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ON THE SCATTERING OF ELECTROMAGNETIC WAVES BY A PERFECTLY CONDUCTING CYLINDER OVER A FINITELY CONDUCTING GROUND

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Abstract

The solution of the three-dimensional scattering of plane electromagnetic waves obliquely incident on an infinitely long perfectly conducting cylinder can be deduced from two-dimensional scattering solutions when the cylinder is in free space or over a perfectly conducting ground. This is not true when the ground is finitely conducting. The problem of the reflection of cylindrical waves at the boundary of a finitely conducting ground is itself a nontrivial problem. In order to develop solutions for these problems we consider first the simpler two-dimensional problems of electric and magnetic line sources over a finitely conducting ground. We calculate the Green's functions and obtain the reflected waves in the form of a reflection operator acting on the image source solution. For large values of the product of wave number and distance from the image source the reflection operators become the well-known Fresnel reflection coefficients. A succinct treatment is given of the problems when the line sources are at infinity and the radiation impinges on a perfectly conducting cylinder lying parallel to the finitely conducting ground. The three-dimensional problems are then considered and the boundary conditions at the surface of the ground are satisfied by introducing transverse waves of both types at the interface, even though the incident waves are either transverse electric or transverse magnetic.

scattering, conduction, cylinders

CONTENTS

ILLUSTRATIONS

I INTRODUCTION

In a previous article [1] we have treated the interaction of electromagnetic waves and pulses with an infinite, conducting cylinder over a perfectly conducting ground. While we assumed that the cylinder was also perfectly conducting in much of that work, we pointed out that the assumption of perfect conductivity was not really physically sound for thin wires at low frequencies. When only the dominant cylindrical wave with cylindrical symmetry ($m = 0$) was kept in the expansion of the scattered field, we were able to treat the problem when the conductivity of the wire was finite in a simple manner, and we showed that it lead to the proper physical behavior in the static limit.

Here we should like to turn our attention to the ground and treat the case of a more realistic medium with a finite conductivity. This problem is much more complicated than the idealized case of a perfectly conducting ground. One of the complicating features of the problem is the fact that the proper resolution of the waves with respect to the cylinder is into transverse electric and transverse magnetic waves, while with respect to the reflection at the ground the waves should be resolved into components with the electric vector in the plane of incidence and into components with the electric vector normal to the plane of incidence. Only in the two-dimensional case when the wave is propagating perpendicularly to the axis of the cylinder (but not in general normally to the ground) do the two resolutions coincide. Because they are simpler than the three-dimensional problems, we shall investigate to begin with the two-dimensional ones. Considering first electric and then magnetic line sources, we obtain the relevant Green's function that satisfies the proper boundary condition at the surface of the ground. The Green's function is in the form of the free-space Green's function plus the image source solution that is acted on by a reflection operator. When the product of the wave number and the distance from the source is very large, that is, when the waves are plane, the reflected waves can be obtained by the saddle point method of integration and the reflection operator becomes simply the appropriate Fresnel reflection coefficient. For small values of kR we obtain, in the case of an electric line source, the reflection operator in the form of a differential operator, while in the case of a magnetic line source we get an integro-differential operator. Applying the Green's functions to the problems where the line source goes to infinity and there is a cylinder with a surface current over the finitely conducting ground, we can express the electromagnetic field in forms that are similar to those in Reference [1] where the ground is perfectly conducting. Upon satisfying the boundary conditions on the surface of the cylinder, which is assumed perfectly conducting for convenience, we obtain two coupled sets of equations for the unknown coefficients, as in Reference [1].

In problems involving the scattering by an infinitely long cylinder in free space, we can get the solution of the three-dimensional problem from the solution of a two-dimensional problem. The relevant equations that express this relation are set down in the Appendix so that we may refer to them in the disucssions that follow. When the cylinder is in the presence of a finitely conducting ground, however, this relationship no longer holds because the boundary conditions at the surface of the ground are not all satisfied in the three-dimensional case, even though they are in the two-dimensional solution. We will show, nevertheless, that we can express the electromagnetic fields in a form that combines the forms of the two independent cases in the Appendix by expressing the reflected and transmitted fields at the interface in terms of both transverse electric and transverse magnetic waves, although the fields from the source are one or the other. The resulting expressions are of course more complicated than in the two-dimensional problems. The explicit calculation of the reflection operators in the three-dimensional case is left for a succeeding note in which we shall calculate the current induced on the cylinder and the potential between the cylinder and the ground for some electromagnetic pulses of interest.

II TWO-DIMENSIONAL PROBLEMS

1. Electric Line Source

Let the plane $y = 0$ separate the homogeneous medium $y < 0$, which is an imperfectly conducting ground (or a lossy dielectric) from the free-space medium $y > 0$ in which there is a line source of electric current parallel to the z - axis at the point $x = x_0$, $y = y_0$, as shown in Figure 1. We shall assume the periodic time variation $e^{-i\omega t}$.

The electromagnetic field is independent of z and may be expressed in terms of E_z. The field components are, in fact, E_z, H_x, H_y, with

$$i\,\omega\,\mu_0\,H_x = \frac{\partial E_z}{\partial y}\quad, \qquad (1)$$

$$i\,\omega\,\mu_0\,H_y = -\frac{\partial E_z}{\partial x}\quad. \qquad (2)$$

E_z satisfies the equations

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_1^2\right) E_z = -\,\delta(x - x_0)\,\delta(y - y_0), \qquad y > 0\quad, \qquad (3)$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_2^2\right) E_z = 0, \qquad y < 0\quad, \qquad (4)$$

