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INTERACTION NOTE

NOTE 136

INVESTIGATION OF THE RESPONSE OF QUASI-TEM MULTIWIRE TRANSMISSION LINES TO EXTERNAL FIELDS BY THE USE OF LAPLACE TRANSFORMS

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SUMMARY

The behavior of a multiwire transmission line, or cable, operating in the TEM, or nearly-TEM mode in response to an external monochromatic electro-magnetic field, is evaluated by solving the inhomogeneous, first-order differential equations of the line with the help of Laplace transforms.

Since, for a uniform line, the singularities of these transforms consist only of poles, the inverse transforms are readily evaluated by the method of residues. The poles of the transforms are the square roots of the eigenvalues of the characteristic matrix of the line, the multiplicity of any pole being generally equal to the multiplicity of the corresponding eigenvalue.

The Laplace-transform attack appears to afford some advantages in directness of solution, and, for at least one important class of problems, in reduction in solution complexity compared to conventional procedures.

End-excitation problems are treated as a special case in which the "distributed" excitation consists of impulse functions at the line terminals.

Multiple poles (i.e., multiple eigenvalues with associated degenerate modes) present no special problems; they are handled in a matrix extension of the standard procedure for such poles.

When a number of poles are nearly equal, computational errors due to matrix ill-conditioning are avoided by expanding transform integrands in Laurent series around an "average" pole. This results in a solution in the form of an infinite series, the rate of convergence increasing with decrease in dispersion of the eigenvalues.

The final solutions for voltage and current at any point along the line each contain four integrals to be evaluated by the residue method, making a total of eight altogether. However, only one of these is basic. When the basic integral for the voltage solution has been evaluated, the remaining three for the voltage solution are obtained by first-order differentiation, and by convolution with the distributed excitation. The integrals for the current solution are the same, with all matrices replaced by their adjoints.

When the line consists of lossless conductors embedded in a homogeneous, isotropic dielectric, the inverse transforms contain only a single, simple pole, whence the solution exhibits the well-known single propagation mode.

CONTENTS

1. INTRODUCTION

It is customary to initiate the steady-state analysis of uniform multiwire transmission lines in source-free regions with a set of homogeneous differential equations of the form[1,2]

$$\left.\begin{aligned} \frac{d\underline{V}}{dx} + \underline{\zeta}\,\underline{I} &= \underline{0} \\ \frac{d\underline{I}}{dx} + \underline{\eta}\,\underline{V} &= \underline{0} \end{aligned}\right\} \qquad (1)$$

If the line is excited continuously along its length, we have the corresponding inhomogeneous equations[1,3]

$$\left.\begin{aligned} \frac{d\underline{V}}{dx} + \underline{\zeta}\,\underline{I} &= \underline{E}^{e}(x) \\ \frac{d\underline{I}}{dx} + \underline{\eta}\,\underline{V} &= \underline{H}^{e}(x) \end{aligned}\right\} \qquad (2)$$

If the line consists of (N+1) conductors of arbitrary cross-section, -- one of the conductors being used as potential reference, -- then the quantities in equations (1) and (2) are defined as follows:

$\underline{V}$, $\underline{I}$, are Nx1 column matrices of conductor potentials and currents respectively:

$$\left.\begin{aligned} \underline{V}^{T} &= \left[V_1, \ldots, V_n\right] \\ \underline{I}^{T} &= \left[I_1, \ldots, I_n\right] \end{aligned}\right\} \qquad (3)$$

where V^T is the transpose of V, etc.; V_k, I_k are potential and current on the kth conductor, $k = 1,\ldots, N$. The current in the zeroth (reference) conductor is the negative of the algebraic sum of the components of $\underline{I}$:

$$\sum_{k=0}^{N} I_k = 0$$

$\underline{\zeta}$ is the line series-impedance NxN matrix:

$$\zeta = \left[\zeta_{ij}\right] \quad \zeta_{ij} = \zeta_{ji} \text{ for every } i,j \tag{4a}$$

$\underline{\eta}$ is the line shunt-admittance NxN matrix:

$$\underline{\eta} = \left[\eta_{ij}\right]; \ \eta_{ij} = \eta_{ji} \text{ for every } i,j \tag{4b}$$

If in the first of equations (1) we specify $I_j \neq 0$ while all other currents are zero, we get, for the potential of the ith conductor

$$\frac{dV_i}{dx} + \zeta_{ij} I_j = 0 \tag{5a}$$

If, in the second of equations (1) we specify $V_j \neq 0$ while all other potentials are zero, we get, for the current in the ith conductor

$$\frac{dI_i}{dx} + \eta_{ij} V_j = 0 \tag{5b}$$

Equations (5a) and (5b) serve as defining equations for ζ_{ij}, η_{ij} respectively. In practice, their components are frequently identified with static capacitances, inductances, dielectric conductances, and conductor skin resistance and internal inductance.

The forcing functions, $\underline{E}^e$ and $\underline{H}^e$ of equations (2) may be identified with the transverse magnetic field and electric field intensities impressed along the line[3,4]. Specifically, we may write

$$\left.\begin{aligned} \underline{E}^e &= j\omega\ \underline{L}^e\ H_z^e \\ \underline{H}^e &= j\omega\ \underline{C}^e\ E_y^e \end{aligned}\right\} \qquad (6)$$

where $\underline{L}^e$, $\underline{C}^e$ are field coupling parameter Nx1 matrices; H_z^e is the transverse impressed magnetic intensity, and E_y^e is the transverse impressed electric intensity.

If all conductors are open-circuited ($\underline{I} = 0$), equations (2) and (6) yield

$$\frac{d\underline{V}}{dx} = j\omega\ \underline{L}^e H_z^e \qquad (7a)$$

while, if all conductors are grounded ($\underline{V} = 0$),

$$\frac{d\underline{I}}{dx} = j\omega\ \underline{C}^e\ E_y^e \qquad (7b)$$

Equations (7a,b) may be taken as defining equations for the coupling parameters. These quantities may frequently be determined from electrostatic considerations[5,6].

If, in equations (2), we write

$$\left.\begin{aligned} \underline{E}^{e}(x) &= \underline{V}_0\ \delta(x) \\ \underline{H}^{e}(x) &= \underline{I}_0\ \delta(x) \end{aligned}\right\} \qquad (8)$$

where $\delta(x)$ is the usual impulse function:

$$\left.\begin{aligned} \delta(x) &= 0,\ x \neq 0 \\ \int_{-\infty}^{\infty} \delta(x)\ dx &= 1 \end{aligned}\right\} \qquad (9)$$

then equations (2) are source-free everywhere except at $x = 0$; that is, they are equivalent to equations (1) with terminal excitation at one end exhibited directly in the equations.

