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ON THE CIRCUMFERENTIAL CURRENT AND CHARGE DISTRIBUTIONS OF CIRCULAR CYLINDERS NEAR A GROUND PLANE

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BY

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ABSTRACT

In free space the current and charge distributions induced on electrically thin circular cylinders are essentially uniform about the circumference. However, if the cylinder is near a ground plane the distributions of current and charge may differ considerably from uniform. In order to determine the effect of the ground plane an infinitely long charged circular cylinder is considered oriented parallel to a ground plane. A simple analytical result is obtained for the circumferential charge distribution. Likewise a circular cylinder supporting a static current is considered oriented parallel to a ground plane and the circumferential current distribution is obtained. In both cases the current and charge tends to accumulate on the side of the cylinder nearest the ground plane.

INTRODUCTION

It is well known that in free space the current and charge distributions induced on electrically thin circular cylinders are essentially uniform about the circumference. However, if the cylinder is near a ground plane the distributions of current and charge may differ considerably from uniform. In order to determine this proximity effect an infinitely long charged circular cylinder is considered oriented parallel to a ground plane. Then conformal mapping is used to obtain a very simple analytical formula for the circumferential charge distribution.[1*] Although this result applies strictly for a static distribution only, it is also valid for electrically thin cylinders separated an electrically short distance from the ground plane.

To determine the effect of the ground plane on the circumferential current distribution, a circular cylinder carrying a static current is also considered oriented parallel to a ground plane and a simple analytical result is obtained for the circumferential current distribution. In both cases conformal mapping is used. And although the results obtained apply strictly for static currents and charges on infinitely long cylinders they are also valid for electrically thin cylinders separated an electrically short distance from the ground plane. Furthermore the results apply for finite length cylinders provided the length of the cylinders is much greater than the height above the ground plane.

It is found that the current and charge tends to accumulate on the side of the cylinder nearest the ground plane. This general behavior may be used to define an effective height for the cylinder above the ground plane that might be used in a wire modeling study.

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*Superscripts refer to the list of references at the end of the paper

ANALYSIS

A right circular cylinder parallel to the z axis is shown in figure 1. The cylinder is oriented over a ground plane that coincides with the xz plane. First the conducting cylinder is considered to support a uniform charge per unit length of q. How this charge distributes about the circumference of the cylinder is to be determined.

The potential field about the cylinder may be obtained by solving Laplace's equation subject to the boundary conditions, $V = 0$ on the ground plane and $V = V_o$ on the cylinder. Solving Laplace's equation directly may be difficult in that the boundary surfaces are not coordinate surfaces. However, the configuration may be mapped into a configuration where the boundary surfaces are coordinate surfaces. A suitable conformal mapping is[2]

$$Z = j\, a \tan \frac{W}{2} \tag{1}$$

where $Z = x + jy$, $W = u + jv$, and $a^2 = d^2 - r^2$. The configuration in the complex w-plane is shown in figure 2.

Solving for the potential in the region between the plates is straight forward. The result is

$$V(v) = V_o \frac{v}{v_o} \qquad -|v_o| \leq v \leq 0 \tag{2}$$

where

$$v_o = -\cosh\left(\frac{d}{r}\right)$$

Using (1) the solution (2) may be transformed back to the original z-plane configuration to obtain

$$V(x,y) = \frac{\mathrm{Re}\{2\ \tanh^{-1}(Z/a)\}}{\cosh^{-1}(d/r)} V_o \qquad (3)$$

The electric field may be obtained from

$$\vec{E} = -\nabla V$$

which yields

$$E_x = \frac{V_o}{v_o} \frac{2a(a^2 - x^2 + y^2)}{(a^2 - x^2 + y^2)^2 + 4x^2y^2} \qquad (4)$$

$$E_y = -\frac{V_o}{v_o} \frac{4axy}{(a^2 - x^2 + y^2)^2 + 4x^2y^2} \qquad (5)$$

Using the following change of variable

$$y = r \sin \phi$$
$$x = (r \cos\phi + d) \qquad (6)$$

and (4) and (5) the surface charge density on the cylinder is obtained.