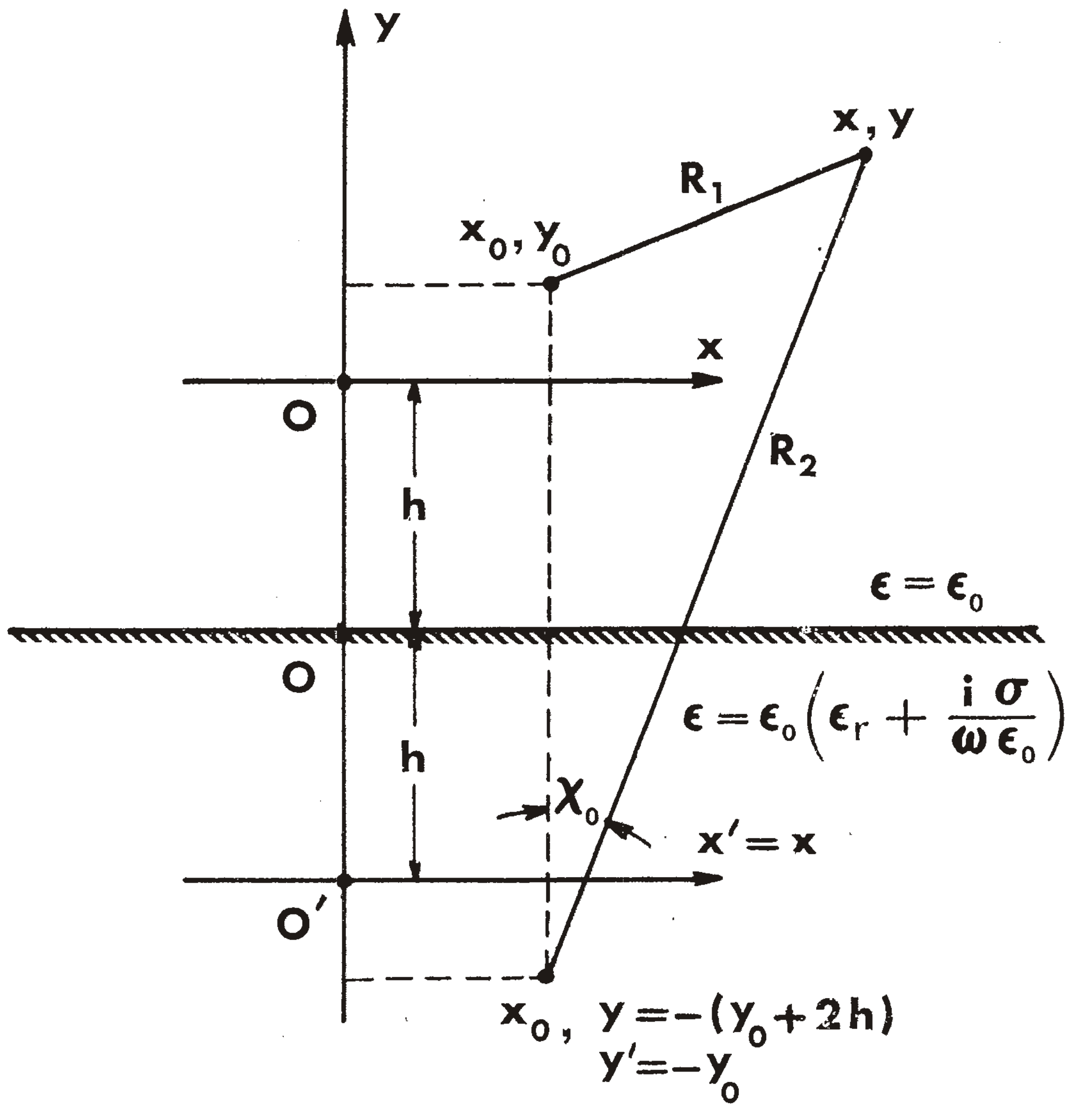

FIGURE 1. THE COORDINATE SYSTEMS AND COORDINATES USED IN THE TEXT

Three different cartesian coordinate systems are used in this work. The (x, y) system most frequently used has its origin a distance h above the ground plane. Another (x, y) system less frequently used has its origin at the ground plane, directly below the former one. Finally, the system (x', y') has the origin O'. In all cases, the z-axis is directed out of the page.

Both electric and magnetic current line sources are the generators of the incident electromagnetic field. These sources are at (x_o, y_o), and the image line sources are determined by reflection from the ground plane as shown.

with

$$k_1^2 = \omega^2 \epsilon_0 \mu_0 \quad , \tag{5}$$

$$k_2^2 = \omega^2 \epsilon \mu_0 \quad , \quad \epsilon = \epsilon_0 \left(\epsilon_r +- i \frac{\sigma}{\omega \epsilon_0} \right) \quad , \tag{6}$$

where ϵ_r and σ are respectively the dielectric constant and conductivity of the medium in $y < 0$. In Eq. (3) we have chosen the strength of the line source such that we are in fact calculating the Green's function. Fourier analyzing in the x- direction, we write

$$E_z(x,y) = \int_{-\infty}^{\infty} u(q,y)\, e^{iq(x-x_0)}\, dq \quad . \tag{7}$$

Substituting this expression in Eqs. (3) and (4) we get

$$\frac{d^2u}{dy^2} + \kappa_1^2 u = -\frac{1}{2\pi}\,\delta(y - y_0), \qquad y > 0 \tag{8}$$

$$\frac{d^2u}{dy^2} + \kappa_2^2 u = 0, \qquad y < 0 \quad , \tag{9}$$

where

$$\kappa_1^2 = k_1^2 - q^2 \quad , \tag{10}$$

$$\kappa_2^2 = k_2^2 - q^2 \quad . \tag{11}$$

For $y > 0$, u is the one-dimensional Green's function, aside from the factor $1/2\pi$, and for $y < 0$, u must represent outgoing waves. Thus

$$u = \frac{i}{4\pi\varkappa_1} e^{i\varkappa_1|y-y_0|} + \frac{i A_1}{4\pi\varkappa_1} e^{i\varkappa_1(y+y_0)} , \qquad y > 0 \qquad (12)$$

$$u = \frac{i B_1}{4\pi\varkappa_2} e^{-i\varkappa_2 y} , \qquad y < 0 , \qquad (13)$$

where A_1 and B_1 are to be determined from the boundary conditions that E_z and $i\omega\mu_0 H_x = \partial E_z/\partial y$ must be continuous at $y = 0$. Thus u and $\partial u/\partial y$ must be continuous at $y = 0$. We therefore have

$$e^{i\varkappa_1 y_0} + A_1 e^{i\varkappa_1 y_0} = \frac{\varkappa_1}{\varkappa_2} B_1 , \qquad (14)$$

$$e^{i\varkappa_1 y_0} - A_1 e^{i\varkappa_1 y_0} = B_1 , \qquad (15)$$

from which we find

$$B_1 = \frac{2\varkappa_2}{\varkappa_1 + \varkappa_2} e^{i\varkappa_1 y_0} , \qquad (16)$$

$$A_1 = \frac{\varkappa_1 - \varkappa_2}{\varkappa_1 + \varkappa_2} . \qquad (17)$$

Thus

$$u(q,y) = \frac{i}{4\pi\sqrt{k_1^2 - q^2}} e^{i\sqrt{k_1^2-q^2}|y-y_0|}$$

$$+ \frac{i\left(\sqrt{k_1^2 - q^2} - \sqrt{k_2^2-q^2}\right)}{4\pi\sqrt{k_1^2 - q^2}\left(\sqrt{k_1^2 - q^2} + \sqrt{k_2^2 - q^2}\right)} e^{i\sqrt{k_1^2-q^2}(y+y_0)} , \qquad (18)$$

$$y \geq 0$$

and

$$u(q,y) = \frac{1}{2\pi} \cdot \frac{ie^{i\sqrt{k_1^2-q^2}\,y_0 - i\sqrt{k_2^2-q^2}\,y}}{\sqrt{k_1^2 - q^2} + \sqrt{k_2^2 - q^2}} \, , \quad y < 0 \quad . \tag{19}$$

We are here only interested in the field for $y \geq 0$. We put the expression in Eq. (18) in the integral Eq. (7) and note that $\exp\left(i\sqrt{k_1^2 - q^2}\right)$ becomes $\exp\left(-\sqrt{q^2 - k_1^2}\right)$ when $q > k_1$. Furthermore when $q > \mathrm{Re}k_2 > k_1$, the factor containing the radicals becomes very small. Thus the main contribution to the integral comes from the range $q < \mathrm{Re}k_2$ where we can develop $\sqrt{k_2^2 - q^2}$ as

$$\sqrt{k_2^2 - q^2} = k_2\left(1 - \frac{q^2}{2k_2^2} - \frac{q^4}{8k_2^4} - \ \cdots\right) \quad . \tag{20}$$