Since equations (1) are linear with constant coefficients, a standard procedure, -- evidently attributable to Carson and Hoyt[2], -- for obtaining a solution, is to assume the solution has the form $Ae^{-\gamma x}$ (for forward waves), so that equations (1) yield

$$\left.\begin{aligned} -\gamma\underline{V} + \underline{\zeta}\underline{I} &= \underline{0} \\ -\gamma\underline{I} + \underline{\eta}\underline{V} &= \underline{0} \end{aligned}\right\} \qquad (10)$$

Elimination of $\underline{I}$ or $\underline{V}$ yields one of the following sets of equations:

$$\left.\begin{aligned} (\gamma^2\underline{I} - \underline{\zeta}\underline{\eta})\underline{V} &= \underline{0} \\ (\gamma^2\underline{I} - \underline{\eta}\underline{\zeta})\underline{I} &= \underline{0} \end{aligned}\right\} \qquad (11)$$

where $\underline{I}$ is the NxN unit matrix.

For these sets to yield non-trivial solutions for $\underline{I}, \underline{V}$, we must have

$$\text{det.}\ (\gamma^2 \underline{I} - \underline{\eta\zeta}) = 0 \qquad (12a)$$

$$\text{det.}\ (\gamma^2 \underline{I} - \underline{\eta}\,\underline{\zeta}) = 0 \qquad (12b)$$

Since $\underline{\eta\zeta}$ is the transpose of $\underline{\zeta\eta}$, and $\underline{I}$ is its own transpose, and since a determinant and its transpose have the same value, equations (12a) and (12b) are equivalent. Since the determinant is of the N<u>th</u> order, the equations represent an N<u>th</u> degree equation in γ^2, with N pairs of roots, $\pm\gamma_i$. The line is characterized by N eigenvalues, γ_i^2, with a pair of propagation modes corresponding to each value.

When the modes are distinct (γ_i all different), each conductor carries N potentials (for a forward wave) which are arbitrary in terms of the terminal excitation, but bear definite ratios to one another in accordance with the first of equations (11). Thus, only one constant has to be determined for each of N modes, corresponding to N terminal potential conditions. If a back wave is also present, N additional arbitrary constants are present for the N back wave modes.

At the other extreme, when all eigenvalues are equal, as, for instance, for a line of lossless conductors in a homogeneous, isotropic dielectric, a choice of methods has been available. One can, in principle, resolve the line potentials and currents into orthogonal modes and proceed as in the previous case[7,8]. In addition to yielding a general procedure for solution, this method also implies certain equivalent circuit concepts which, in special cases, result in powerful analytical and design tools[7]. While certain special situations readily yield the required mode sets[9], the determination of such sets appears, in general, to be a difficult one.

On the other hand, whenever conductor losses can be ignored, this special case can be solved completely in terms of Maxwell's capacitance coefficients for the line, if lossless, or by a minor analytical variation if the dielectric is significantly lossy. The variation consists of replacing the dielectric permittivity of the first case with complex permittivity that includes the loss tangent in the second. This is surely the simpler of the two methods in terms of computational complexity.

When some -- but not all -- of the eigenvalues are equal, some of the degenerate modes must be resolved into orthogonal components to be treated by the first method. The alternate approach is a hybrid of the two methods described for the completely degenerate case.

In many problems of practical interest, the eigenvalues may be only slightly different, at most. This could be true, for instance, for a compact multiwire cable in which materials of differing permittivities and losses are used in various regions of the cable cross-section. It could also occur in some problems where one wishes to take conductor losses into account, and, for the sake of simplicity, is willing to accept an approximate solution. In such cases it would seem appropriate to seek solution methods that are only a little more complicated than the simple solution for the completely degenerate case. This type of problem is discussed in Section 2.3.

Up to now we have considered only methods of attack for the homogeneous equations (1). We have yet to address ourselves to the general case of equations (2) involving forcing functions distributed along the line. This problem was also investigated by Carson and Hoyt (loc.cit.) for a number of special cases which permitted formulation of a solution in terms of a small number of independent modes. The subject is also treated in the previously cited comprehensive report by Strawe[8].

In this report we formulate the solution through use of Laplace transforms. We obtain results for the general case in which any number of the eigenmodes may be degenerate, and we review the special cases of complete degeneracy and of nearly equal eigenvalues which have been treated previously[3,11].

2. ANALYSIS: FORMAL SOLUTION

The extension of Laplace transform methods to matrices is straightforward. Use the general designation

$$\underline{\tilde{F}}(p) = \text{Laplace transform of } \underline{F}(x)$$

$$\underline{\tilde{F}}(p) = \int_0^\infty \underline{F}(\lambda) e^{-p\lambda}\, d\lambda \ ; \ p = c + j\eta,\ c>0 \qquad (13a)$$

and, for the inverse transform,

$$\underline{F}(x) = \mathcal{L}^{-1} \underline{\tilde{F}}(p) = \frac{1}{j2\pi} \int_{c-j\infty}^{c+j\infty} \underline{\tilde{F}}(p)\, e^{px}\, dp \qquad (13b)$$

Taking transforms in equations (2) leads to[16]

$$\left.\begin{aligned} p\underline{\tilde{V}} + \underline{\zeta}\underline{\tilde{I}} &= \underline{V}(0) + \underline{\tilde{E}}^e \\ \underline{\eta}\underline{\tilde{V}} + p\underline{\tilde{I}} &= \underline{I}(0) + \underline{\tilde{H}}^e \end{aligned}\right\} \qquad (14)$$

Multiply the first of these by p, pre-multiply the second by $\underline{\zeta}$, and subtract

$$(p^2\underline{I} - \underline{\zeta}\underline{\eta})\, \underline{\tilde{V}} = p\underline{V}(0) - \underline{\zeta}\underline{I}(0) + p\underline{\tilde{E}}^e - \underline{\zeta}\underline{\tilde{H}}^e \qquad (15)$$

whence

$$\underline{\tilde{V}} = (p^2\underline{I} - \underline{\zeta}\underline{\eta})^{-1} \left[p\underline{V}(0) - \underline{\zeta}\underline{I}(0) + p\underline{\tilde{E}}^e - \underline{\zeta}\underline{\tilde{H}}^e\right] \qquad (16a)$$

Similarly, elimination of $\underline{\tilde{V}}$ in equations (14) yields the dual equation

$$\underline{\tilde{I}} = (p^2\underline{I} - \underline{\eta}\underline{\zeta})^{-1} \left[p\underline{I}(0) - \underline{\eta}\underline{V}(0) + p\underline{\tilde{H}}^e - \underline{\eta}\underline{\tilde{E}}^e\right] \qquad (16b)$$

To obtain the inverse transforms, write

$$\underline{Q} = (p^2\underline{I} - \underline{\zeta\eta}) = \left[q_{ij}\right] \tag{17a}$$

$$\underline{Q}^T = (p^2\underline{I} - \underline{\eta\zeta}) = \left[q_{ji}\right] \tag{17b}$$

$$\underline{Q}^{-1} = \frac{\tilde{\underline{Q}}}{|\underline{Q}|} \tag{18}$$

where $|\underline{Q}|$ is the determinant of $\underline{Q}$, and $\hat{\underline{Q}}$ is its adjoint:

$$\hat{\underline{Q}} = \left[Q_{ji}\right] = \begin{bmatrix} Q_{n}, \ldots, Q_{ni} \\ \ldots, \quad \ldots, \quad \ldots, \\ Q_{in}, \ldots, Q_{nn} \end{bmatrix} = \left[Q_{ij}\right]^T \tag{19}$$

and Q_{ij} is the cofactor of q_{ij} in $|\underline{Q}|$.