$$\rho(\phi) = -\frac{\varepsilon_o V_o}{v_o r} f(\phi) \qquad (7)$$

where

$$f(\phi) = \frac{\sqrt{1 - (r/d)^2}}{1 + (r/d)\cos\phi} \qquad (8)$$

Since the charge per unit length, q, of the cylinder is

$$q = \int_o^{2\pi} \rho(\phi) r d\phi \qquad (9)$$

then from (7)

$$q = -\frac{2\pi\varepsilon_o V_o}{v_o} \qquad (10)$$

Using (10) in (7) yields

$$\rho(\phi) = \frac{q}{2\pi r} f(\phi) , \tag{11}$$

a surprisingly simple result that is exact for a charged infinite circular cylinder oriented parallel to a ground plane.

Now the conducting cylinder is considered to support a constant axial current in its configuration above the ground plane. Again Laplace's equation is to be solved, but this time the magnetic scalar potential is obtained. The boundary condition is that the normal derivative of the magnetic scalar potential must be zero at the surface of the conductor. A cut line is introduced along the positive x axis from x = ∞ to the surface of the cylinder. This insures a unique solution for the scalar potential. Furthermore the following condition is imposed: across the cut line the magnetic scalar potential is discontinuous by

$$V_m(x,0+) - V_m(x,0-) = I \qquad x \geq d + r \tag{12}$$

The foregoing insures that the magnetic field satisfies Ampere's law where I is the total axial current on the cylinder.

For the magnetic field problem (1) is also a suitable conformal mapping. Using the aforementioned boundary conditions and solving Laplace's equation in the complex w-plane one obtains

$$V_m(u) = \frac{I}{2\pi} u \tag{13}$$

Transforming (13) back to the original z-plane using (1) yields

$$V_m(x,y) = -\frac{I}{2\pi} \mathrm{Re}\{\tan^{-1}(j \frac{Z}{a})\} \tag{14}$$

Since

$$\vec{H} = - \nabla V_m$$

the components of the magnetic field are

$$H_x = - \frac{I}{2\pi} \frac{4axy}{(a^2 + y^2 - x^2)^2 + 4x^2y^2} \qquad (15)$$

$$H_y = - \frac{I}{2\pi} \frac{2a(a^2 + y^2 - x^2)}{(a^2 + y^2 - x^2)^2 + 4x^2y^2} \qquad (16)$$

The component of the magnetic field tangent to the surface of the cylinder is

$$H_\phi = - H_x \sin \phi + H_y \cos \phi \qquad (17)$$

where ϕ is defined in figure 1 and (6). Using (6), (15) and (16) in (17) one obtains

$$H_\phi = \frac{I}{2\pi r} f(\phi) \qquad (18)$$

a remarkably simple, yet exact, result for the variation of the magnetic field about the surface of the cylinder. Since the surface current density is

$$\vec{J} = \hat{n} \times \vec{H}$$

then

$$\vec{J} = \frac{I}{2\pi r} f(\phi)\hat{z} \qquad (19)$$

The normalized current and charge distributions are found to be exactly the same, $f(\phi)$. And for small r/d (8) yields

$$f(\phi) \approx 1 - (r/d)\cos\phi + (1/2)(r/d)^2 \cos 2\phi$$

The foregoing expression differs less than 10% from the exact expression for $r/d < 0.45$. Note that if the cylinder is a large distance from the ground plane the surface current and charge densities are uniform, i. e.

$$f(\phi) \xrightarrow[r/d \to 0]{} 1$$

Thus as expected the variation of the surface current and charge densities is strictly due to the interaction with the ground plane. Furthermore the interaction with the ground plane results in larger current and charge densities on the side of the cylinder nearer the ground plane.

