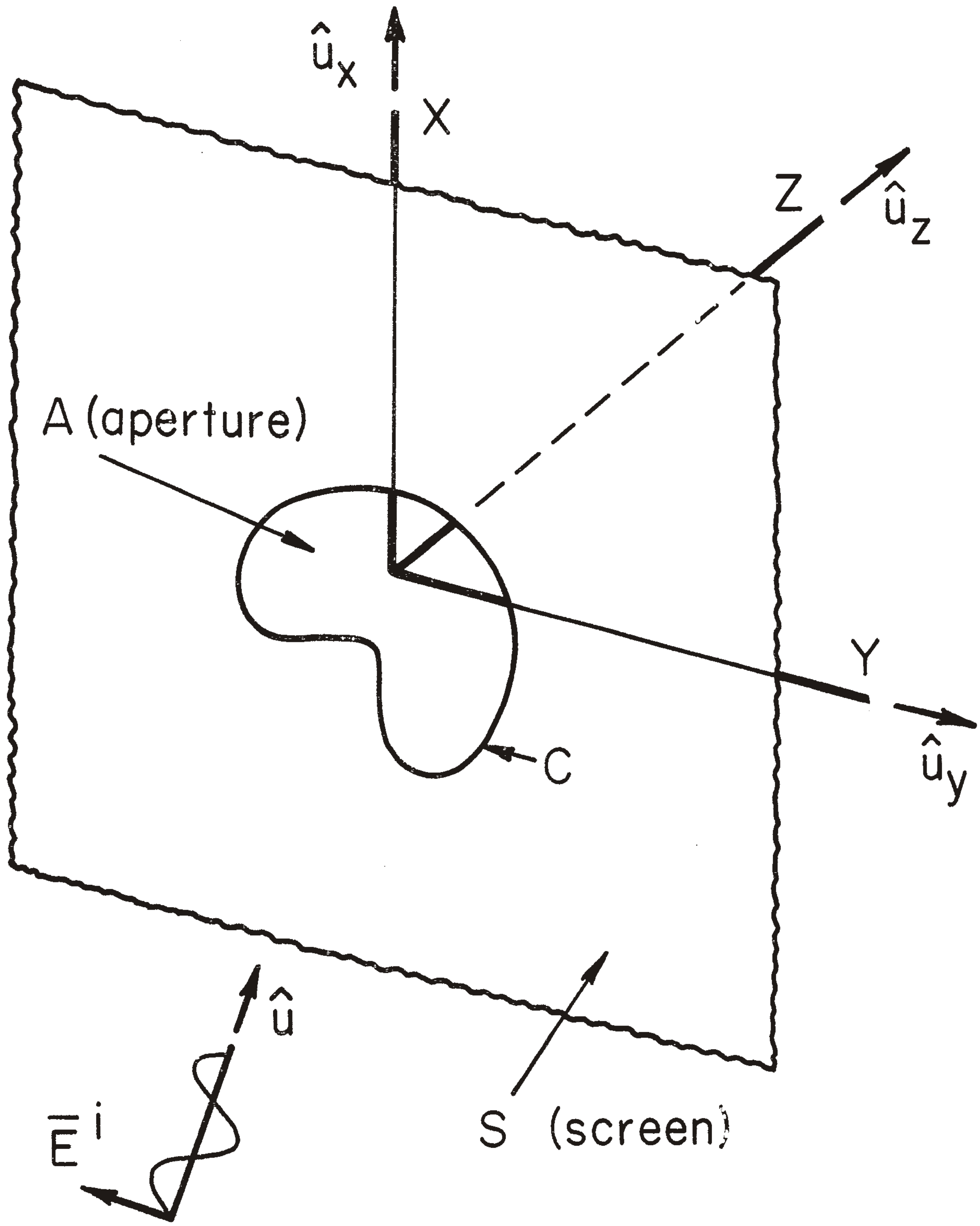

FIGURE 1. APERTURE IN CONDUCTING SCREEN ILLUMINATED BY INCIDENT FIELD

solution procedure highly analogous to that commonly applied to Hallén's equation.

The traditional equations of the aperture/screen problem are briefly reviewed in this note and a Rayleigh series analysis is presented. Equations pertinent to the zeroth and first order Rayleigh series coefficients are contrasted with corresponding equations based upon Bethe's [1,5] small aperture theory. Finally, these equations are converted to new forms of integral equations satisfactory for efficient moment method analysis.

II

APERTURE EQUATIONS

Properties of Fields Introduced by Presence of Aperture

Preliminary to the actual formulation of equations pertaining to the aperture problem under consideration here, it is desirable to review certain characteristics of the electromagnetic fields scattered by the infinite, perfectly conducting, perforated planar screen. There are several schemes whereby one can establish the basic properties of the fields scattered; one may investigate the nature of the currents and charges induced on the conducting plane and directly infer the behavior of the fields which these induced sources produce, but possibly a more illuminating approach, one which is compatible with a partitioning of the total field into two parts having the advantage that such leads to a useful set of equations, is one which is founded upon a theorem set forth by Schelkunoff [6] in 1951. Applied to the aperture problem suggested in Figure 1, the theorem simply says that the total fields $\overline{E}$ and $\overline{H}$, due to a prescribed incident electromagnetic wave $\overline{E}^i$ and $\overline{H}^i$ upon the screen/aperture, can be viewed as the sum of two partial fields $\overline{E}^a + \overline{E}^{sc}$ and $\overline{H}^a + \overline{H}^{sc}$, where $\overline{E}^{sc}$ and $\overline{H}^{sc}$ are the so-called short-circuit electric and magnetic fields which would exist, if the aperture were not present in the screen, i.e., if the plane