Then, using equation (13b), the solution for $\underline{V}$ is, formally,

$$\underline{V} = \frac{1}{j2\pi}\int_{c-j\infty}^{c+j\infty} \frac{\hat{\underline{Q}}(p)}{|\underline{Q}|}\Big\{ p\underline{V}(0) - \underline{\zeta I}(0) + P\underline{\tilde{E}}^{e}(p) - \underline{\zeta \tilde{H}}^{e}(p)\Big\}\, e^{px}dp \tag{20}$$

This may be written

$$\underline{V} = \frac{d\underline{T}}{dx}\,\underline{V}(0) - \underline{T\zeta I}(0) + \frac{d\underline{T}}{dx} * \underline{E}^{e}(x) \quad \underline{T} * \zeta\underline{H}^{e}(x) \tag{21}$$

where

$$\underline{T} = \frac{1}{j2\pi}\int_{c-j\infty}^{c+j\infty} \frac{\hat{\underline{Q}}(p)\, e^{px}dp}{|\underline{Q}|} \tag{22a}$$

and terms of the form $\underline{F}*\underline{G}$ represent the convolution convention

$$\underline{F}*\underline{G} = \frac{1}{j2\pi}\int_{c-j\infty}^{c+j\infty} \tilde{\underline{F}}(p)\tilde{\underline{G}}(p)\; e^{px}\; dp$$
$$= \int_0^x \underline{F}(\lambda)\underline{G}(x-\lambda)d\lambda = \int_0^x \underline{F}(x-\lambda)\underline{G}(\lambda)d\lambda \tag{23}$$

To obtain the current from equation (16b), we have, using equation (17b),

$$\underline{I} = \frac{d\underline{T}^T}{dx}\,\underline{I}(0) - \underline{T}^T\underline{\eta}\underline{V}(0) + \frac{d\underline{T}^T}{dx} * \underline{H}^e(x)$$
$$- \underline{T}^T * \underline{\eta}\underline{E}^e(x) \tag{24}$$

where

$$\underline{T}^T = \frac{1}{j2\pi}\int_{c-j\infty}^{c+j\infty} \frac{\hat{\underline{Q}}^T e^{px} dp}{|\underline{Q}|} \tag{22b}$$

From these results it is clear that the essential problem is the evaluation of the inverse transform, $\underline{T}(x)$, equation (22a). Before investigating various aspects of this problem, we complete the formal solutions for $\underline{V}$ and $\underline{I}$ by eliminating the terminal quantities, $\underline{V}(0)$ and $\underline{I}(0)$, in favor of arbitrary terminal admittances.

We proceed essentially as in Reference 3, Appendix B.

In conformity with previously established notation (loc. cit.) we introduce the following symbol changes:

$$\underline{V}(0) \rightarrow \underline{V}^i$$

$$\underline{V}(\ell) \rightarrow \underline{V}^0$$

$$\underline{I}(0) \rightarrow \underline{I}^i$$

$$\underline{I}(\ell) \rightarrow \underline{I}^0$$

where ℓ is the length of the line.

In addition we have the terminal conditions

$$\left.\begin{aligned} \underline{I}^i + \underline{Y}^i\underline{V}^i &= \underline{0} \\ \underline{I}^0 - \underline{Y}^0\underline{V}^0 &= \underline{0} \end{aligned}\right\} \qquad (25)$$

where $\underline{Y}^i$, $\underline{Y}^0$ are the line termination admittance matrices at $x = 0, \ell$, respectively[12].

Using these conditions, the following results are obtained in Appendix A for the potential and current at any point along the line:

$$\left.\begin{aligned} \underline{V}(x) &= \left\{ \underline{T}^{T'}(x) + \underline{T}(x)\ \zeta\underline{Y}^i \right\} \underline{S}_0^{-1}\underline{K}_0(\ell) + \underline{U}(x) \\ I(x) &= -\left\{ \underline{T}^{T'}(x)\ \underline{Y}^i + \underline{T}^T(x)\ \underline{\eta} \right\} \underline{S}_0^{-1}\underline{K}_0(\ell) + \underline{W}(x) \end{aligned}\right\} \qquad (26)$$

where

$$\underline{T}'(x) = \frac{d\underline{T}}{dx} \; ; \; \underline{T}^{T'}(x) = \frac{d\underline{T}^T}{dx}$$

and

$$\left.\begin{aligned}
\underline{S}_0 &= \left\{ \underline{T}^{T'}(\ell)\underline{Y}^i + \underline{T}^T(\ell)\underline{\eta} \right\} + \underline{Y}^0 \left\{ \underline{T}'(\ell) + \underline{T}(\ell)\underline{\zeta}\underline{Y}^i \right\} && \text{(a)} \\
\underline{U}(x) &= \underline{T}'(x) * \underline{E}^e(x) - \underline{T}(x) * \underline{\zeta}\underline{H}^e(x) && \text{(b)} \\
W(x) &= \underline{T}^{T'}(x) * \underline{H}^e(x) - \underline{T}^T(x) * \underline{\eta}\underline{E}^e(x) && \text{(c)} \\
\underline{K}_0(\ell) &= \underline{W}(\ell) - \underline{Y}^0\underline{U}(\ell) && \text{(d)}
\end{aligned}\right\} \quad (27)$$

Special Excitation Conditions

Certain excitation conditions are of special interest: (1) impulse excitation (2) linear phase excitation,

1. Impulse excitation. Excitation is localized at a point, -- or a number of isolated points -- along the line. Thus, at some point, x_0, let

$$\left.\begin{aligned}
\underline{E}^e(x) &= \underline{V}^e \delta(x-x_0) \\
\underline{H}^e(x) &= \underline{I}^e \delta(x-x_0)
\end{aligned}\right\} \quad (28)$$

where $\delta(x)$ is defined in equations (9). Then for any function $\underline{J}(x)$, we have

$$\begin{aligned}
\underline{J}(x) * \underline{E}^e(x) &= \int_0^x \underline{J}(x-\zeta) \cdot \underline{V}^e \delta(\xi - x_0)d\xi \\
&= \underline{J}(x-x_0)\underline{V}^e u(x-x_0)
\end{aligned} \quad (29a)$$

where u(x) is a unit step:

$$\left.\begin{aligned} u(x) &= 0, \; x<0 \\ &= 1, \; x>0 \end{aligned}\right\} \qquad (30)$$