Thus

$$\frac{\sqrt{k_1^2 - q^2} - \sqrt{k_2^2 - q^2}}{\sqrt{k_1^2 - q^2} + \sqrt{k_2^2 - q^2}} = \frac{\left\{\sqrt{k_1^2 - q^2} - \sqrt{k_2^2 - q^2}\right\}^2}{k_1^2 - k_2^2}$$

$$= \frac{k_1^2 - q^2 + k_2^2 - q^2 - 2\sqrt{k_1^2 - q^2}\,\sqrt{k_2^2 - q^2}}{k_1^2 - k_2^2}$$

$$= -1 - \frac{2\left(k_1^2 - q^2\right)}{k_2^2 - k_1^2} + \frac{2\sqrt{k_1^2 - q^2}}{k_2^2 - k_1^2}\left\{k_2 - \frac{q^2}{2k_2} - \frac{q^4}{8k_2^3} - \ \cdots\right\}$$

$$= -1 + \left(1 + \frac{k_2^2}{k_2^2 - k_1^2}\right)\frac{\sqrt{k_1^2 - q^2}}{k_2} - \frac{2\left(k_1^2 - q^2\right)}{k_2^2 - k_1^2}$$

$$+ \frac{\left(k_1^2 - q^2\right)^{3/2}}{k_2\left(k_2^2 - k_1^2\right)} + O\left(\frac{k_1}{k_2}\right)^5 \quad . \tag{21}$$

We substitute this result in the expression for E_z given by Eqs. (7) and (18). Let us make a transformation to a coordinate system centered at $x = 0$, $y = h$:

$$x \rightarrow x \quad , \quad y \rightarrow y + h \quad ,$$
$$x_0 \rightarrow x_0 \quad , \quad y_0 \rightarrow y_0 + h \quad . \qquad (22)$$

We thereby obtain

$$E_z(x,y) = \frac{i}{4\pi}\int_{-\infty}^{\infty} \frac{e^{iq(x-x_0)+i\sqrt{k_1^2-q^2}\,|y-y_0|}}{\sqrt{k_1^2-q^2}}\,dq$$

$$+ \frac{i}{4\pi}\int_{-\infty}^{\infty}\left\{-1+\left(1+\frac{k_2^2}{k_2^2-k_1^2}\right)\frac{\sqrt{k_1^2-q^2}}{k_2}\right.$$

$$\left. - \frac{2\left(k_1^2-q^2\right)}{k_2^2-k_1^2} + \frac{\left(k_1^2-q^2\right)^{3/2}}{k_2\left(k_2^2-k_1^2\right)} + \ldots\right\}\frac{e^{iq(x-x_0)+i\sqrt{k_1^2-q^2}\,(y+y_0+2h)}}{\sqrt{k_1^2-q^2}}\,dq \; ,$$

$$y, y_0 > -h \quad . \qquad (23)$$

We now note that this may be written as

$$E_z(x,y) = \frac{i}{4\pi}\int_{-\infty}^{\infty} \frac{e^{iq(x-x_0)+i\sqrt{k_1^2-q^2}\,|y-y_0|}}{\sqrt{k_1^2 - q^2}}\,dq$$

$$+ \frac{i}{4\pi}\, 0_{RE}\int_{-\infty}^{\infty} \frac{e^{iq(x-x_0)+i\sqrt{k_1^2-q^2}\,(y+y_0+2h)}}{\sqrt{k_1^2 - q^2}}\,dq \quad ,$$

$$y,\; y_0 > -h \quad , \qquad (24)$$

where the operator O_{RE}, which we shall call the reflection operator, is given by*

$$O_{RE} = - 1 - \frac{i}{k_2}\left(1 + \frac{k_2^2}{k_2^2 - k_1^2}\right)\frac{\partial}{\partial(2h)}$$

$$+ \frac{2}{k_2^2 - k_1^2}\frac{\partial^2}{\partial(2h)^2} + \frac{i}{k_2\left(k_2^2 - k_1^2\right)}\frac{\partial^3}{\partial(2h)^3} + \cdots \quad . \tag{25}$$

O_{RE} reduces to - 1 when the conductivity of the lower medium becomes infinite.

The integrals in Eq. (24) are recognized to be the integral representations of the two-dimensional Green's function of the scalar Helmholtz equation:

$$\frac{i}{4} H_0^{(1)}(k_1R_1) = \frac{i}{4\pi}\int_{-\infty}^{\infty} \frac{e^{iq(x-x_0)+i\sqrt{k_1^2-q^2}\,|y-y_0|}}{\sqrt{k_1^2 - q^2}}\, dq \quad , \tag{26}$$

where

$$R_1 = \sqrt{(x - x_0)^2 + (y - y_0)^2} \quad . \tag{27}$$

We shall accordingly now write $G_E(x,y;x_0,y_0) = G_E(\vec{\rho},\vec{\rho}_0)$ in place of E_z. From Eq. (24) we have

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*An operator like O_{RE} with terms up to the second derivative, but with a simplified form of the coefficient of the first derivative term has been given previously by Kelly and Schultz [3] without any derivation.

$$G_E(\vec{\rho},\vec{\rho}_0) = \frac{i}{4} H_0^{(1)}(k_1R_1) + \frac{i}{4} O_{RE} H_0^{(1)}(k_1R_2), \qquad y, y_0 > -h, \quad (28)$$

where

$$R_2 = \sqrt{(x - x_0)^2 + (y + y_0 + 2h)^2}$$

$$= \sqrt{(x' - x_0)^2 + [y' - (- y_0)]^2} \quad . \quad (29)$$

The primed as well as the unprimed coordinates are depicted on Figure 1.

The above formulation is essentially a low frequency one. When the frequency becomes very high, or, to be more exact, when k_1R_2 is large, a simpler result is valid. We write Eq. (7) with the use of Eq. (18) in the form

$$G_E(\vec{\rho},\vec{\rho}_0) = \frac{i}{4} H_0^{(1)}(k_1R_1)$$

$$+ \frac{i}{4\pi}\int_{-\infty}^{\infty} \frac{\sqrt{1-q^2} - \sqrt{\left(\frac{k_2}{k_1}\right)^2 - q^2}}{\sqrt{1-q^2} + \sqrt{\left(\frac{k_2}{k_1}\right)^2 - q^2}} \; \frac{e^{ik_1(x-x_o)q+ik_1\sqrt{1-q^2}(y+y_0+2h)}}{\sqrt{1-q^2}} \, dq \quad (30)$$

where we have let $q \to k_1q$. For large values of k_1R_2 this integral can be evaluated quite accurately by the saddle point method. The saddle point of interest is at $q = (x - x_o)/R_2$. We find that

$$G_E(\vec{\rho},\vec{\rho}_0) = \frac{i}{4} H_0^{(1)}(k_1R_1) + R_\perp \frac{i}{4} H_0^{(1)}(k_1R_2), \quad (31)$$

$$k_1R_2 \gg 1; \; y, y_0 > -h$$

where $R_\perp$ is the reflection coefficient

$$R_\perp = \frac{k_1 \cos \chi_0 - \sqrt{k_2^2 - k_1^2 \sin^2 \chi_0}}{k_1 \cos \chi_0 + \sqrt{k_2^2 - k_1^2 \sin^2 \chi_0}} \quad , \tag{32}$$

in which χ_0 is the angle defined by

$$\sin \chi_0 = \frac{x - x_0}{R_2} \tag{33}$$

Let us apply the Green's function to the problem with an electric line source, that we let be at infinity, from which radiation impinges on a perfectly conducting cylinder of radius a over a finitely conducting ground. We get

$$E_z(\vec{\rho}) = E_0 e^{ikr \cos \Theta_i} + R_\perp E_0' e^{ikr \cos \Theta_r}$$

$$+ ik\sqrt{\frac{\mu_0}{\epsilon_0}} \int_S G_E(\vec{\rho}, \vec{\rho}_{0S}) \; J_z(\vec{\rho}_{0S}) \; d\sigma_0 \quad , \tag{34}$$

where J_z is the density of the current induced on the surface of the cylinder.* We express the surface current in the form

$$J_z(\vec{\rho}_{0S}) = \sum_n [a_{en} \cos n\varphi_0 + a_{0n} \sin n\varphi_0] \tag{35}$$