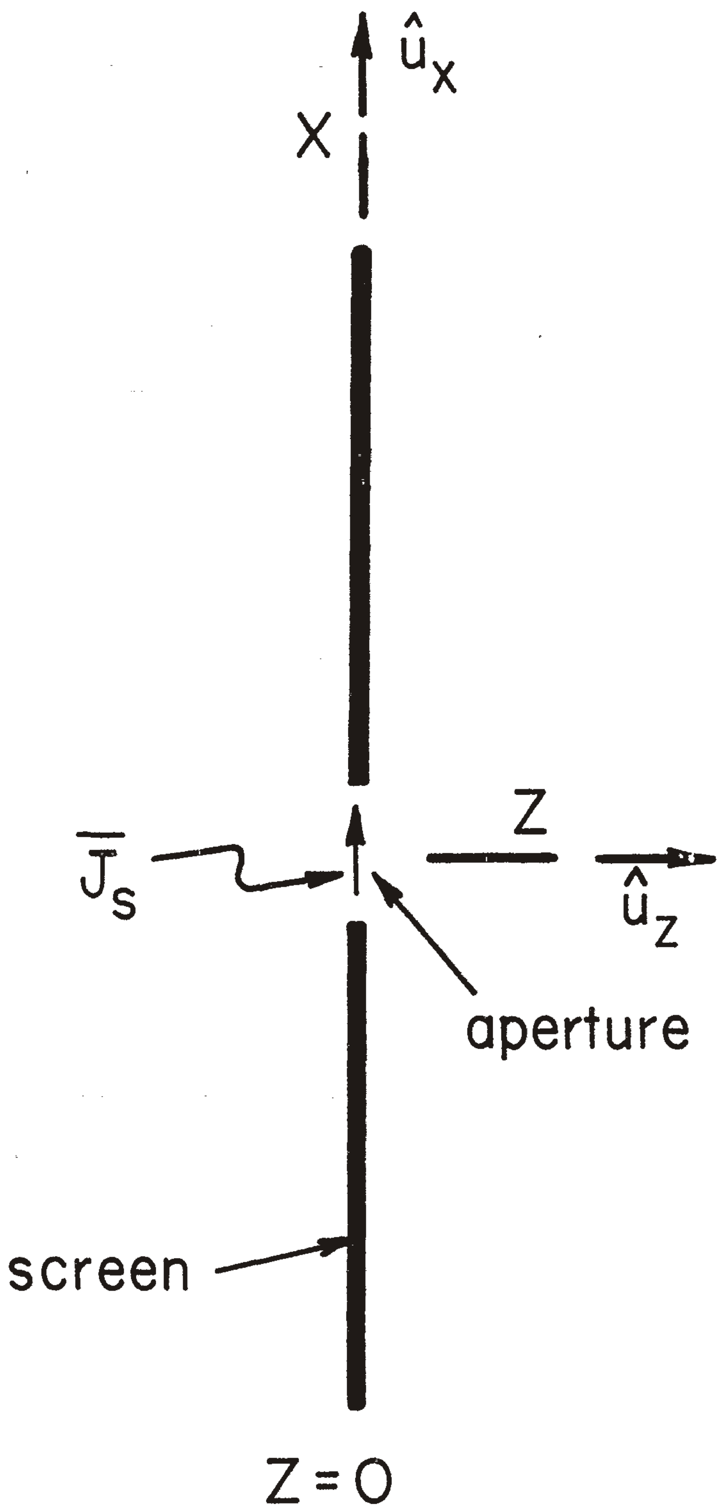

FIGURE 2. EQUIVALENT PROBLEM

were unperforated, and where $\overline{E}^a$ and $\overline{H}^a$ are the fields which would be radiated by a surface electric current $\overline{J}_s$ in the aperture A. The theorem further specifies that the electric surface current be $\overline{J}_s = -\overline{H}^{sc} \times \hat{u}_z$ which for the present situation becomes $\overline{J}_s = -2\overline{H}^i \times \hat{u}_z$ on the aperture A. In other words $\overline{E}^a$ and $\overline{H}^a$ are produced by $\overline{J}_s$ impressed in the aperture and radiating in the presence of the perforated conducting plane as depicted in Figure 2. From geometric symmetry and the fact that $\overline{J}_s$ is coplanar with the conducting screen, one readily observes the following properties* of $\overline{E}^a$ and $\overline{H}^a$:

$$E_x^a(x,y,z) = E_x^a(x,y,-z) \tag{1a}$$

$$E_y^a(x,y,z) = E_y^a(x,y,-z) \tag{1b}$$

$$E_z^a(x,y,z) = -E_z^a(x,y,-z) \tag{1c}$$

and

$$H_x^a(x,y,z) = -H_x^a(x,y,-z) \tag{2a}$$

$$H_y^a(x,y,z) = -H_y^a(x,y,-z) \tag{2b}$$

$$H_z^a(x,y,z) = H_z^a(x,y,-z). \tag{2c}$$

*A fuller discussions of symmetry in electromagnetics is found in [11].

In addition to the above symmetry properties, one observes that $\hat{u}_z \times \overline{E}^a$ is continuous through the aperture while $\hat{u}_z \times \overline{H}^a$ suffers a jump discontinuity through A proportional to $\overline{J}_s$.

The above theorem due to Schelkunoff is a simple way to establish desired properties but pursuant to explicit calculations, one abandons the theorem and expresses $\overline{E}^a$ and $\overline{H}^a$ in terms of an electric vector potential $\overline{F}$ in the usual way,

$$\overline{E}^a(\overline{r}) = \frac{1}{\varepsilon} \operatorname{curl} \overline{F}(\overline{r}), \qquad z \geq 0 \tag{3a}$$

$$\overline{H}^a(\overline{r}) = j\frac{\omega}{k^2}\left[k^2\overline{F}(\overline{r}) + \operatorname{grad}(\operatorname{div} \overline{F}(\overline{r}))\right], \qquad z \geq 0 \tag{3b}$$

and

$$\overline{E}^a(\overline{r}) = -\frac{1}{\varepsilon} \operatorname{curl} \overline{F}(\overline{r}), \qquad z \leq 0 \tag{4a}$$

$$\overline{H}^a(\overline{r}) = -j\frac{\omega}{k^2}\left[k^2\overline{F}(\overline{r}) + \operatorname{grad}(\operatorname{div} \overline{F}(\overline{r}))\right], \qquad z \leq 0, \tag{4b}$$

where, of course, the time variation $e^{j\omega t}$ has been suppressed. In (3) and (4) the angular frequency is denoted by ω and $k = \omega\sqrt{\mu\varepsilon}$ where ε and μ are the permittivity and permeability, respectively, of the medium into which the perforated screen is immersed. A Lorentz-type condition has been incorporated in (3) and (4), and these equations ensure that $\overline{E}^a$ and $\overline{H}^a$ satisfy both Maxwell's equations and the symmetry conditions (1) and (2), if one specifies the electric vector potential to be