Similarly,

$$\underline{J}(x) * \underline{H}^{e}(x) = \underline{J}(x-x_0)\underline{I}^{e}\, u(x-x_0) \qquad (29b)$$

Thus, U(x) and W(x) of equations (27) become

$$\left.\begin{aligned} \underline{U}(x) &= \left\{ \underline{T}'(x-x_0)\underline{V}^{e} - \underline{T}(x-x_0)\underline{\zeta}\underline{I}^{e} \right\} u(x-x_0) \\ \underline{W}(x) &= \left\{ \underline{T}^{T'}(x-x_0)\underline{I}^{e} - \underline{T}^{T}(x-x_0)\underline{\eta}\underline{V}^{e} \right\} u(x-x_0) \end{aligned}\right\} \qquad (31)$$

and

$$\begin{aligned} \underline{K}_0(\ell) &= \underline{W}(\ell) - \underline{Y}^{0}\underline{U}(\ell) \\ &= \left\{ \underline{T}^{T'}(\ell-x_0) + \underline{Y}^{0}\underline{T}(\ell-x_0)\underline{\zeta} \right\} \underline{I}^{e} \\ &\quad - \left\{ \underline{T}^{T}(\ell-x_0)\underline{\eta} + \underline{Y}^{0}\underline{T}'(\ell-x_0) \right\} \underline{V}^{e} \end{aligned} \qquad (32)$$

In particular, for $x_0 = 0$, we have <u>end excitation</u> of the line:

$$\left.\begin{aligned} \underline{U}(x) &= \left\{ \underline{T}'(x)\underline{V}^{e} - \underline{T}(x)\underline{\zeta}\underline{I}^{e} \right\} u(x) \\ \underline{W}(x) &= \left\{ \underline{T}^{T'}(x)\underline{I}^{e} - \underline{T}^{T}(x)\underline{\eta}\underline{V}^{e} \right\} u(x) \\ \underline{K}_0(\ell) &= \left\{ \underline{T}^{T'}(\ell) + \underline{Y}^{0}\underline{T}(\ell)\underline{\zeta} \right\} \underline{I}^{e} \\ &\quad - \left\{ \underline{T}^{T}(\ell)\underline{\eta} + \underline{Y}^{0}\underline{T}'(\ell) \right\} \underline{V}^{e} \end{aligned}\right\} \qquad (33)$$

If, in the latter case, $\underline{I}^e \equiv 0$, we have a generalized Thevenin source[13,10], with

$$\left.\begin{aligned} \underline{U}(x) &= \underline{T}'(x)\underline{V}^e u(x) \\ \underline{W}(x) &= -\underline{T}^T(x)\underline{\eta}\underline{V}^e u(x) \\ \underline{K}_0(\ell) &= -\left\{ \underline{T}^T(\ell)\underline{\eta} + \underline{Y}^0\underline{T}'(\ell) \right\} \underline{V}^e \end{aligned}\right\} \quad (34)$$

If, on the other hand, $\underline{V}^e \equiv \underline{0}$, for end excitation, we have a generalized Norton source, with

$$\left.\begin{aligned} \underline{U}(x) &= -\underline{T}(x)\underline{\zeta}\underline{I}^e u(x) \\ \underline{W}(x) &= \underline{T}^{T'}(x)\underline{I}^e u(x) \\ \underline{K}_0(\ell) &= \left\{ \underline{T}^{T'}(\ell) + \underline{Y}^0\underline{T}(\ell)\underline{\zeta} \right\} \underline{I}^e \end{aligned}\right\} \quad (35)$$

If we have a shielded cable with a narrow circumferential break at $x = x_0$, an external field can produce a gradient $V^e\,\delta(x-x_0)$ across the break. The effect is the same as exciting the internal conductors with a series emf, $-V^e\delta(x-x_0)\underline{I}_c$, where $\underline{I}_c$ is a unit column vector. In that case, equations (31) become

$$\left.\begin{aligned} \underline{U}(x) &= -\underline{T}'(x-x_0)\underline{I}_c V^e u(x-x_0) \\ \underline{W}(x) &= \underline{T}^T(x-x_0)\eta \underline{I}_c V^e u(x-x_0) \end{aligned}\right\} \quad (36)$$

and equation (32) becomes

$$K_0(\ell) = \{\underline{T}^T(\ell-x_0)\underline{\eta} + \underline{Y}^0\underline{T}'(\ell-x_0)\}\underline{I}_c V^e \quad (37)$$

2. Linear phase excitation. In a large class of problems, the line may be considered to be excited by a wave incident at an angle, such that the impressed field has a linear phase variation along the line. The forcing functions may then be expressed as

$$\left.\begin{aligned} \underline{E}^{e}(x) &= \underline{E}^{e}(0)\exp(-j\beta_e x) \\ \underline{H}^{e}(x) &= \underline{H}^{e}(0)\exp(-j\beta_e x) \end{aligned}\right\} \qquad (38)$$

For any function, $\underline{J}(x)$, we have

$$\underline{J}(x) * \underline{E}^{e}(x) = \left\{ \int_0^x \underline{J}(x-\xi)\exp(-j\beta_e \xi)\,d\xi \right\} \underline{E}^{e}(0)$$

and similarly for $H^{e}(x)$. Equations (27b,c) become

$$\left.\begin{aligned} \underline{U}(x) &= \underline{\Phi}(x)\underline{E}^{e}(0) - \underline{\Psi}(x)\underline{\zeta}\underline{H}^{e}(0) \\ \underline{W}(x) &= \underline{\Phi}^{T}(x)\underline{H}^{e}(0) - \underline{\Psi}^{T}(x)\underline{E}^{e}(0) \end{aligned}\right\} \qquad (39)$$

where

$$\left.\begin{aligned} \underline{\Phi}(x) &= \int_0^x \underline{T}'(x-\xi)\exp(-j\beta_e \xi)\,d\xi \\ \underline{\Psi}(x) &= \int_0^x \underline{T}(x-\xi)\exp(-j\beta_e \xi)\,d\xi \end{aligned}\right\} \qquad (40)$$

$\underline{\Phi}$ and $\underline{\Psi}$ are not unrelated. By integration by parts one obtains

$$\underline{\Phi}(x) + j\beta_e\underline{\Psi}(x) = \underline{T}(x) - \underline{T}(0)\exp(-j\beta_e x) \qquad (40a)$$

We now turn our attention to the important problem of evaluating $\underline{T}(x)$, equation (22a), for a number of special cases and, finally, a general case involving any number of multiple poles. First, we note that since both $|\underline{Q}|$ and the elements of $|\hat{\underline{Q}}|$ are polynomials in p^2, the singularities, if any, of the integrand in the right member of equation (22a) are poles in the finite region of the complex plane. Thus, the integral is zero for $x<0$, and is equal to $(j2\pi)$ times the sum of the residues at the poles of the integrand for $x>0$. The possible poles, in turn, are the roots of

$$\begin{aligned} |\underline{Q}| &= \det.(p^2\underline{I} - \underline{\zeta\eta}) \\ &= \left\{ \det.(\gamma^2\underline{I} - \underline{\zeta\eta}) \right\}_{\gamma^2=p^2} = 0 \qquad (41) \end{aligned}$$

Comparing equation (41) with equation (12a), we see that the poles of the integrand are identical with the square roots of the eigenvalues of the line matrix, $\underline{\zeta\eta}$, and of its transpose, $\underline{\eta\zeta}$. The transform approach affords a routine method for determining the response of the system in terms of these poles.