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*Θ_i, Θ_r, and E_0' are as defined in Reference [1] with $\gamma = \frac{1}{2}\pi$.

and utilize the well-known expansions of the Green's functions:

$$\frac{i}{4} H_0^{(1)}(k_1 R_1) = \frac{i}{4} \sum_m (2 - \delta_{0m}) H_m^{(1)}(k_1 \rho_>) J_m(k_1 \rho_<) \cos m(\varphi - \varphi_0) \ , \tag{36}$$

$$\frac{i}{4} H_0^{(1)}(k_1 R_2) = \frac{i}{4} \sum_m (2 - \delta_{0m}) H_m^{(1)}(k\rho') J_m(k\rho_1) \cos m(\varphi + \varphi_0) \ . \tag{37}$$

We obtain

$$ik\sqrt{\frac{\mu_0}{\epsilon_0}} \int_S G_E(\vec{\rho}, \vec{\rho}_{0S}) \, J_z(\vec{\rho}_{0S}) \, d\sigma_0$$

$$= -\frac{\pi ka}{2} \sqrt{\frac{\mu_0}{\epsilon_0}} \Bigg\{ \sum_m \Big[a_{em} H_m^{(1)}(k\rho) J_m(ka) \cos m\varphi$$

$$+ a_{0m} H_m^{(1)}(k\rho) J_m(ka) \sin m\varphi \Big]$$

$$+ O_{RE} \sum_m \Big[a_{em} H_m^{(1)}(k\rho') J_m(ka) \cos m\varphi$$

$$- a_{0m} H_m^{(1)}(k\rho') J_m(ka) \sin m\varphi \Big] \Bigg\} \tag{38}$$

where the primed coordinates are centered on the image cylinder.

If we utilize the fact that

$$\vec{E} = \hat{e}_z E_z = \frac{1}{k^2} \text{curl curl } \hat{e}_z E_z \tag{39}$$

and set

$$- \frac{\pi a}{2k} \sqrt{\frac{\mu_0}{\epsilon_0}} J_m(ka)\, a_{\substack{e\\o}m} = c_{\substack{e\\o}m} \begin{matrix} \cos m\alpha \\ \sin m\alpha \end{matrix} \quad , \tag{40}$$

we can see that we will obtain the expansions of the incident, reflected, and scattered waves in the same form as in Reference 1 except for the effect of the operator O_{RE}. In fact, when the ground becomes perfectly conducting, so that $R_{\perp} = O_{RE} = -1$, we get the identical expansions in vector wave functions that we used in Reference 1, when $\gamma = \pi/2$, in the case of transverse magnetic waves. There is, of course, no advantage in expanding the field in vector wave functions in this two-dimensional case. We mentioned it only to establish the equivalence of the above expression for the electric field with the one in Reference 1. The solution of the present problem is obtained by using the scalar addition theorem for cylindrical waves that was given in Reference 1 to obtain the cylindrical functions in the primed coordinates in terms of the unprimed ones. Setting $E_z = 0$ on the cylinder, we then obtain two coupled sets of equations for the coefficients $c_{\substack{e\\o}m}$. We shall not write them down here.

2. Magnetic Line Source

Let us consider now the case of a (fictitious) line source of magnetic current parallel to the z-axis at $x = x_0$, $y = y_0$ in the unprimed coordinate system of Figure 1. The electromagnetic field is again independent of z and can be expressed in terms of H_z. The field components are H_z, E_x, and E_y with

$$i \omega \epsilon E_x = - \frac{\partial H_z}{\partial y} , \qquad (41)$$

$$i \omega \epsilon E_y = \frac{\partial H_z}{\partial x} \qquad (42)$$

where

$$\epsilon = \epsilon_0 = \frac{k_1^2}{\omega^2 \mu_0} , \qquad y > - h , \qquad (43)$$

$$\epsilon = \epsilon_0 \left(\epsilon_r + \frac{i\sigma}{\omega\epsilon_0} \right) = \frac{k_2^2}{\omega^2 \mu_0} , \qquad y < - h . \qquad (44)$$

H_z satisfies the equations

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_1^2 \right) H_z = - \delta(x - x_0)\delta(y - y_0), \qquad y > - h , \qquad (45)$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_2^2 \right) H_z = 0, \qquad y < - h . \qquad (46)$$

Setting

$$H_z = \int_{-\infty}^{\infty} v(q,y) e^{iq(x-x_0)} \, dq \qquad (47)$$

we get

$$\frac{d^2 v}{dy^2} + \varkappa_1^2 v = - \frac{1}{2\pi} \delta(y - y_0), \qquad y > - h , \qquad (48)$$

$$\frac{d^2v}{dy^2} + \varkappa_2^2 v = 0 , \qquad y < -h , \tag{49}$$

where $\varkappa_1$ and $\varkappa_2$ are defined in Eqs. (10) and (11).

The solutions of Eqs. (48) and (49) must be of the form

$$v = \frac{i}{4\pi\,\varkappa_1} e^{i\varkappa_1|y-y_0|} + \frac{i\,A_2}{4\pi\,\varkappa_1} e^{i\varkappa_1(y+y_0+2h)} , \quad y > -h \tag{50}$$

$$v = \frac{i\,B_2}{4\pi\,\varkappa_2} e^{-i\varkappa_2(y+h)} , \qquad y < -h . \tag{51}$$

Now H_z and E_x must be continuous at the interface. Therefore v must be continuous at $y = -h$ and

$$\frac{1}{k_1^2}\left(\frac{\partial v}{\partial y}\right)_{y=-h} = \frac{1}{k_2^2}\left(\frac{\partial v}{\partial y}\right)_{y=-h} , \tag{52}$$

where Eqs. (50) and (51) are to be used in evaluating the left and right hand sides, respectively, of Eq. (52). These boundary conditions lead to the equations

$$e^{i\varkappa_1(y_0+h)} + A_2\, e^{i\varkappa_1(y_0+h)} = \frac{\varkappa_1}{\varkappa_2} B_2 , \tag{53}$$

$$e^{i\varkappa_1(y_0+h)} - A_2\, e^{i\varkappa_1(y_0+h)} = \frac{k_1^2}{k_2^2} B_2 . \tag{54}$$

Whence

$$A_2 = \frac{k_2^2 \varkappa_1 - k_1^2 \varkappa_2}{k_2^2 \varkappa_1 + k_1^2 \varkappa_2} , \tag{55}$$

$$B_2 = \frac{2\, k_2^2 \varkappa_2}{k_2^2 \varkappa_1 + k_1^2 \varkappa_2} e^{i\varkappa_1 (y_0+h)} . \tag{56}$$

Thus

$$v(q,y) = \frac{i}{4\pi \sqrt{k_1^2-q^2}} e^{i\sqrt{k_1^2-q^2}\,|y-y_0|}$$

$$+ \frac{i}{4\pi \sqrt{k_1^2-q^2}} \left(\frac{k_2^2 \sqrt{k_1^2-q^2} - k_1^2 \sqrt{k_2^2-q^2}}{k_2^2 \sqrt{k_1^2-q^2} + k_1^2 \sqrt{k_2^2-q^2}} \right) e^{i\sqrt{k_1^2-q^2}(y+y_0+2h)} , \quad y \geq -h , \tag{57}$$

and

$$v(q,y) = \frac{i}{2\pi} \frac{k_2^2\, e^{i\sqrt{k_1^2-q^2}(y_0+h) - i\sqrt{k_2^2-q^2}(y+h)}}{k_2^2 \sqrt{k_1^2 - q^2} + k_1^2 \sqrt{k_2^2 - q^2}} , \quad y < -h . \tag{58}$$