$$\overline{F}(\overline{r}) = \frac{\varepsilon}{4\pi} \iint_A \overline{M}(\overline{r}') \frac{e^{-jk|\overline{r}-\overline{r}'|}}{|\overline{r}-\overline{r}'|} ds', \tag{5}$$

where

$$\overline{r} = x\hat{u}_x + y\hat{u}_y + z\hat{u}_z$$

and

$$\overline{r}' = x'\hat{u}_x + y'\hat{u}_y, \qquad x',y'\varepsilon A,$$

and where one may interpret $\overline{M} = M_x\hat{u}_x + M_y\hat{u}_y$ as a magnetic surface current density in the aperture. Equations (3a) and (4a), subject to (5), characterize an electric field $\overline{E}^a$ whose component parallel to the screen is zero over the entire xy-plane at $z = 0$ but which "jumps" to the correct value of $\overline{E}^a$ at $z = 0^+$ and $z = 0^-$ in the aperture A. An alternate expression for $\overline{F}$ which shows explicitly its relationship to the transverse component of the electric field introduced by the presence of the aperture follows immediately from (5):

$$\overline{F}(\overline{r}) = \frac{\varepsilon}{2\pi} \iint_A \left(\overline{E}^a(\overline{r}')\times\hat{u}_z\right)_{z=0} \frac{e^{-jk|\overline{r}-\overline{r}'|}}{|\overline{r}-\overline{r}'|} ds'. \tag{6}$$

Equations Governing the Electric Vector Potential $\overline{F}$

The expressions for the fields $\overline{E}^a$ and $\overline{H}^a$, introduced by the presence of the aperture in the screen illuminated by a specified incident field, together with the above-mentioned theorem [6] provide a basis for formulating equations governing $\overline{F}$. In particular, if one writes the total fields $\overline{E}$ and $\overline{H}$ as sums of short-circuit fields $\overline{E}^{sc}$ and $\overline{H}^{sc}$ and aperture-produced fields $\overline{E}^a$ and $\overline{H}^a$ as follows,

$$\overline{E}(\overline{r}) = \frac{1}{\varepsilon} \operatorname{curl} \overline{F}, \qquad z \geq 0 \tag{7a}$$

$$\overline{H}(\overline{r}) = j\frac{\omega}{k^2}\left\{k^2\overline{F} + \operatorname{grad}(\operatorname{div} \overline{F})\right\}, \qquad z \geq 0 \tag{7b}$$

and

$$\overline{E}(\overline{r}) = \overline{E}^{sc}(\overline{r}) - \frac{1}{\varepsilon} \operatorname{curl} \overline{F}(\overline{r}), \qquad z \leq 0 \tag{8a}$$

$$\overline{H}(\overline{r}) = \overline{H}^{sc}(\overline{r}) - j\frac{\omega}{k^2} \left\{k^2\overline{F}(\overline{r}) + \operatorname{grad}(\operatorname{div} \overline{F}(\overline{r}))\right\}, \qquad z \leq 0, \tag{8b}$$

and enforces continuity of $\hat{u}_z \times \overline{H}$ in the aperture, one arrives at

$$\frac{\partial^2}{\partial x^2}F_x + k^2F_x + \frac{\partial^2}{\partial x\partial y}F_y = -j\frac{k^2}{\omega} H_x^i, \qquad z = 0 \tag{9a}$$

and

$$\frac{\partial^2}{\partial y^2}F_y + k^2F_y + \frac{\partial^2}{\partial y \partial x}F_x = -j\frac{k^2}{\omega} H_y^i, \qquad z = 0, \tag{9b}$$

which hold in the aperture A. Use is made above of the fact that, on the illuminated side of the conducting plane at $z = 0$, $\overline{H}^{sc} \times \hat{u}_z = 2\overline{H}^i \times \hat{u}_z$ where $\overline{E}^i$ and $\overline{H}^i$ denote the known incident fields. It is worth noting that Equations (7) and (8), subject to the particular integral (5) for $\overline{F}$, are entirely compatible with Maxwell's equations in the two half-spaces and, further, they imply $\overline{E} \times \hat{u}_z = 0$ at $z = 0$; Equations (9) imply, in addition, that $\overline{H} \times \hat{u}_z$ and $\overline{E} \cdot \hat{u}_z$ be continuous through the aperture, and they serve to relate in equation form the electric vector potential associated with $\overline{E}^a$ and $\overline{H}^a$ to the prescribed incident fields $\overline{E}^i$ and $\overline{H}^i$ in the aperture A.

Although Equations (9) imply continuity of $\overline{E} \cdot \hat{u}_z$ through the aperture, this condition, which is equivalent in A to

$$\left(\frac{\partial}{\partial x}F_y - \frac{\partial}{\partial y}F_x\right) = \varepsilon E_z^i, \qquad z = 0, \tag{10}$$

can be substituted into (9) to achieve in A

$$\left(\nabla_t^2 + k^2\right)F_x = -\varepsilon\frac{\partial}{\partial z}E_y^i , \qquad z = 0 \tag{11a}$$

and

$$\left(\nabla_t^2 + k^2\right)F_y = \varepsilon\frac{\partial}{\partial z}E_x^i , \qquad z = 0 \tag{11b}$$

where $\nabla_t^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. Equations (11) do not of themselves ensure continuity of $\overline{E}\cdot\hat{u}_z$ through the aperture and, hence, are not compatible with Maxwell's equations. However, (11) and (10) together are entirely equivalent to (9) and they, of course, guarantee Maxwell's equations as well as appropriate boundary conditions on the screen and in the aperture. Notice that both F_x and F_y appear explicitly in (9a) and (9b) whereas only one component appears in each of Equations (11) with coupling between the two components being provided in the latter set through (10). Equations (9) and (11) (with (10)) are equally applicable to the aperture problem but the capacity for the uncoupled equations (11) and their coupling to be identified and handled apart from one another leads to a highly desirable equation formulation. For convenience in subsequent discussions, (11) and (10) are written below in vector form:

$$\left(\nabla_t^2 + k^2\right)\overline{F} = \varepsilon\frac{\partial}{\partial z}\left(\hat{u}_z \times \overline{E}^i\right), \quad \text{on } A \tag{12}$$

and

$$\mathrm{div}_t(\overline{F} \times \hat{u}_z) = \varepsilon \hat{u}_z \cdot \overline{E}^i, \quad \text{on } A \tag{13}$$

where, of course, $\mathrm{div}_t\ \overline{V} = \frac{\partial}{\partial x} V_x + \frac{\partial}{\partial y} V_y$.