2.1. Lossless Conductors in a Homogeneous, Isotropic Dielectric (Pure TEM Case); Single Multiple-Pole Pair; Completely Degenerate Mode.

This case has been discussed previously as an isolated problem[3]. Here we relate the results to the general formulation. For lossless conductors we have

$$\underline{\zeta} = j\omega\underline{L} = j\omega \left[L_{ij}\right] \tag{42a}$$

and for a homogeneous, isotropic dielectric,

$$\begin{aligned}\underline{\eta} &= \underline{G} + j\omega\underline{C} = \left(\frac{g_d}{j\omega\varepsilon} + 1\right) j\omega\underline{C} \\ &= (1 - j \tan \delta_d)\ j\omega \left[C_{ij}\right]\end{aligned} \tag{42b}$$

where

$\underline{L}$ is the line inductance matrix, henry/meter

$\underline{C}$ is the line capacitance matrix, farad/meter

g_d is the dielectric conductivity, mho/meter

ε is the dielectric permittivity, farad/meter

$\tan \delta_d$ is the dielectric loss tangent,

$$\tan \delta_d = gd/\omega\varepsilon \tag{43}$$

Furthermore,

$$\underline{L}\underline{C} = \left[\mu\varepsilon\delta_{ij}\right] = \left[v^{-2}\delta_{ij}\right] \tag{44}$$

where v is the velocity of wave propagation in the dielectric (meter/sec.)

Thus we have

$$\underline{Q} = (p^2\underline{I} - \zeta\eta) = \left[q_{ij}\right]$$

$$= p^2\underline{I} - (j\omega\underline{L})(1 - \tan\delta_d)(j\omega\underline{C})$$

$$= p^2\underline{I} + \omega^2(1 - j\tan\delta_d)\underline{LC} \tag{45}$$

$$= \left[\left\{p^2 + \frac{\omega^2}{v^2}(1 - j\tan\delta_d)\right\}\delta_{ij}\right]_N$$

$$= \left[(p^2 - k^2)\delta_{ij}\right]_N = (p^2 - k^2)\underline{I}$$

where

$$\left.\begin{aligned} k &= \pm j(1 - j\tan\delta_d)^{\frac{1}{2}}\beta \\ \beta &= \frac{\omega}{v} \end{aligned}\right\} \tag{46}$$

The cofactor of Q_{ij} is

$$q_{ij} = (p^2 - k^2)^{N-1}\delta_{ij}$$

so that

$$\hat{\underline{Q}} = \left[(p^2 - k^2)^{N-1}\delta_{ij}\right] = (p^2 - k^2)^{N-1}\underline{I} \tag{47}$$

while

$$|\underline{Q}| = (p^2 - k^2)^{N}\underline{I} \tag{48}$$

Then equations (47) and (48) in equation (22a) yield

$$\left.\begin{aligned}\underline{T}(x) &= \frac{\underline{I}}{j2\pi}\int_{c-j\infty}^{c+j\infty}\frac{e^{px}dp}{p^2-k^2} = \underline{I}\,\frac{\sinh kx}{k} \\ &= \underline{T}^{T}(x)\end{aligned}\right\} \tag{49a}$$

$$\underline{T}'(x) = \underline{I}\cosh kx = \underline{T}^{T'}(x) \tag{49b}$$

Equations (26) become

$$\left.\begin{aligned}\underline{V}(x) &= \left[\underline{I}\cosh kx + \frac{\sinh kx}{k}\,\underline{\zeta}\underline{Y}'\right]\underline{S}_0{}^{-1}\underline{K}_0(\ell)\ \underline{U}(x) \\ \underline{I}(x) &= -\left[\underline{Y}^{i}\cosh kx + \frac{\sinh kx}{k}\,\underline{\eta}\right]\underline{S}_0{}^{-1}\underline{K}_0(\ell) + \underline{W}(x)\end{aligned}\right\} \tag{50}$$

But

$$\frac{\underline{\zeta}}{k} = \frac{j\omega\underline{L}}{j\frac{\omega}{v}(1-j\tan\delta_d)^{1/2}} = \underline{Z}(1-j\tan\delta_d)^{-1/2} = \underline{Z}' \tag{51a}$$

where $\underline{Z}'$ is the line characteristic impedance matrix and $\underline{Z}$ is the characteristic impedance matrix in the absence of dielectric loss:

$$\underline{Z} = v\underline{L} \tag{52a}$$

Furthermore

$$\frac{\underline{\eta}}{k} = \frac{j\omega\underline{C}(1-j\tan\delta d)}{j\frac{\omega}{v}(1-j\tan\delta d)^{1/2}} = \underline{Y}\,(1-j\tan\delta_d)^{1/2} = \underline{Y}' \tag{51b}$$

where $\underline{Y}'$ is the line characteristic admittance matrix and $\underline{Y}$ is the characteristic admittance matrix in the absence of dielectric loss:

$$\underline{Y} = v\underline{C} \tag{52b}$$

We have

$$\underline{Z}'\ \underline{Y}' = \underline{Z}\underline{Y} = v^2\underline{L}\underline{C} = \left[\delta_{ij}\right] = \underline{I} \tag{53a}$$

by equation (44).

In many situations, $\tan \delta_d$ is so small that its effect on $\underline{Z}'$ and $\underline{Y}'$ is frequently ignored, even though it may be important in the value of the propagation constant, k.

Using equations (51a,b) in equations (50), and writing

$$\underline{P}^i = \underline{Z}'\underline{Y}^i \tag{53b}$$

$$\left.\begin{aligned} \underline{V}(x) &= (\underline{I} \cosh kx + \underline{P}^i \sinh kx)\ \underline{S}_0{}^{-1}\underline{K}_0(\ell) + \underline{U}(x) \\ \underline{I}(x) &= -\underline{Y}'(\underline{P}^i \cosh kx + \underline{I} \sinh kx)\ \underline{S}_0{}^{-1}\underline{K}_0(\ell) + \underline{W}(x) \end{aligned}\right\} \tag{54}$$

where, using these various results in equations (27a-d), and writing $\underline{P}^0 = \underline{Z}'\underline{Y}^0$,

$$\left.\begin{aligned}
\underline{U}(x) &= \int_0^x \left\{ \underline{E}^e(\xi) \cosh[k(x-\xi)] - \underline{Z}'\underline{H}^e(\xi) \sinh[k(x-\xi)] \right\} d\xi \\
\underline{W}(x) &= \int_0^x \left\{ \underline{H}^e(\xi) \cosh[k(x-\xi)] - \underline{Y}'\underline{E}^e(\xi) \sinh[k(x-\xi)] \right\} d\xi \\
\underline{S}_0 &= \underline{Y}' \left\{ (\underline{P}^i + \underline{P}^0) \cosh k\ell + (\underline{I} + \underline{P}^0\underline{P}^i) \sinh k\ell \right\} \\
\underline{K}_0(\ell) &= \underline{Y}' \left\{ \int_0^\ell \left[\underline{I} \cosh[k(\ell-\xi)] + \underline{P}^0 \sinh[k(\ell-\xi)] \right] \underline{Z}'\underline{H}^e(\xi) d\xi \right. \\
&\quad \left. - \int_0^\ell \left[\underline{P}^0 \cosh[k(\ell-\xi)] + \underline{I} \sinh[k(\ell-\xi)] \right] \underline{E}^e(\xi) d\xi \right\}
\end{aligned}\right\} \quad (55)$$