At certain values of ϕ the distribution function $f(\phi)$ assumes simple forms. For example,

$$f(0) = 1/\sqrt{s}$$

$$f(\pi/2) = \sqrt{1 - (r/d)^2}$$

$$f(\pi) = \sqrt{s}$$

where s is the ratio of the maximum value of $f(\phi)$ to the minimum value,

$$s \equiv \frac{f(\pi)}{f(0)} = \frac{1 + r/d}{1 - r/d}$$

SAMPLE CALCULATIONS

As pointed out in the foregoing the distribution of the current and charge about the periphary of a finite cylinder parallel to a ground plane should be essentially that of an infinite cylinder provided the length of the finite cylinder is much greater than the height above the ground plane while the height above the ground plane must be much less than the operating wave length. Also it is shown in the foregoing that the normalized current and charge densities have the same aximuthal variation, $f(\phi)$.

The primary motivation for this study is the determination of the influence of the ground on the circumferential variation of the current and charge densities induced on a parked aircraft. For example, considering the fuselage of a B-52 parked one obtains $r/d \simeq 0.59$. A table of the normalized distribution function for the circumferential variation of the current and charge densities is given for illustration.

To illustrate the dependence of the normalized distribution function, $f(\phi)$, on the ratio r/d, figure 3 exhibits graphs of $f(\phi)$ versus r/d for several values of ϕ. The $\phi = 0$ point is furthermost from the ground plane and $\phi = \pi$ point is nearest the ground plane.

EFFECTIVE HEIGHT ABOVE GROUND PLANE

In solving for the current induced on an aircraft over a ground plane thin wire approximations are used in the low frequency regime. An image of the aircraft is used to replace the ground plane. Determining the interaction between the aircraft and its image they are considered to be formed out of current filaments located at the geometrical centers of the wings and fuselage. The fuselage is more nearly a circular cylinder and the foregoing approximation is reasonable provided the charge and current distributions are uniform about the cross section. However, in the foregoing development it is shown that a cylinder in proximity to the ground does not have the uniform current and charge densities about the cross section. In fact there is an accumulation of current and charge on the side of the cylinder nearer the ground. Thus the centers of the current and charge shifts nearer the ground.

To account for the shifting of the current and charge densities an effective height above the ground plane is defined. It is

$$d_e = \frac{1}{\pi}\int_0^{\pi} f(\phi)(r\cos\phi + d)d\phi \qquad (20)$$

The evaluation of (20) yields

$$d_e = d\sqrt{1 - (r/d)^2} \qquad (21)$$

This effective height should be used to determine the interaction of the aircraft with its image using thin wire approximations. Note that this effective height is needed only because thin wire approximations assume uniform currents and charges about the wire sections.

For the B-52 over a ground plane $d_e = d(.807)$

TABLE

The Variation of the Normalized Current and Charge Densities about the Periphery of a Cylinder Over a Ground Plane

ϕ	$f(\phi)$			
	r/d = 0	r/d = 0.25	r/d = 0.59	r/d = 0.75
0°	1.0	0.7746	0.5071	0.3780
20°	1.0	0.7841	0.5187	0.3880
40°	1.0	0.8126	0.5553	0.4201
60°	1.0	0.8607	0.6227	0.4810
80°	1.0	0.9280	0.7316	0.5852
90°	1.0	0.9682	0.8067	0.6614
100°	1.0	1.012	0.8990	0.7605
120°	1.0	1.107	1.145	1.058
140°	1.0	1.198	1.474	1.555
160°	1.0	1.266	1.814	2.240
180°	1.0	1.291	1.972	2.646

REFERENCES

1. F. M. Tesche, "Charge Distribution on a Two-Dimensional Air Foil," Interaction Note 75, May 1971.

2. C. E. Baum, "Impedances and Field Distributions for Symmetrical Two Wire and Four Wire Transmission Line Simulators," Sensor and Simulation Note 27, October 1966.