III

INCIDENT FIELD

The field incident upon the aperture A in the infinite screen is specialized here to be a uniform, plane electromagnetic wave whose electric field can be represented by

$$\overline{E}^i = \left[e_x^i \hat{u}_x + e_y^i \hat{u}_y + e_z^i \hat{u}_z\right] e^{-jk(x \cos\alpha + y \cos\beta + z \cos\gamma)}$$

or

$$\overline{E}^i = \overline{e}^i \, e^{-jk(\hat{u} \cdot \overline{r})} \tag{14a}$$

where the direction cosines above are

$$\begin{aligned} \cos\alpha &= \hat{u} \cdot \hat{u}_z \\ \cos\beta &= \hat{u} \cdot \hat{u}_y \\ \cos\gamma &= \hat{u} \cdot \hat{u}_z \end{aligned} \tag{14b}$$

and where $\hat{u}$ is the direction in which the incident wave propagates. In (14a), e_x^i, e_y^i, and e_z^i are constants and represent the designated components of the incident electric field at the origin of coordinates (0,0,0). Furthermore, since $\mathrm{div}\overline{E}^i = 0$, $\overline{e}^i$ must possess the property, $\hat{u} \cdot \overline{e}^i = 0$.

Equations (12) and (13) characterizing the electric vector potential involve $\left(\hat{u}_z \cdot \overline{E}^i\right)_{z=0}$ and $\frac{\partial}{\partial z}\left(\hat{u}_z \times \overline{E}^i\right)_{z=0}$, which are given below:

$$\left(\hat{u}_z \cdot \overline{E}^i\right)_{z=0} = e_z^i \, e^{-jk(\hat{u}\cdot\overline{r})_0} \tag{15a}$$

and

$$\frac{\partial}{\partial z}\left(\hat{u}_z \times \overline{E}^i\right)_{z=0} = -jk \cos\gamma(\hat{u}_z \times \overline{e}^i)e^{-jk(\hat{u}\cdot\overline{r})_0}, \tag{15b}$$

where

$$(\hat{u}\cdot\overline{r})_0 = (\hat{u}\cdot\overline{r})_{z=0} = (x \cos\alpha + y \cos\beta). \tag{15c}$$

IV

RAYLEIGH SERIES ANALYSIS

In this section is outlined a procedure for converting Equations (12) and (13) to integral equations valid for apertures whose dimensions are small relative to the wavelength. The procedure is founded upon a power series expansion in $k\left(\frac{2\pi}{\lambda}\right)$ of pertinent quantities and leads, in principle, to a sequence of simple integral equations like those occurring in potential theory. Rayleigh [2] first proposed such a series expansion scheme for solving scattering and diffraction problems and Stevenson [7] later developed a systematic way to treat scattering from small conducting bodies. Kleinman [4] corrected and greatly simplified Stevenson's theory, and he presented an alternate technique based on Stevenson's ideas but simpler to apply. Eggimann [8] used a Rayleigh series approach to obtain a differential equation description of scattering from a small disk and small circular aperture but neither he nor any of the other authors mentioned above provides useful numerical data. In this note, a Rayleigh series analysis is used to convert Equations (12) and (13) into integral equations necessitating, in principle, that one solve only the type equations which occur in electrostatics.

Pursuant to a Rayleigh series analysis of the aperture equations, one expands quantities in (5), (12), and (13) each in a power series in k. If one writes such a power series expansion for the magnetic current $\overline{M}$ in the integrand of (5) and thereby defines the vector coefficients $\overline{M}_n$,

$$\overline{M} = \overline{M}_0 + \overline{M}_1 k + \overline{M}_2 k^2 + \ldots + \overline{M}_n k^n + \ldots$$

or

$$\overline{M}(\overline{r}') = \sum_{n=0}^{\infty} \overline{M}_n(\overline{r}')k^n, \qquad \overline{r}' \varepsilon A, \tag{16}$$

then he may obtain from (16) and

$$\frac{1}{R} e^{-jkR} = \frac{1}{R} \sum_{n=0}^{\infty} \frac{1}{n!}(-jR)^n k^n,$$

a useful series representation of the total integrand of $\overline{F}$ evaluated in the aperture A (at z = 0).

$$\overline{M} \frac{e^{-jkR}}{R} = \sum_{n=0}^{\infty} k^n \left\{ \sum_{p=0}^{n} \frac{(-j)^p}{p!} R^{p-1} \overline{M}_{n-p} \right\} \tag{17}$$

where

$$R(\overline{r},\overline{r}') = \Big[\left| \overline{r} - \overline{r}' \right| \Big]_{z=0} = \Big[(x-x')^2 + (y-y')^2 \Big]^{1/2}.$$

In a similar manner the inhomogeneous terms of (12) and (13) may be expanded also (see (15)):

$$\frac{\partial}{\partial z}\left(\hat{u}_z \times \bar{E}^i\right)_{z=0} = \cos\gamma\left(\hat{u}_z \times \bar{e}^i\right) \sum_{n=0}^{\infty} \frac{(-j)^{n+1}}{n!}\left[(\hat{u}\cdot\bar{r})_0\right]^n k^{n+1} \tag{18a}$$

and

$$\left(\hat{u}_z \cdot \bar{E}^i\right)_{z=0} = e_z^i \sum_{n=0}^{\infty} \frac{(-j)^n}{n!}\left[(\hat{u}\cdot\bar{r})_0\right]^n k^n. \tag{18b}$$

In terms of the series above, Equation (15a) now can be written

$$\left(\nabla_t^2 + k^2\right) \iint_A \sum_{n=0}^{\infty} k^n \left\{\sum_{p=0}^{n} \frac{(-j)^p}{p!} R^{p-1} \bar{M}_{n-p}\right\} ds'$$

$$= 4\pi \cos\gamma\left(\hat{u}_z \times \bar{e}^i\right) \sum_{n=0}^{\infty} \frac{(-j)^{n+1}}{n!}\left[(\hat{u}\cdot\bar{r})_0\right]^n k^{n+1} \tag{19}$$

which can be cast into the following form

$$\nabla_t^2 \iint_A \bar{M}_0 R^{-1} ds' + k\, \nabla_t^2 \iint_A \bar{M}_1 R^{-1} ds'$$

$$+ \sum_{n=2}^{\infty} k^n \left\{\sum_{p=0}^{n} \frac{(-j)^p}{p!} \nabla_t^2 \iint_A \bar{M}_{n-p} R^{p-1} ds'$$

$$+ \sum_{p=0}^{n-2} \frac{(-j)^p}{p!} \iint_A \overline{M}_{n-2-p} \, R^{p-1} ds' \Bigg\}$$

$$= 4\pi \cos\gamma (\hat{u}_z \times \overline{e}^i) \sum_{n=1}^{\infty} \frac{(-j)^n}{(n-1)!} \left[(\hat{u} \cdot \overline{r})_0 \right]^{n-1} \kappa^n .$$