When it is justifiable to assume that both dielectric and conductors are lossless, we have $k = j\beta$ (equations (46)), and the foregoing results become

$$\left.\begin{aligned}
\underline{V}(x) &= (\underline{I} \cos \beta x + j\underline{P}^i \sin \beta x)\, \underline{S}^{-1}\underline{K}(\ell) + \underline{U}(x) \\
\underline{I}(x) &= -\underline{Y}(\underline{P}^i \cos \beta x + j\underline{I} \sin \beta x)\, \underline{S}^{-1}\underline{K}(\ell) + \underline{W}(x)
\end{aligned}\right\} \quad (56)$$

where

$$\left.\begin{aligned}
\underline{U}(x) &= \int_0^x \{\underline{E}^{e}(x)\ \cos[\beta(x-\xi)] - j\underline{Z}\underline{H}^{e}(\xi)\ \sin[\beta(x-\xi)]\}\, d\xi \\
\underline{W}(x) &= \underline{Y}\int_0^x \{\underline{Z}\underline{H}^{e}(\xi)\ \cos[\beta(x-\xi)] - j\underline{E}^{e}(\xi)\sin[\beta(x-\xi)]\, d\xi \\
\underline{S} &= \underline{Z}\underline{S}_0 = (\underline{P}^{i}+\underline{P}^{0})\ \cos\ \beta\ell + j(\underline{I} + \underline{P}^{0}\underline{P}^{i})\ \sin\ \beta\ell \\
\underline{K}(\ell) &= \underline{Z}\underline{K}_0(\ell) = \int_0^{\ell} \{\underline{I}\ \cos[\beta(\ell-\xi)] \\
&\qquad + j\underline{P}^{0}\ \sin[\beta(\ell-\xi)]\}\ \underline{Z}\underline{H}^{e}(\xi)\, d\xi \\
&\qquad - \int_0^{\ell} \{\underline{P}^{0}\ \cos[\beta(\ell-\xi)] \\
&\qquad + j\underline{I}\ \sin[\beta(\ell-\xi)]\}\underline{E}^{e}(\xi)\, d\xi
\end{aligned}\right\} \quad (57)$$

These are the results previously obtained for the lossless TEM case[3].

2.2 <u>Mixed Dielectrics; Poles of Integrand Simple and Widely Separated (Non-Degenerate-Mode Case)</u>.

In equation (22a) write

$$|\underline{Q}| = \prod_{k=1}^{N} (p^2 - p_k^2),\ p_k \text{ all different} \qquad (58)$$

Also, let

$$R(p_i) = \prod_{\substack{k=1 \\ (i)}}^{n} (p_i^2 - p_k^2) = R(-p_i) \tag{59}$$

where the symbol, (i), in the product operator indicates that the factor corresponding to k = i has been omitted.

Then the standard solution of equation (22a) is

$$\underline{T}(x) = \frac{1}{2}\sum_{i=1}^{n} \left\{ \frac{\hat{\underline{Q}}(p_i)\exp(p_i x) - \hat{\underline{Q}}(-p_i)\exp(-p_i x)}{p_i R(p_i)} \right\} \tag{60}$$

Clearly, since the elements of $\hat{\underline{Q}}(p)$ are functions of p^2,

$$\hat{\underline{Q}}(-p_i) = \hat{\underline{Q}}(p_i) \tag{61}$$

and equation (60) becomes

$$\underline{T}(x) = \sum_{i=1}^{n} \frac{\hat{\underline{Q}}(p_i) \sinh p_i x}{p_i \; R(p_i)} \tag{62}$$

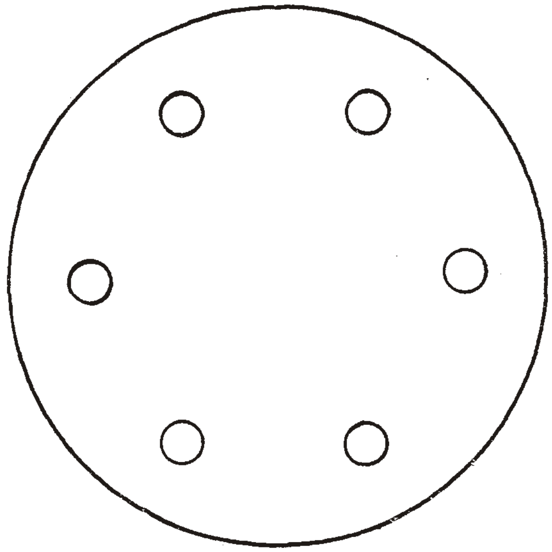
The remaining quantities needed to complete the solution given in equations (26) follow immediately from equation (62), and are not detailed here.

2.2.1 Number of Different Elements in $\hat{\underline{Q}}(p)$.

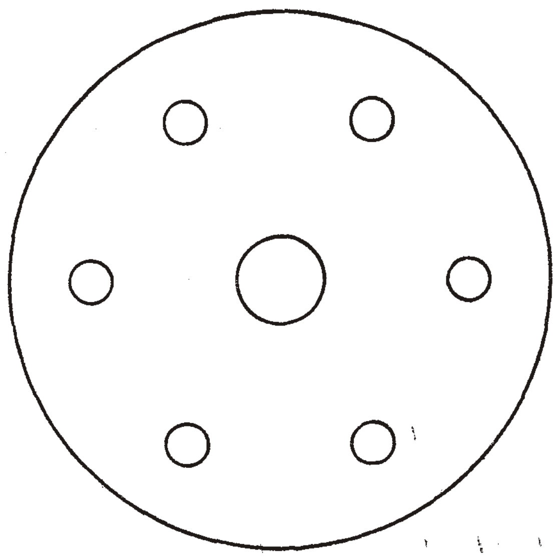
In general, $\underline{Q}$, therefore $\hat{\underline{Q}}$, is non-symmetric, and therefore contains N^2 different elements, each of which must be evaluated for N (generally) different values of p, making a total of N^3 computations on $\hat{\underline{Q}}$ alone. For some special configurations this number may be dramatically reduced. Alternatively, if it turns out that the relative dispersion in eigenvalues is small, the approximate method of Section 2.3 affords a great simplification in computation for large N.