This problem differs considerably from the preceding one of the electric line source because here the reflection coefficient has a pole at $q = q_p$ where the denominator $k_2^2 \sqrt{k_1^2 - q^2} + k_1^2 \sqrt{k_2^2 - q^2}$ vanishes. q_p is given by

$$q_p^2 = \frac{k_1^2\, k_2^2}{k_1^2 + k_2^2} . \tag{59}$$

From Eqs. (5) and (6) we obtain

$$q_p^2 = \frac{\omega^2 \mu_0 \epsilon_0 \epsilon_r (\epsilon_r + 1) + \frac{\mu_0}{\epsilon_0} \sigma^2 + i\,\omega\,\mu_0 \sigma}{(\epsilon_r + 1)^2 + \left(\frac{\sigma}{\omega \epsilon_0}\right)^2} \quad . \tag{60}$$

Approximate values of q_p are:

$$q_p^2 \simeq k_1^2 + i \frac{k_1^2 \sigma}{(\epsilon_r + 1)^2 \omega \epsilon_0} , \qquad \frac{\sigma}{\omega \epsilon_0} \ll 1 \quad , \tag{61}$$

$$q_p^2 \simeq k_1^2 + i\, k_1^2 \frac{\omega \epsilon_0}{\sigma} , \qquad \frac{\sigma}{\omega \epsilon_0} \gg 1 \quad . \tag{62}$$

Let us return to the expression for the magnetic field in the upper free-space medium as given by Eqs. (47) and (57). Since we are again actually calculating the Green's function, we now write $G_H(\vec{\rho},\vec{\rho}_0)$ for H_z. Using Eq. (26) we write the expression for G_H in the form

$$G_H(\vec{\rho},\vec{\rho}_0) = \frac{i}{4} H_0^{(1)}(k_1 R_1) + \frac{i}{4} R_{\parallel}(\chi_0)\, H_0^{(1)}(k_1 R_2)$$

$$+ \frac{i}{4\pi} \int_{-\infty}^{\infty} F(q) \frac{e^{iq(x-x_0)+i\sqrt{k_1^2-q^2}(y+y_0+2h)}}{\sqrt{k_1^2 - q^2}} dq \quad , \tag{63}$$

where

$$R_{\parallel}(\chi_0) = \frac{k_2^2 \cos \chi_0 - k_1 \sqrt{k_2^2 - k_1^2 \sin^2 \chi_0}}{k_2^2 \cos \chi_0 + k_1 \sqrt{k_2^2 - k_1^2 \sin^2 \chi_0}} \quad , \tag{64}$$

and

$$F(q) = \frac{k_2^2 \sqrt{k_1^2 - q^2} - k_1^2 \sqrt{k_2^2 - q^2}}{k_2^2 \sqrt{k_1^2 - q^2} + k_1^2 \sqrt{k_2^2 - q^2}}$$

$$- \frac{k_2^2 \sqrt{k_1^2 - q_0^2} - k_1^2 \sqrt{k_2^2 - q_0^2}}{k_2^2 \sqrt{k_1^2 - q_0^2} + k_1^2 \sqrt{k_2^2 - q_0^2}} \quad , \tag{65}$$

in which

$$q_0 = k_1 \sin \chi_0 \quad . \tag{66}$$

R_1, R_2, and χ_0 are defined in Eqs. (27), (29), and (33), respectively.

When $k_1 R_2$ is large, we can evaluate the integral in Eq. (63) by the saddle point method. But F(q) vanishes at the saddle point q_0 and therefore G_H is given quite accurately by the first two terms of Eq. (63) when $k_1 R_2 \gg 1$.

With samller values of $k_1 R_2$, we regroup the terms in Eq. (63) and write

$$G_H(\vec{\rho}, \vec{\rho}_0) = \frac{i}{4} H_0^{(1)}(k_1 R_1) - \frac{i}{4} H_0^{(1)}(k_1 R_2)$$

$$+ \frac{k_2^2}{2\pi} \frac{\partial}{\partial(2h)} \int_{-\infty}^{\infty} \frac{e^{iq(x-x_0)+i\sqrt{k_1^2-q^2}(y+y_0+2h)}}{f(q) \sqrt{k_1^2 - q^2}} \, dq \tag{67}$$

with

$$f(q) = k_2^2 \sqrt{k_1^2 - q^2} + k_1^2 \sqrt{k_2^2 - q^2}$$

$$= f'(q_\rho)(q - q_\rho) + \ldots$$

$$= -\frac{\left(k_1^4 + k_2^4\right)}{k_1 k_2}(q - q_\rho) + \ldots \quad , \qquad (68)$$

where we have expanded $f(q)$ about the zero q_ρ. The intergral in Eq. (67) becomes

$$-\frac{k_1 k_2^3}{2\pi\left(k_1^4 + k_2^4\right)} \frac{\partial}{\partial(2h)} \int_{-\infty}^{\infty} \frac{e^{iq(x-x_0)+i\sqrt{k_1^2-q^2}(y+y_0+2h)}}{\sqrt{k_1^2 - q^2}(q - q_\rho)} dq \qquad (69)$$

and the integral in Eq. (69) can be put into the form

$$i \int_{-\infty}^{\infty} dq \frac{e^{iq(x-x_0)+i\sqrt{k_1^2-q^2}(y+y_0+2h)}}{\sqrt{k_1^2 - q^2}} \int_0^{\infty} e^{-i(q-q_p)s} ds \quad . \qquad (70)$$

Interchanging the order of integration, we get

$$i \int_0^{\infty} ds\, e^{iq_p s} \int_{-\infty}^{\infty} \frac{e^{iq(x-x_0-s)+i\sqrt{k_1^2-q^2}(y+y_0+2h)}}{\sqrt{k_1^2 - q^2}} dq$$

$$= i\pi \int_0^{\infty} e^{iq_p s} H_0^{(1)}(k_1 R_s)\, ds \quad , \qquad (71)$$

where

$$R_s = \sqrt{(x - x_0 - s)^2 + (y + y_0 + 2h)^2} \quad . \tag{72}$$

Putting this back into Eq. (67) we have

$$G_H(\vec{\rho},\vec{\rho}_0) = \frac{i}{4} H_0^{(1)}(k_1R_1) + \frac{i}{4} O_{RH} H_0^{(1)}(k_1R_2) \tag{73}$$

where the reflection operator O_{RH} is given by

$$O_{RH} = -1 - \frac{2\,k_1k_2^3}{k_1^4 + k_2^4} \frac{\partial}{\partial(2h)} \int_0^\infty ds\, e^{iq_p s}\, S \tag{74}$$

in which S is the shift operator:

$$S\, f(x,y) = f(x - s,y) \quad . \tag{75}$$

When the conductivity of the conducting medium becomes infinite, the correct form of the Green's function is best recovered from Eq. (67). It is seen that the integral term there becomes $\frac{1}{2}\, i\, H_0^{(1)}(k,R_2)$ as $k_2 \to \infty$ and therefore $O_{RH} \to 1$.