Within the radius of convergence of the series on the left and right hand sides above, one may equate like powers of k to arrive, after index manipulation, at

$$\nabla_t^2 \iint_A \overline{M}_0 \, R^{-1} \, ds' = \overline{0}, \qquad (n = 0); \qquad (20)$$

$$\nabla_t^2 \iint_A \overline{M}_1 \, R^{-1} \, ds' = -j4\pi \cos\gamma (\hat{u}_z \times \overline{e}^i), \qquad (n = 1); \qquad (21)$$

and

$$\nabla_t^2 \iint_A \overline{M}_n R^{-1} ds' = 4\pi \cos\gamma (\hat{u}_z \times \overline{e}^i) \frac{(-j)^n}{(n-1)!} \left[(\hat{u} \cdot \overline{r})_0 \right]^{n-1}$$

$$- \sum_{p=0}^{n-2} \iint_A \Bigg\{ \frac{(-j)^{p+2}}{(p+2)!} \nabla_t^2 \left(\overline{M}_{n-2-p} \, R^{p+1} \right)$$

$$+ \frac{(-j)^p}{p!} \overline{M}_{n-2-p} \, R^{p-1} \Bigg\} ds', \qquad (n = 2,3,\ldots). \qquad (22)$$

Expansion of both members of the auxiliary condition (13) leads to

$$\sum_{n=0}^{\infty} k^n \left\{ \sum_{p=0}^{n} \frac{(-j)^p}{p!} \operatorname{div}_t \iint_A \left(\overline{M}_{n-p} \times \hat{u}_z \right) R^{p-1} ds' \right\}$$

$$= 4\pi e_z^i \sum_{n=0}^{\infty} \left\{ \frac{(-j)^n}{n!} \left[(\hat{u}\cdot\overline{r})_0 \right]^n \right\} k^n$$

which, upon equating coefficients of like powers of k, enables one to obtain

$$\operatorname{div}_t \iint_A \left(\overline{M}_0 \times \hat{u}_z \right) R^{-1} ds' = 4\pi e_z^i, \qquad (n=0); \qquad (23)$$

$$\operatorname{div}_t \iint_A \left(\overline{M}_1 \times \hat{u}_z \right) R^{-1} ds' = -j4\pi e_z^i (\hat{u}\cdot\overline{r})_0, \qquad (n=1); \qquad (24)$$

and

$$\operatorname{div}_t \iint_A \left(\overline{M}_n \times \hat{u}_z \right) R^{-1} ds' = 4\pi \frac{(-j)^n}{n!} e_z^i \left[(\hat{u}\cdot\overline{r})_0 \right]^n - \sum_{p=0}^{n-2} \frac{(-j)^{p+2}}{(p+2)!} \operatorname{div}_t \iint_A \left(\overline{M}_{n-2-p} \times \hat{u}_z \right) R^{p+1} ds'$$

$$(n=2,3,\ldots). \qquad (25)$$

Equations (20), (21), and (22) plus the necessary auxiliary conditions (23), (24), and (25) constitute the equations which one must solve for the coefficients $\overline{M}_n$ of (16) from which, of course, one can construct the Rayleigh series solution for the magnetic current density in the aperture. For apertures small relative to the wavelength (16) converges to the correct value of the magnetic current once the coefficients are known; said differently, the power series has a finite radius of convergence depending upon the electrical size of the aperture and within a circle of this radius (16) does, indeed, converge to the correct magnetic current.

The equations which one must solve for the zeroth, first and second order coefficients, $\overline{M}_0$, $\overline{M}_1$, and $\overline{M}_2$, are repeated on the next page for convenience. Notice that the zeroth order coefficient $\overline{M}_0$ can be determined (by solving the vector Equation (26)) from knowledge only of the z-component of the incident electric field evaluated in the aperture. Also, the first order coefficient of magnetic current $\overline{M}_1$ can be determined from a knowledge of the three components of the incidence electric field. But one sees that the second order coefficient depends not only upon the incident electric field but also upon the zeroth order coefficient $\overline{M}_0$; this pattern of dependence upon knowledge of lower order coefficients is observed

$$\nabla_t^2 \iint_A \overline{M}_0(\overline{r}')R^{-1}(\overline{r},\overline{r}')ds' = \overline{0} \tag{26a}$$

$$,\overline{r}\varepsilon A, \quad (n=0)$$

$$\mathrm{div}_t \iint_A \left[\overline{M}_0(\overline{r}') \times \hat{u}_z\right]R^{-1}(\overline{r},\overline{r}')ds' = 4\pi e_z^i \tag{26b}$$

$$\nabla_t^2 \iint_A \overline{M}_1(\overline{r}')R^{-1}(\overline{r},\overline{r}')ds' = -j4\pi \cos\gamma(\hat{u}_z \times \overline{e}^i) \tag{27a}$$

$$,\overline{r}\varepsilon A, \quad (n=1)$$

$$\mathrm{div}_t \iint_A \left[\overline{M}_1(\overline{r}') \times \hat{u}_z\right]R^{-1}(\overline{r},\overline{r}')ds' = -j4\pi e_z^i(\hat{u}\cdot\overline{r})_0 \tag{27b}$$

$$\nabla_t^2 \iint_A \overline{M}_2(\overline{r}')R^{-1}(\overline{r},\overline{r}')ds' = -4\pi \cos\gamma(\hat{u}_z \times \overline{e}^i)(\hat{u}\cdot\overline{r})_0 - \iint_A \left\{\overline{M}_0(\overline{r}')R^{-1}(\overline{r},\overline{r}') - \frac{1}{2}\nabla_t^2\left(\overline{M}_0(\overline{r}')R(\overline{r},\overline{r}')\right)\right\}ds', \tag{28a}$$