Cables consisting of a number of conductors in a circular sheath frequently enjoy a degree of symmetry which reduces the number of different values of the elements of $\hat{\underline{Q}}$. Two of the simpler cases are indicated in Figure 1. In both of these cases the total cable cross-section has a circular symmetry such that rotation of the cable on its axis by any integral multiple of some fixed angle yields the original configuration exactly. The first case, (a), consisting of a single ring of symmetrically disposed conductors of equal diameter has a characteristic matrix containing $(\frac{N}{2} + 1)$ different elements if N is even, $(\frac{N+1}{2})$ different elements if N is odd[9]. The inverse matrix has the same property, whence the same is true of the adjoint.

The second case, having a symmetric ring of N_2 conductors, as in the preceding, plus a central conductor of any diameter, has a characteristic matrix with $(\frac{N_2}{2} + 3)$ unique elements for N_2 even, $(\frac{N_2+1}{2} + 2)$ unique elements for N_2 odd. Furthermore, using a bordering technique for matrix inversion[14], it is easily seen that the inverse, therefore the adjoint, has the



(a)



(b)

FIGURE 1. TWO SIMPLE EXAMPLES OF SYMMETRIC CABLE CONFIGURATIONS .

same number of unique elements. For example, if $N_2 = 20$, the number of unique element values is 13, whereas, in general, the matrix of a 21-conductor cable could have as many as $(21)^2$ or 441 different values.

Strawe[9] has derived explicit analytic expressions in terms of fundamental line parameters for end excitation of configurations of the general form of Figure 1a.

2.3. Pole-Pairs (Eigenvalues) Nearly Equal (Nearly Degenerate Mode Case): Average-Pole Expansion.

This case has been treated previously[11]; therefore only an outline of the analysis is given here. The poles, $\pm p_i$, not necessarily all different, are assumed to be contained in regions in the complex plane bounded by two small circles of radius, ρ, such that

$$\left|\frac{\rho}{p_i}\right| << 1, \; i = 1, \ldots, N$$

An "average" pole, p_a is defined by

$$p_a = \frac{1}{N}\sum_{i=1}^{N} p_i \tag{63}$$

and the pole variations, δ_i, by

$$\delta_i = p_i - p_a, \quad \left|\frac{\delta_i}{p_a}\right| << 1 \tag{64}$$

Clearly,

$$\sum_{i=1}^{N} \delta_i = 0 \tag{65}$$

Let

$$|\underset{\sim}{Q}| = \Pi_+(p) \cdot \Pi_-(p)$$

where

$$\Pi_{\pm}(p) = \prod_{i=1}^{N} (p \pm p_i)$$

In the small circle of radius, ρ, containing the poles around $p = p_a$, write

$$\Pi_+(p) = \prod_{i=1}^{N} (p - p_a - \delta_i) = (p - p_a)^N \prod_{i=1}^{N} (1 - \frac{\delta i}{p-pa})$$

$$\Pi_-(p) = \prod_{i=1}^{N} (p + p_a + \delta_i) \approx (p+p_a)^N$$

This leads to

$$\begin{aligned} |\underset{\sim}{Q}|^{-1} &\approx \left[(p^2 - p_a{}^2)^N \prod_{i=1}^{N} (1 - \frac{\delta_i}{p-p_a}) \right]^{-1} \\ &= (p^2 - p_a{}^2)^{-N} \sum_{s=0}^{\infty} \frac{S_s}{(p-p_a)^s} \end{aligned} \tag{66}$$

where

$$\left.\begin{aligned} S_0 &= 1 \\ S_1 &= \sum_{i=1}^{N} \delta_i = 0 \\ S_2 &= \sum_{i=1}^{N} \sum_{j=1}^{i} \delta_i \delta_j \\ &\cdot\cdot \quad \cdot\cdot \quad \cdot\cdot \end{aligned}\right\} \tag{67}$$

A recursion formula for S_s, $s>2$, is derived in Appendix B.

A little study shows that the cofactors of the diagonal elements of $\underline{\hat{Q}}$ are of degree (N-1) in p^2, while off-diagonal elements are of degree (N-2) in p^2, at most. Thus,

$$\underline{\hat{Q}} = \left[Q_{ji}\right] = \sum_{q=1}^{N} \underline{C}^{(q)} (p^2)^{N-q} \tag{68}$$

where

$$\underline{C}^{(q)} = \left[c_{ij}{}^{(q)}\right]; \ \underline{C}^{(1)} = \left[\delta_{ij}\right] = \underline{I} \tag{69}$$

$\underline{\hat{Q}}$ may be expanded in powers of $(p^2 - p_a{}^2)$ by Taylor's theorem:

$$\underline{\hat{Q}} = \sum_{q=1}^{N} \underline{A}^{(q)} (p^2 - p_a{}^2)^{N-q} \tag{70}$$

where

$$\left.\begin{aligned}
&\underline{A}^{(q)} = \left[a_{ij}^{(q)}\right] \\
&a_{ij}^{(1)} = \delta_{ij} \\
&a_{ij}^{(2)} = (N-1)p_a^{\,2}\,\delta_{ij} + c_{ij}^{(2)} \\
&\cdots \quad \cdots \quad \cdots
\end{aligned}\right\} \tag{71}$$

The exponential, e^{px}, expanded in a Taylor's series around $p = p_a$, is

$$e^{px} = e^{p_a x} \sum_{r=0}^{\infty} \frac{(p-p_a)^r x^r}{r!} \quad . \tag{72}$$

Substituting equations (66), (68), and (72) in equation (22a) yields, for the poles around p_a:

$$\mathrm{Res}(p_a) = e^{p_a x} \sum_{q=1}^{N} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \underline{A}^{(q)} S_s \left(\frac{x^r}{r!}\right) \left\{ \frac{1}{j2\pi} \int_{(p_a)} \frac{(p-p_a)^{r-q-s}}{(p+p_a)^q}\, dp \right\} \tag{73}$$

Next, expand $(p + p_a)^{-q}$ in powers of $(p - p_a)$;

$$(p + p_a)^{-q} = (2p_a)^{-q} \sum_{m=0}^{\infty} (-1)^m \binom{q+m-1}{m} \left(\frac{p-p_a}{2p_a}\right)^m$$

where

$$\binom{q+m-1}{m} = \text{binomial coefficient}$$

$$= \frac{(q+m-1)(q+m-2)\ldots.(q+1)q}{m!}$$

$$= \binom{q+m-1}{q-1}$$

Equation (73) becomes

$$\text{Res}(p_a) = e^{p_a x} \sum_{q=1}^{N} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \sum_{m=0}^{\infty} (-1)^m \binom{q+m-1}{q-1} \frac{\underline{A}^{(q)}}{(2p_a)^{q+m}} S_s \left(\frac{x^r}{r!}\right) C_\alpha(p_a) \qquad (74)$$

where

$$\left.\begin{aligned} C_\alpha &= \frac{1}{j2\pi} \int_{(p_a)} (p-p_a)^\alpha dp \\ \alpha &= r + m - q - s \end{aligned}\right\} \qquad (75)$$