If we now consider the problem of a magnetic line source at infinity from which radiation impinges on a perfectly conducting cylinder, we have

$$H_z(\vec{\rho}) = H_0\, e^{ikr \cos \Theta_i} + R_{\parallel}\, H_0\, e^{ikr \cos \Theta_r}$$

$$- \int_S J_\varphi(\vec{\rho}_{0S}) \frac{\partial G(\vec{\rho},\vec{\rho}_{0S})}{\partial \rho_0}\, d\sigma_0 \quad . \tag{76}$$

If we expand the surface current J_φ as in Eq. (34) and employ the expansions of Eqs. (36) and (37), we get for the above integral a result that is similar to Eq. (38) but with the Hankel functions replaced by their derivatives and O_{RE} replaced by O_{RH}. By means of the relation of Eq. (39) for $\vec{H} = \hat{e}_z H_z$, we can put the formal solution for the field in the same form as the one in Reference 1, when $\gamma = \frac{1}{2}\pi$, in the case of transverse electric waves, except for the effect of O_{RH}. They become identical when $O_{RH} = 1$.

III THREE-DIMENSIONAL PROBLEMS

In the presence of a finitely conducting ground, the solutions of the two-dimensional problems of scattering by an infinitely long perfectly conducting cylinder cannot be generalized to the solutions of the three-dimensional scattering problems as simply as when the cylinder is in free space. The main reason is that the transverse waves do not retain the characteristic of being polarized either parallel or perpendicular to the plane of incidence as they are in the two-dimensional problems. The effect of this may be seen by looking at the expressions for the field given in the Appendix for the case of transverse magnetic waves, for example, by Eqs. (A-5), (A-6), and (A-7). Now if u_1 satisfies the proper boundary conditions at the surface of the finitely conducting ground, then so will the three-dimensional fields E_z and H_x. But E_x will not. It follows, therefore, that in order to satisfy the boundary conditions we must introduce transverse electric waves in addition to transverse magnetic waves at the boundary and satisfy the boundary conditions of the continuity of E_z, H_x, H_z, and E_x at the interface. Similarly when the incident waves are transverse electric we must also introduce transverse magnetic waves at the surface of the ground.

Taking our cue from the fact that the formalism in the Appendix completely solves the three-dimensional scattering of a plane wave in free space, we shall express the field by equations that are similar to those in the Appendix but normalized differently. We let all quantities, including the sources, vary as $\exp[ikz \cos \gamma]$. We thereby reduce the problems to two-dimensional ones but we must determine the field functions so that the three-dimensional field satisfies the boundary conditions at the surface of the ground.

As in the previous section on two-dimensional problems, we shall concentrate on the effects at the surface of the ground and simply consider line source solutions. The explicit effects of a perfectly conducting cylinder are easily taken into account by current distributions on the cylinder as in our previous two-dimensional treatments.

1. Electric Line Source

Let us write to start with

$$\psi_1 = \begin{cases} \psi_1^s + \psi_1^r \ , & y > 0 \ , \\ \psi_1^t \ , & y < 0 \ , \end{cases} \tag{77}$$

$$\psi_2 = \begin{cases} \psi_2^r \ , & y > 0 \\ \psi_2^t & y < 0 \ , \end{cases} \tag{78}$$

where s denotes a wave from a source, r a reflected wave, and t a transmitted wave. We shall work in a coordinate system in which the origin is on the surface of the ground. The electric and magnetic fields are now expressed in the form

$$\vec{E} = \frac{1}{k^2 \sin^2 \gamma} \nabla \times \nabla \times \hat{e}_z \psi_1 + \frac{i \omega \mu_0}{k^2 \sin^2 \gamma} \nabla \times \hat{e}_z \psi_2$$

$$= \frac{1}{k^2 \sin^2 \gamma} \left[ik \cos \gamma \nabla \psi_1 + \hat{e}_z k^2 \psi_1 \right]$$

$$+ \frac{i \omega \mu_0}{k^2 \sin^2 \gamma} \nabla \psi_2 \times \hat{e}_z \ , \tag{79}$$

$$\vec{H} = \frac{1}{k^2 \sin^2 \gamma} \nabla \times \nabla \times \hat{e}_z \psi_2 + \frac{1}{i \omega \mu_0 \sin^2 \gamma} \nabla \times \hat{e}_z \psi_1$$

$$= \frac{1}{k^2 \sin^2 \gamma} \left[ik \cos \gamma \nabla \psi_2 + \hat{e}_z k^2 \psi_2 \right]$$

$$+ \frac{1}{i \omega \mu_0 \sin^2 \gamma} \nabla \psi_1 \times \hat{e}_z \quad , \tag{80}$$

where

$$k = \begin{cases} k_1 & , y > 0 \quad , \\ k_2 & , y < 0 \quad , \end{cases} \tag{81}$$

and where we have used the fact that ψ_1 and ψ_2 are of the form

$$\psi_1 = u_1(\vec{\rho}; k \sin \gamma) e^{ikz \cos \gamma} \quad , \tag{82}$$

$$\psi_2 = v_1(\vec{\rho}; k \sin \gamma) e^{ikz \cos \gamma} \quad . \tag{83}$$

The field components tangential to the ground are explicitly

$$E_z = \psi_1 \quad , \tag{84}$$

$$E_x = \frac{i}{k} \frac{\cos \gamma}{\sin^2 \gamma} \frac{\partial \psi_1}{\partial x} + \frac{i \omega \mu_0}{k^2 \sin^2 \gamma} \frac{\partial \psi_2}{\partial y} \quad , \tag{85}$$

$$H_z = \psi_2 \quad , \tag{86}$$

$$H_x = \frac{i}{k}\frac{\cos\gamma}{\sin^2\gamma}\frac{\partial\psi_2}{\partial x} + \frac{1}{i\,\omega\,\mu_0\,\sin^2\gamma}\frac{\partial\psi_1}{\partial y} \tag{87}$$

The functions ψ_1 and ψ_2 satisfy the scalar Helmholtz equation. For a unit electric line source we have

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_1^2\sin^2\gamma\right) u_1 = -\,\delta(x - x_0)\,\delta(y - y_0) \quad , \quad y > 0 \quad , \tag{88}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_2^2\sin^2\gamma\right) u_1 = 0 \quad , \qquad y < 0 \quad , \tag{89}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_1^2\sin^2\gamma\right) v_1 = 0 \quad , \qquad y > 0 \quad , \tag{90}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_2^2\sin^2\gamma\right) v_1 = 0 \quad , \qquad y < 0 \quad . \tag{91}$$