$$\mathrm{div}_t \iint_A \left[\overline{M}_2(\overline{r}') \times \hat{u}_z\right]R^{-1}(\overline{r},\overline{r}')ds' = -\pi e_z^i\left[(\hat{u}\cdot\overline{r})_0\right]^2 + \frac{1}{2}\,\mathrm{div}_t \iint_A \left[\overline{M}_0(\overline{r}') \times \hat{u}_z\right]R(\overline{r},\overline{r}')ds', \tag{28b}$$

$$\overline{r}\varepsilon A, \quad (n=2).$$

in the equations governing each coefficient of order equal to or higher than two. In principle, then, one finds that he must solve an equation constrained by an auxiliary condition for each coefficient where the coefficient equations plus auxiliary conditions differ only in that each possesses a different but known inhomogeneous term. Notice that systematically one can solve a single operator equation with different inhomogeneous terms to obtain $\overline{M}_0$ and $\overline{M}_1$. Then, knowing $\overline{M}_0$, one can determine $\overline{M}_2$ by solving again the same operator equation but with a still different inhomogeneous term. The same pattern is exhibited by the equations governing the higher order magnetic current coefficients: for all $\overline{M}_n$ the same operator equation must be solved, the inhomogeneous term of each being a function of both the known incident electric field and the previously calculated lower order magnetic current coefficients.

V

BETHE THEORY [5]

It is well known [1] that Bethe's theory for electromagnetic penetration through small apertures yields correct zero order results but incorrect first order results. In the vernacular of Equations (12) and (13) the essence of Bethe's theory can be reduced to four simple steps: first, one replaces $\varepsilon \frac{\partial}{\partial z}\left(\hat{u}_z \times \overline{E}^i\right)$ in (12) and $\varepsilon \hat{u}_z \cdot \overline{E}^i$ in (13) by their respective values at (0,0,0) the "center" of the aperture; second, the Helmholtz operator of (12) is replaced by the Laplacian operator; third, one replaces $\overline{F}$ of (5) by its Rayleigh series expansion; and, fourth, in (12) and (13) the first and second order coefficients in k are equated. These steps lead to

$$\nabla_t^2 \iint_A \overline{M}_0(\overline{r}')R^{-1}(\overline{r},\overline{r}')ds' = \overline{0} \tag{29a}$$

$$,\overline{r}\varepsilon A \qquad (n=0)$$

$$\text{div}_t \iint_A \left(\overline{M}_0(\overline{r}') \times \hat{u}_z\right) R^{-1}(\overline{r},\overline{r}')ds' = 4\pi e_z^i \tag{29b}$$

and

$$\nabla_t^2 \iint_A \overline{M}_1(\overline{r}')R^{-1}(\overline{r},\overline{r}')ds' = -j4\pi \cos\gamma(\hat{u}_z \times \overline{e}^i) \tag{30a}$$

$$,\overline{r}\varepsilon A \qquad (n=1)$$

$$\text{div}_t \iint_A \left(\overline{M}_1(\overline{r}') \times \hat{u}_z\right) R^{-1}(\overline{r},\overline{r}')ds' = 0 \tag{30b}$$

which, subject to comparison with (26) and (27), clearly support the correctness of Bethe's zero order results and the incorrectness of his first order results. Fortunately, far fields calculated on the basis of Bethe's first order theory are correct even though the corresponding near fields are in error.

VI

NEW INTEGRAL EQUATIONS FOR $\overline{M}_0$ AND $\overline{M}_1$

At this point, attention is turned to the formulation of new integral equations from which one can calculate the zeroth and first order magnetic current distributions in the aperture. Equations (26) and (27) are not well suited for numerical solution methods and, therefore, it is desirable to convert them to other integral equations which more readily yield to approximate solution techniques.

Preliminary to formulating well-behaved integral equations for $\overline{M}_0$ and $\overline{M}_1$, it is desirable to recall a few simple principles of potential theory. Namely, if one considers the two-dimensional, scalar equation below

$$\nabla_t^2 \ \phi(\overline{r}) = v(\overline{r}) \tag{31}$$

valid for $\overline{r}$ in s_{in} and s_{ex} of Figure 3 but not necessarily on their common boundary c, he recalls that

$$\phi(\overline{r}) = \iint\limits_{s_{in}+s_{ex}} v(\overline{r}')g(\overline{r}',\overline{r})ds(\overline{r}') + \oint\limits_{c} \left[\phi(\overline{r}_c^{in}) - \phi(\overline{r}_c^{ex})\right]\frac{\partial}{\partial n(\overline{r}_c)}g(\overline{r}_c,\overline{r})d\ell(\overline{r}_c)$$
$$- \oint\limits_{c} \left[\frac{\partial}{\partial n(\overline{r}_c)}\phi(\overline{r}_c^{in}) - \frac{\partial}{\partial n(\overline{r}_c)}\phi(\overline{r}_c^{ex})\right]g(\overline{r}_c,\overline{r})d\ell(\overline{r}_c) \tag{32}$$