3. G. Arfken, Mathematical Methods for Physicists, (Academic Press, New York, 1970) pp. 333-336.

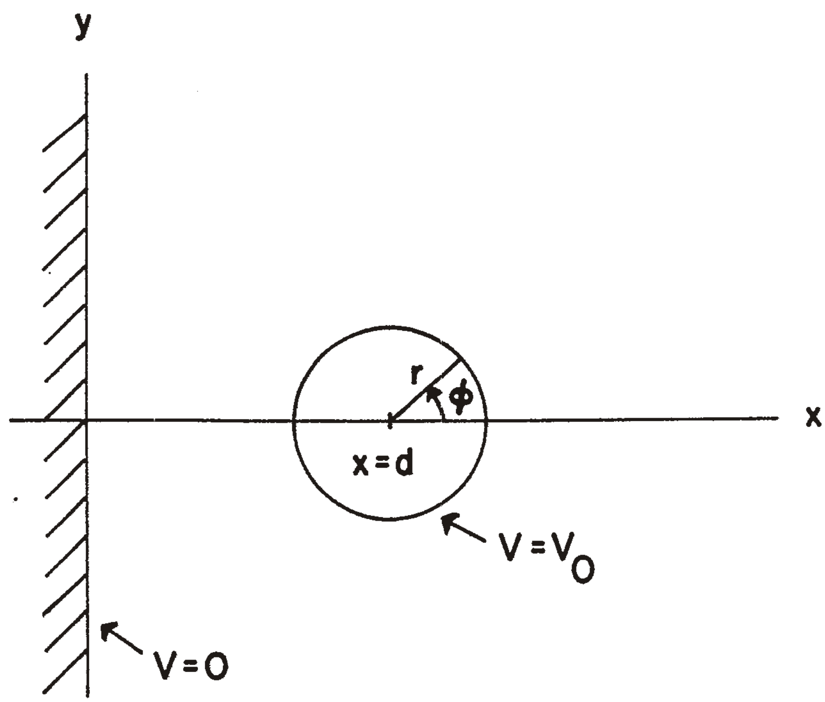

Figure 1: Right circular cylinder oriented parallel to a ground plane.

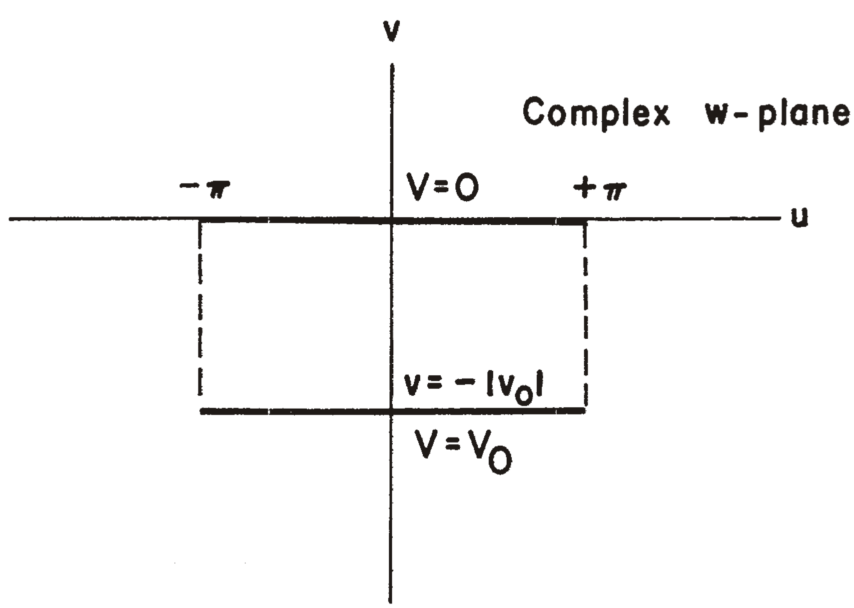

Figure 2: Parallel plate configuration obtained by applying conformal mapping to the configuration of figure 1.