We let

$$u_1(x,y;k\,\sin\gamma) = \int_{-\infty}^{\infty} u_1(q,y)\; e^{iq(x-x_0)}\, dq \quad , \tag{92}$$

$$v_1(x,y;k\,\sin\gamma) = \int_{-\infty}^{\infty} v_1(q,y)\; e^{iq(x-x_0)}\, dq \quad , \tag{93}$$

and obtain

$$\frac{d^2u_1}{dy^2} + \varkappa_1^2 u_1 = -\frac{1}{2\pi}\,\delta(y - y_0) \ , \ y > 0 \ , \tag{94}$$

$$\frac{d^2u_1}{dy^2} + \varkappa_2^2 u_1 = 0 \ , \ y < 0 \ , \tag{95}$$

$$\frac{d^2v_1}{dy^2} + \varkappa_1^2 v_1 = 0 \ , \ y > 0 \ , \tag{96}$$

$$\frac{d^2v_1}{dy^2} + \varkappa_2^2 v_1 = 0 \ , \ y < 0 \ , \tag{97}$$

where

$$\varkappa_1^2 = k_1^2 \sin^2\gamma - q^2 \ , \tag{98}$$

$$\varkappa_2^2 = k_2^2 \sin^2\gamma - q^2 \ . \tag{99}$$

The solutions of these equations are

$$u_1 = \frac{i}{4\pi\,\varkappa_1}\, e^{i\varkappa_1 |y-y_0|} + \frac{i\,A_1}{4\pi\,\varkappa_1}\, e^{i\varkappa_1(y+y_0)} \ , \ y > 0 \ , \tag{100}$$

$$u_1 = \frac{i\,B_1}{4\pi\,\varkappa_2}\, e^{-i\varkappa_2 y} \ , \ y < 0 \ , \tag{101}$$

$$v_1 = \frac{i\,C_1}{4\pi\,\varkappa_1}\, e^{i\varkappa_1 y} \ , \ y > 0 \ , \tag{102}$$

$$v_1 = \frac{i D_1}{4\pi \varkappa_2} e^{-i\varkappa_2 y} \quad , \; y < 0 \quad . \tag{103}$$

Applying the boundary conditions of the continuity of the tangential components of the electric and magnetic fields at y = 0, we get the system of equations

$$\frac{e^{i\varkappa_1 y_0}}{\varkappa_1} A_1 - \frac{1}{\varkappa_2} B_1 = - \frac{e^{i\varkappa_1 y_o}}{\varkappa_1} \quad , \tag{104}$$

$$\frac{i\, q \cos\gamma}{k_1} \frac{e^{i\varkappa_1 y_0}}{\varkappa_1} A_1 - \frac{i\, q \cos\gamma}{k_2 \varkappa_2} B_1$$

$$+ \frac{i\,\omega\,\mu_0}{k_1^2} C_1 + \frac{i\,\omega\,\mu_o}{k_2^2} D_1 = - \frac{i\, q \cos\gamma\, e^{i\varkappa_1 y_0}}{k_1 \varkappa_1} \quad , \tag{105}$$

$$\frac{1}{\varkappa_1} C_1 = \frac{1}{\varkappa_2} D_1 \quad , \tag{106}$$

$$\frac{1}{i\,\omega\,\mu_0} e^{i\varkappa_1 y_0} A_1 + \frac{1}{i\,\omega\,\mu_0} B_1 + \frac{i\, q \cos\gamma}{k_1 \varkappa_1} C_1$$

$$- \frac{i\, q \cos\gamma}{k_2 \varkappa_2} D_1 = \frac{e^{i\varkappa_1 y_0}}{i\,\omega\,\mu_0} \quad , \tag{107}$$

from which we obtain

$$A_1 = \frac{(\varkappa_1 - \varkappa_2)\left(k_2^2 \varkappa_1 + k_1^2 \varkappa_2\right) - q^2 \cos^2\gamma\, (k_2 - k_1)^2}{(\varkappa_1 + \varkappa_2)\left(k_2^2 \varkappa_1 + k_1^2 \varkappa_2\right) + q^2 \cos^2\gamma\, (k_2 - k_1)^2} \quad , \tag{108}$$

$$B_1 = \frac{2\,\varkappa_2\left(k_2^2\varkappa_1 + k_1^2\varkappa_2\right)e^{i\varkappa_1 y_0}}{(\varkappa_1 + \varkappa_2)\left(k_2^2\varkappa_1 + k_1^2\varkappa_2\right) + q^2\cos^2\gamma\,(k_2 - k_1)^2} \quad , \tag{109}$$

$$C_1 = -\frac{2\,k_1k_2(k_2 - k_1)\,\varkappa_1\,q\cos\gamma\,e^{i\varkappa_1 y_0}}{\omega\,\mu_0\left\{(\varkappa_1 + \varkappa_2)\left(k_2^2\varkappa_1 + k_1^2\varkappa_2\right) + q^2\cos^2\gamma\,(k_2 - k_1)^2\right\}} \tag{110}$$

$$D_1 = \frac{\varkappa_2}{\varkappa_1}\,C_1 \quad . \tag{111}$$

When $\gamma \to \frac{1}{2}\pi$, so that the problem becomes two dimensional, the coefficients reduce to the values obtained previously:

$$A_1 \longrightarrow \frac{\varkappa_1 - \varkappa_2}{\varkappa_1 + \varkappa_2} \quad , \tag{112}$$

$$B_1 \longrightarrow \frac{2\,\varkappa_2\,e^{i\varkappa_1 y_0}}{\varkappa_1 + \varkappa_2} \quad , \tag{113}$$

$$C_1 = \frac{\varkappa_1}{\varkappa_2}\,D_1 \longrightarrow 0 \quad . \tag{114}$$

2. <u>Magnetic Line Source</u>

We let

$$\psi_2 = \begin{cases} \psi_2^s + \psi_2^r \ , & y > 0 \ , \\ \psi_2^t \ , & y \leqslant 0 \ , \end{cases} \tag{115}$$

$$\psi_1 = \begin{cases} \psi_1^r , & y > 0 , \\ \psi_1^t , & y < 0 , \end{cases} \tag{116}$$

with

$$\psi_2 = v_2(\vec{\rho};k \sin \gamma)\ e^{ikz\cos\gamma} , \tag{117}$$

$$\psi_1 = u_2(\vec{\rho};k \sin \gamma)\ e^{ikz\cos\gamma} . \tag{118}$$

The equations for v_2 and u_2 are the same as those for u_1 and v_1, respectively, in Eqs. (88) to (91). We thus have

$$v_2 = \frac{i}{4\pi \varkappa_1} e^{i\varkappa_1|y-y_0|} + \frac{i A_2}{4\pi \varkappa_1} e^{i\varkappa_1(y+y_0)} , \quad y > 0 , \tag{119}$$

$$v_2 = \frac{i B_2}{4\pi \varkappa_2} e^{-i\varkappa_2 y} , \quad y < 0 , \tag{120}$$

$$u_2 = \frac{i C_2}{4\pi \varkappa_1} e^{i\varkappa_1 y} , \quad y > 0 , \tag{121}$$

$$u_2 = \frac{i D_2}{4\pi \varkappa_2} e^{-i\varkappa_2 y} , \quad y < 0 . \tag{122}$$

Applying the boundary conditions at y = 0, we get the equations

$$\frac{e^{i\varkappa_1 y_0}}{\varkappa_1} A_2 - \frac{1}{\varkappa_2} B_2 = - \frac{1}{\varkappa_1} e^{i\varkappa_1 y_0} , \tag{123}$$

$$\frac{i\,q\cos\gamma}{k_1}\,\frac{e^{i\varkappa_1 y_0}}{\varkappa_1}\,A_2 - \frac{i\,q\cos\gamma}{k_2\varkappa_2}\,B_2$$

$$+ \frac{1}{i\,\omega\,\mu_0}\,C_2 + \frac{1}{i\,\omega\,\mu_0}\,D_2 = -\frac{i\,q\cos\gamma}{k_1\varkappa_1}\,e^{i\varkappa_1 y_0} , \tag{124}$$

$$\frac{1}{\varkappa_1}\,C_2 = \frac{1}{\varkappa_2}\,D_2 , \tag{125}$$

$$\frac{i\,\omega\,\mu_0}{k_1^2}\,e^{i\varkappa_1 y_0}\,A_2 + \frac{i\,\omega\,\mu_0}{k_2^2}\,B_2 + \frac{i\,q\cos\gamma}{k_1\varkappa_1}\,C_2$$

$$-\frac{i\,q\cos\gamma}{k_2\varkappa_2}\,D_2 = \frac{i\,\omega\,\mu_0}{k_1^2}\,e^{i\varkappa_1 y_0} . \tag{126}$$