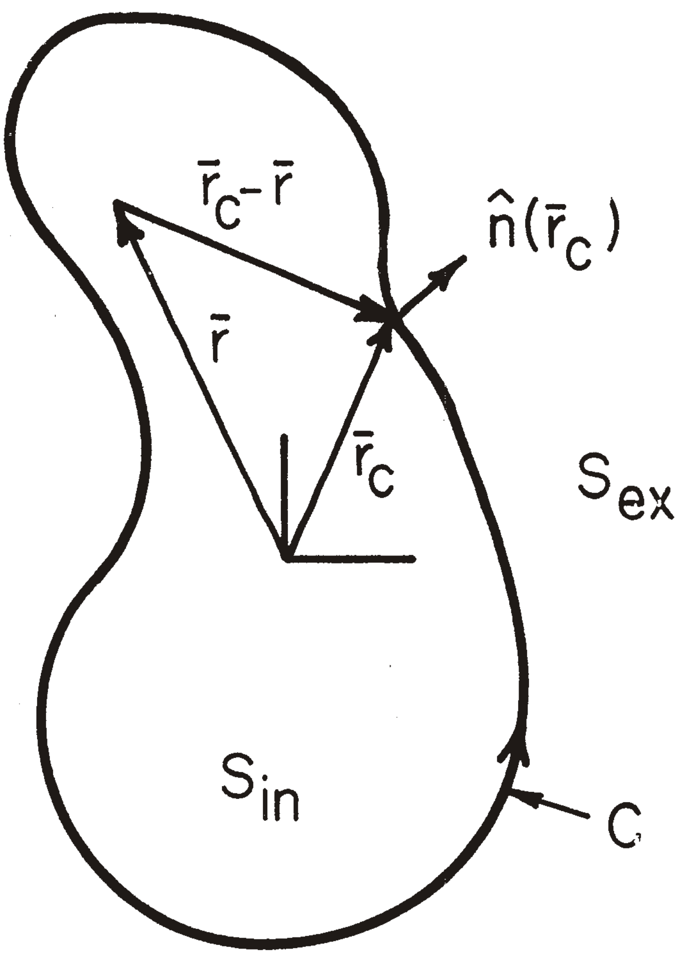

FIGURE 3. PLANAR REGIONS, CONTOUR, AND COORDINATE SYSTEM

where $\overline{r}_c$ designates a point on the contour c and where $\overline{r}_c^{in}$ and $\overline{r}_c^{ex}$ are the limits as $\overline{r}_c$ is approached from within the open regions s_{in} and s_{ex}, respectively. Also $n(\overline{r}_c)$ is the normal to c at $\overline{r}_c$ outward from s_{in} and $g(\overline{r}',\overline{r})$ is the two-dimensional Green's function,

$$g(\overline{r}',\overline{r}) = \frac{1}{2\pi} \ell n \; |\overline{r}'-\overline{r}|. \tag{33}$$

If ϕ is assumed continuous across c but its normal derivative discontinuous, then in s_{in}

$$\phi(\overline{r}) = \iint_{s_{in}} v(\overline{r}') \; g(\overline{r}',\overline{r}) \; ds(\overline{r}') + \oint_c \xi(\overline{r}_c) \; g(\overline{r}_c,\overline{r}) d\ell(\overline{r}_c) \tag{34}$$

where

$$\xi(\overline{r}_c) = \left\{ \frac{\partial}{\partial n(\overline{r}_c)} \phi(\overline{r}_c^{ex}) - \frac{\partial}{\partial n(\overline{r}_c)} \phi(\overline{r}_c^{in}) \right\}. \tag{35}$$

At this point, it is expedient to identify the surface integral term in (34) as the particular solution of (31) and the contour integral term as the homogeneous solution, even though in subsequent utilization of (34) a different form of the particular solution is to be employed and ξ is to be treated as merely some function defined on the contour c. Expression (34) is only one of several equivalent representations for ϕ in potential theory and it is found to be useful in the following analysis.

Zeroth Order Equation

Equations (26) govern the zeroth order magnetic current coefficient $\overline{M}_0$; they can be written more compactly as

$$\nabla_t^2\ \overline{f}_0(\overline{r}) = \overline{0} \qquad , \ \overline{r}\varepsilon A \tag{36a}$$

and

$$\mathrm{div}_t(\overline{f}_0 \times \hat{u}_z) = 4\pi e_z^i \tag{36b}$$

where the vector $\overline{f}_0$ represents

$$\overline{f}_0(\overline{r}) = \iint_A \overline{M}_0(\overline{r}')R^{-1}(\overline{r},\overline{r}')ds', \ \overline{r}\varepsilon A. \tag{36c}$$

Based on (34) the solution for the homogeneous equation (36a) can be written directly as

$$\overline{f}_0(\overline{r}) = 2\pi e_z^i(\hat{u}_z \times \overline{r}) + \oint_c \overline{\psi}_0(\overline{r}_c)\ g(\overline{r}_c,\overline{r})d\ell(\overline{r}_c) \tag{37}$$

where the first term above is seen to be harmonic and is added to the contour integral homogeneous solution to lessen the complexity of enforcing the auxiliary condition

(36b). Applying the auxiliary condition, one sees that $\overline{\psi}_0$ must be a vector on the contour c which satisfies the following for all points $\overline{r}$ in the open region $\overline{A}$:

$$\oint_c \left[\overline{\psi}_0(\overline{r}_c) \times \hat{u}_z\right] \cdot \mathrm{grad}_t \; g(\overline{r}_c, \overline{r}) \; d\ell(\overline{r}_c) = 0,$$

$$\text{for all } \overline{r} \varepsilon A. \tag{38}$$

To the set, (36c), (37), and (38), one appends the boundary condition that the component of the magnetic current coefficient normal to c be zero all along c,

$$\hat{n}(\overline{r}_c) \cdot \overline{M}_0(\overline{r}_c) = 0, \tag{39}$$

which is, of course, equivalent to requiring that the zeroth order electric field tangential to the aperture/screen edge be zero. Such an additional requirement is expected, since the introduction of the arbitrary boundary vector function $\overline{\psi}_0$ effectively increased the unknowns in the problem.

First Order Integral Equation

The first order coefficient equations (27) are written

$$\nabla_t^2 \; \overline{f}_1(\overline{r}) = -j4\pi \; \cos\gamma(\hat{u}_z \times \overline{e}^i) \tag{40a}$$

and

$$\mathrm{div}_t(\overline{f}_1 \times \hat{u}_z) = -j4\pi e_z^i(\hat{u}_z \cdot \overline{r}) \tag{40b}$$

$$, \; \overline{r} \varepsilon A$$

where, of course,

$$\overline{f}_1(\overline{r}) = \iint_A \overline{M}_1(\overline{r}')R^{-1}(\overline{r},\overline{r}')ds'. \tag{40c}$$

Here one writes the sum of the homogeneous and particular solutions for (40a) in the form

$$\overline{f}_1(\overline{r}) = -j\frac{2\pi}{1-\cos^2\gamma}\left\{\cos\gamma\left[(\hat{u}\cdot\overline{r})_0\right]^2(\hat{u}_z \times \overline{e}^i) + 2xye_z^i\left[\cos\beta\hat{u}_y-\cos\alpha\hat{u}_x\right]\right\}$$
$$+ \oint_C \overline{\psi}_1(\overline{r}_c)\; g(\overline{r}_c,\overline{r})d\ell(\overline{r}_c), \quad \overline{r}\varepsilon A \tag{41}$$

where again a harmonic term is appended to the homogeneous solution to lessen the difficulties of enforcing the auxiliary condition (40b) which when applied to (41) reduces to the requirement,

$$\oint_C \left(\overline{\psi}_1(\overline{r}_c) \times \hat{u}_z\right)\cdot\mathrm{grad}_t\; g(\overline{r}_c,\overline{r})d\ell(\overline{r}_c) = 0, \tag{42}$$

for all $\overline{r}\varepsilon A$.