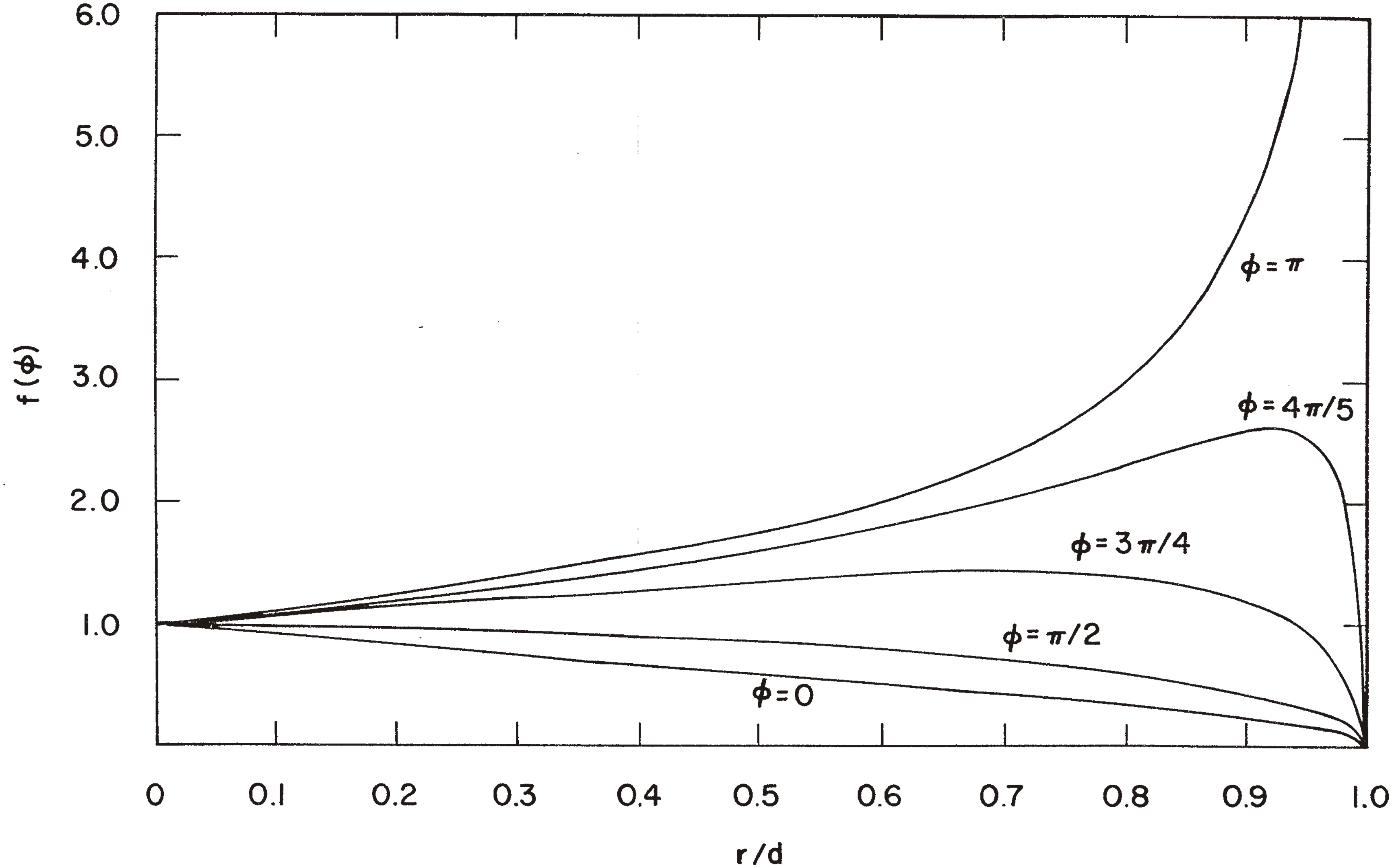

Figure 3: Normalized distribution function for the circumferential variation of the current and charge on a circular cylinder over a ground plane.