The solutions of these equations are

$$A_2 = \frac{\left(k_2^2\varkappa_1 - k_1^2\varkappa_2\right)(\varkappa_1 + \varkappa_2) - q^2\cos^2\gamma(k_2 - k_1)^2}{\left(k_2^2\varkappa_1 + k_1^2\varkappa_2\right)(\varkappa_1 + \varkappa_2) + q^2\cos^2\gamma(k_2 - k_1)^2} , \tag{127}$$

$$B_2 = \frac{2\,k_2^2\varkappa_2(\varkappa_1 + \varkappa_2)\,e^{i\varkappa_1 y_0}}{\left(k_2^2\varkappa_1 + k_1^2\varkappa_2\right)(\varkappa_1 + \varkappa_2) + q^2\cos^2\gamma(k_2 - k_1)^2} , \tag{128}$$

$$C_2 = \frac{2(k_2/k_1)(k_2 - k_1)\ \omega\ \mu_0\ q\ \cos\gamma\ \varkappa_1\ e^{i\varkappa_1 y_0}}{\left(k_2^2\varkappa_1 + k_1^2\varkappa_2\right)(\varkappa_1 + \varkappa_2) + q^2\cos^2\gamma(k_2 - k_1)^2}\ , \qquad (129)$$

$$D_2 = \frac{\varkappa_2}{\varkappa_1} C_2 \quad . \qquad (130)$$

These coefficients reduce correctly to the previously obtained two-dimensional ones when $\gamma \rightarrow \pi/2$:

$$A_2 \longrightarrow \frac{k_2^2\varkappa_1 - k_1^2\varkappa_2}{k_2^2\varkappa_1 + k_1^2\varkappa_2}\ , \qquad (131)$$

$$B_2 \longrightarrow \frac{2k_2^2\varkappa_2\ e^{i\varkappa_1 y_0}}{k_2^2\varkappa_1 + k_1^2\varkappa_2}\ , \qquad (132)$$

$$C_2 = \frac{\varkappa_1}{\varkappa_2} D_2 \longrightarrow 0 \quad . \qquad (133)$$

3. <u>The Green's Functions</u>

Explicit evaluation of the Green's functions in both cases just treated of electric and magnetic line sources is readily carried out when k_1R_2 is large, as before, by the saddle point method. The saddle point is now at $q_0 = k_1 \sin\gamma \sin\chi_0$. We shall save writing down the results for a future note in which we hope to obtain expressions for the reflection operators when k_1R_2 is not large.

Appendix

THREE-DIMENSIONAL SCATTERING SOLUTIONS DEDUCED FROM TWO-DIMENSIONAL ONES

If we have an infinitely long perfectly conducting cylinder in free space, the solution of the scattering of a plane electromagnetic wave, which is incident obliquely on the cylinder, can be deduced from a two-dimensional problem when the plane wave is incident normally on the cylinder.* Although this relationship does not hold when the cylinder is situated above a finitely conducting ground, we shall refer in the text to the equations that express the three-dimensional electromagnetic field in terms of scalar functions which are obtained from two-dimensional solutions. For this reason we deem it advantageous to set down the relevant equations in this appendix.

We let

$$\psi^{inc} = e^{ikr \cos \Theta_i}$$

$$\equiv e^{ik(x \sin \gamma \cos \alpha + y \sin \gamma \sin \alpha + z \cos \gamma)}$$

$$\equiv u_0(x,y;k \sin \gamma)\, e^{ikz \cos \gamma} \qquad \text{(A-1)}$$

where

$$u_0(x,y;k) = e^{ik(x \cos \alpha + y \sin \alpha)} \qquad \text{(A-2)}$$

---

*Cf. Jones [2].

is the incident wave in the two-dimensional scattering problems, and γ, α are the spherical coordinate angles of the propagation vector. We easily find that

$$(\hat{e}_x \sin\alpha - \hat{e}_y \cos\alpha)\, e^{ikr\cos\Theta_i} = \frac{1}{ik\sin\gamma} \nabla\psi^{inc} \times \hat{e}_z \quad . \tag{A-3}$$

<u>Transverse magnetic waves.</u>* Let

$$u_1 = u_0 + u_1^{scat} \tag{A-4}$$

be the solution to the two-dimensional scattering problem when $u_0(x,y;k)$ is incident on the cylinder and the boundary condition is that $u_1 = 0$ on the surface of the cylinder. Set

$$\psi_1 = \psi^{inc} + \psi_1^{scat}$$

$$= u_1(x,y;k\sin\gamma)\, e^{ikz\cos\gamma} \quad . \tag{A-5}$$

Then the total electromagnetic field in the three-dimensional problem in which the waves are transverse magnetic is given by

$$\vec{H} = \frac{H_0}{ik\sin\gamma} \nabla\psi_1 \times \hat{e}_z$$

$$= \frac{E_0}{i\,\omega\,\mu_0\sin\gamma} \nabla\psi_1 \times \hat{e}_z \, , \tag{A-6}$$

---

*Transverse here is always with respect to the axis of the cylinder, which is assumed to lie along the z-axis.

and

$$\vec{E} = \frac{E_0}{k^2 \sin \gamma} \nabla \times \nabla \times \hat{e}_z \psi_1$$

$$= \frac{E_0}{k^2 \sin \gamma} \left[ik \cos \gamma \, \nabla\psi_1 + \hat{e}_z \, k^2 \psi_1 \right]$$

$$\xrightarrow[\gamma \to \pi/2]{} \hat{e}_z \, E_0 \, u_1 \quad , \tag{A-7}$$

where E_0 is the amplitude of the incident electric field. It is easily verified that the field of Eqs. (A-6) and (A-7) satisfies the boundary conditions of the vanishing of the tangential components of the electric field and of the normal component of the magnetic field on the surface of the cylinder.

<u>Transverse electric waves</u>. Now let $u_2(x,y;k)$ be the solution to the two-dimensional scattering problem when $u_0(x,y;k)$ is incident on the cylinder and the boundary condition is $\partial u_2/\partial n = 0$ on the surface of the cylinder. With

$$\psi_2 = u_2(x,y;k \sin \gamma) \, e^{ikz \cos \gamma} \quad , \tag{A-8}$$

the total electromagnetic field in the three-dimensional problem in which the waves are transverse electric is given by

$$\vec{E} = \frac{E_0}{ik \sin \gamma} \nabla\psi_2 \times \hat{e}_z \tag{A-9}$$

and

$$\vec{H} = - \sqrt{\frac{\epsilon_0}{\mu_0}} \frac{E_0}{k^2 \sin \gamma} \nabla \times \nabla \times \hat{e}_z \psi_2$$

$$= - \sqrt{\frac{\epsilon_0}{\mu_0}} \frac{E_0}{k^2 \sin \gamma} \left[ik \cos \gamma \nabla \psi_2 + \hat{e}_z k^2 \psi_2 \right]$$

$$\xrightarrow[\gamma \to \pi/2]{} - H_0 \hat{e}_z u_2 \tag{A-10}$$

where $H_0 = \sqrt{\epsilon_0/\mu_0}\, E_0$ is the amplitude of the incident magnetic field. It is again easily verified that the boundary conditions are satisfied on the surface of the cylinder.

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