As with the zeroth order equations, one adds the boundary condition,

$$\hat{n}(\overline{r}_c)\cdot\overline{M}_1(\overline{r}_c) = 0. \tag{43}$$

In the special but important case in which the excitation is normally incident upon the screen/aperture, (41) becomes indeterminant and hence is replaced by

$$\overline{f}_1^{\perp}(\overline{r}) = -j\pi\left\{(\overline{r}\cdot\overline{r})(\hat{u}_z \times \overline{e}^i) + 2xy(e_x^i\hat{u}_x - e_y^i\hat{u}_y)\right\}$$

$$+ \oint_C \overline{\psi}_1^{\perp}(\overline{r}_c)\; g(\overline{r}_c,\overline{r})d\ell(\overline{r}_c), \quad \overline{r}\varepsilon A \tag{44}$$

whereas the auxiliary condition retains the same form,

$$\oint_C \left(\overline{\psi}_1^{\perp}(\overline{r}_c) \times \hat{u}_z\right)\cdot \mathrm{grad}_t\; g(\overline{r}_c,\overline{r})d\ell(\overline{r}_c) = 0, \tag{45}$$

for all $\overline{r}$ in A. The superscript $\perp$ identifies quantities peculiar to the normal incidence case.

Auxiliary Conditions

Each of the auxiliary conditions (38), (42), and (45) requires that a contour integral, having an integrand involving an arbitrary vector defined on the contour, be zero over the entire open region A. In keeping with the proffered objective of seeking a set of integral equations wieldy for numerical solution techniques, it is demonstrated here that satisfaction of an auxiliary condition over the bounding contour c is equivalent to enforcing it over the entirety of A.

Following Smythe [9], one expands the Green's function (33) in circular harmonics and performs the indicated operations to arrive at

$$\sum_{n=1}^{\infty} \frac{1}{r} \oint_{c} \left(\frac{r}{r_c}\right)^n \left\{ \psi_\phi(\overline{r}_c) \cos n(\phi - \phi_c) + \psi_r(\overline{r}_c) \sin n(\phi - \phi_c) \right\} d\ell(\overline{r}_c) = 0, \quad r < r_c \qquad (46)$$

which is representative of any of the auxiliary conditions if $\overline{\psi} = \psi_r \hat{u}_r + \psi_\phi \hat{u}_\phi$ in polar coordinates is appropriately interpreted. In (46) (r, ϕ) and (r_c, ϕ_c) are the circular coordinate variables for $\overline{r}$ and $\overline{r}_c$, respectively.

Observe that, since the left-hand side of (46) is a Fourier series in ϕ, the requirement that (46) hold on any circle of radius r, $r < r_c$, ensures that it be true over the entirety of any open disk whose radius is less than r_c. Furthermore, by analytic continuation one recognizes that requiring satisfaction of the auxiliary conditions over the contour c is equivalent to requiring them to be true for all $\bar{r} \varepsilon A$. That the auxiliary conditions can be enforced over the contour rather than over the entire region greatly enhances the numerical attractiveness of the new integral equations presented here. Lastly, it is pointed out that on the contour c the integrals in (38), (42), and (45) are inproper (but convergent) [10] and care must be exercised in evaluating them.

Summary

Integral equations for $\overline{M}_0$ follow directly by equating (36c) and (37), subject to the boundary condition (39) and to the auxiliary condition (38) enforced on c. Similarly, equations for $\overline{M}_1$ follow from (40c) and (41) subject to (42) and (43) with the modifications established in (44) for the special case of normal incidence. These equations are recorded on the next page.

$$\iint_A \overline{M}_0(\overline{r}')R^{-1}(\overline{r},\overline{r}')ds' = 2\pi e_z^i(\hat{u}_z \times \overline{r}) + \oint_c \overline{\psi}(\overline{r}_c)\; g(\overline{r}_c,\overline{r})\; d\ell(\overline{r}_c), \quad \overline{r}\varepsilon A, \tag{47a}$$

(n=0)

$$\hat{n}(\overline{r}_c)\cdot\overline{M}_0(\overline{r}_c) = 0\;, \qquad \overline{r}_c\varepsilon c \tag{47b}$$

$$\begin{aligned}\iint_A \overline{M}_1(\overline{r}')R^{-1}(\overline{r},\overline{r}')ds' = -j\frac{2\pi}{1-\cos^2\gamma} &\left\{\cos\gamma\left[(\hat{u}\cdot\overline{r})_0\right]^2(\hat{u}_z \times \overline{e}^i)\right.\\ &\left.+ 2xye_z^i\left[\hat{u}_y \cos\beta - \hat{u}_x \cos\alpha\right] + \oint_c \overline{\psi}(\overline{r}_c)\; g(\overline{r}_c,\overline{r})\; d\ell(\overline{r}_c)\right\}, \quad \overline{r}\varepsilon A, \; (\gamma\neq 0)\end{aligned} \tag{48a}$$

(n=1)

$$\hat{n}(\overline{r}_c)\cdot\overline{M}_1(\overline{r}_c) = 0, \; \overline{r}_c\varepsilon c \tag{48b}$$

$$\lim_{\substack{\overline{r}\to\overline{r}_c \\ (\overline{r}\varepsilon A)}} \left\{\mathrm{div}_t \oint_c \overline{\psi}(\overline{r}_c) \times \hat{u}_z\; g(\overline{r}_c,\overline{r})\; d\ell(\overline{r}_c)\right\} = 0, \quad \text{all } \overline{r}_c\varepsilon c, \qquad (n=0 \text{ and } n=1). \tag{49}$$

VII

CONCLUSIONS

Based on preliminary considerations the new integral equations (47) - (49) presented here for the zeroth and first order magnetic current coefficients $\overline{M}_0$ and $\overline{M}_1$ should be well suited for moment method analysis, but converting Equations (28), or corresponding equations for higher order coefficients, to a form similar to that of (47) - (49) does not appear promising. However, a procedure paralleling that outlined in this note can be applied directly to (12) and (13) to obtain integral equations which are valid for the full dynamic aperture/ screen problem and which are similar in form to (47) - (49).

VIII

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IX

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