The residue at $p = p_a$ is the coefficient of C_{-1}; that is the coefficient corresponding to the constraint

$$\alpha = r + m - q - s = -1 \qquad (76)$$

As a result we get for the residue at p_a:

$$\mathrm{Res}(p_a) = e^{p_a x} \sum_{q=1}^{N} \sum_{r=0}^{q+s-1} \sum_{s=0}^{\infty} \frac{\underline{A}^{(q)}}{(2p_a)^q} S_s\left(\frac{x^r}{r!}\right) \frac{(-1)^{q+s-r-1}}{(2p_a)^{q+s-r-1}} \binom{2q+s-r-2}{q-1} \quad (77)$$

Following the same procedure in the region around $-p_a$ yields

$$\mathrm{Res}(-p_a) = e^{-p_a x} \sum_{q=1}^{N} \sum_{r=0}^{q+s-1} \sum_{s=0}^{\infty} \frac{\underline{A}^{(q)}}{(2p_a)^q} \quad S_s\left(\frac{x^r}{r!}\right) \frac{(-1)^{q+s}}{(2p_a)^{q+s-r-1}} \binom{2q+s-r-2}{q-1} \quad (78)$$

The solution for $\underline{T}$ is

$$\underline{T}(x) = \mathrm{Res}(p_a) + \mathrm{Res}(-p_a)$$

$$= \sum_{q=1}^{N} \frac{\underline{\underline{A}}^{(q)}}{(4p_a{}^2)^{q-1}} \quad F(x;p_a;q) \quad (79)$$

where

$$F = \sum_{r=0}^{q+s-1} \sum_{s=0}^{\infty} \frac{(-1)^{q+s-r-1}}{(2p_a)^{s-r+1}} \quad S_s\left(\frac{x^r}{r!}\right) \binom{2q+s-r-2}{q-1} \left[e^{p_a x} - (-1)^r e^{-p_a x} \right] \quad (80)$$

Because off-diagonal elements of $\underline{Q}$ are expected to be small (reference 11,Section 4), it follows that

$$\left| \frac{a_{ij}{}^{(r+t)}}{p_a{}^{2(r+t-1)}} \right| << \left| \frac{a_{ij}{}^{(r)}}{p_a{}^{2(r-1)}} \right| << 1, \; r>1, \; t \text{ pos. int.}$$

As a result, we are particularly interested in values of F for $q = 1,2$; with the help of equations (67) we get

$$F(x;p_a;1) = \frac{1}{p_a}\left\{\sinh p_a x + \frac{S_2}{(2p_a{}^2)}\left[(1+2p_a{}^2x^2)\sinh p_a x - 2\,p_a x \cosh p_a x\right] + \ldots\right. \quad (81)$$

$$F(x;p_a;2) = -\frac{2}{p_a}\left\{\left[\sinh p_a x - p_a x \cosh p_a x\right] - \frac{S_2}{p_a{}^3}\left[(1 + p_a{}^2x^2)\sinh p_a x - 3p_a x(1+\tfrac{1}{9}p_a{}^2x^2)\cosh p_a x\right] + \ldots\right. \quad (82)$$

For low-loss conductors in a homogeneous isotropic dielectric, we expect applications of these results to show that the dispersion effects are accounted for adequately by the $\underline{A}^{(q)}$, $(q>1)$, and that their multiplication by factors containing S_s, $(s>1)$, introduces only second-order corrections to the dispersion calculations. In that case, F reduces to the finite single summation:

$$F = F_0(x;p_a;q) = \sum_{r=0}^{q-1} \frac{(-1)^{q-r-1}}{(2p_a)^{1-r}}\left(\frac{x^r}{r!}\right)\binom{2q-r-2}{q-1}\left[e^{p_a x} - (-1)^r e^{-p_a x}\right] \quad (83)$$

In particular,

$$\left.\begin{aligned} F_0(x;p_a;1) &= \frac{1}{p_a} \sinh p_a x \\ F_0(x;p_a;2) &= -\frac{2}{p_a} (\sinh p_a x - p_a x \cosh p_a x) \end{aligned}\right\} \qquad (84)$$

2.4. General Case: Any Number of Multiple Poles (Any Number of Degenerate Modes); Pole-Pairs (Eigenvalues) Not Nearly Equal.

Recall that, generally, $|\underline{Q}|$ is a polynomial of the N$\underline{th}$ degree in p^2:

$$|\underline{Q}| = \prod_{i=1}^{N} (p^2-p_i^2) = \sum_{r=0}^{N} d_r p^{2(N-r)};\ d_0 = 1$$

Let factors, $(p - p_k)$ have multiplicities, r_k, $(k = 1, \ldots, m;\ m \leq N)$, so that

$$|\underline{Q}| = \prod_{k=1}^{m} (p^2-p_k^2)^{r_k};\ \sum_{k=1}^{m} r_k = N \qquad (85)$$

Then

$$\underline{T}(x) = \frac{1}{j2\pi} \int_{c-j\infty}^{c+j\infty} \frac{\hat{\underline{Q}}\ e^{px} dp}{\prod_{k=1}^{m} (p^2-p_k^2)^{r_k}} \qquad (86)$$

To find the residue at p_i, the factor in the integrand not containing $(p - p_i)^{r_i}$, namely,

$$\frac{\hat{\underline{Q}}\, e^{px}}{(p+p_i)^{r_i} \prod_{\substack{k=1 \\ (i)}}^{m} (p^2-p_k{}^2)^{r_k}}$$

must be expanded in a Taylor's series around p_i.

The denominator function

$$\phi_+(p;p_i) = (p+p_i)^{r_i} \prod_{\substack{k=1 \\ (i)}}^{m} (p^2-p_k{}^2)^{r_k} \tag{87}$$

is a polynomial of degree $(2N - r_i)$ in p. It may be inverted to become a power series in the numerator; more generally, therefore, it may be expressed as a Taylor's series in $(p - p_i)$:

$$\left[\phi_+(p;p_i)\right]^{-1} = \sum_{s=0}^{\infty} b_{s,i}(p-p_i)^s \tag{88}$$

Assume this done. Also, as in the preceding section, expand $\hat{\underline{Q}}$ and exp(px) in Taylor's series around P_i:

$$\hat{\underline{Q}}(p) = \sum_{n=0}^{\infty} \hat{\underline{Q}}^{(n)}(p_i)\frac{(p-p_i)^n}{n!} \; ; \; \hat{\underline{Q}}^{(n)}(p) = \frac{d^n\hat{\underline{Q}}}{dp^n} \tag{89}$$

$$e^{px} = e^{p_i x} \sum_{r=0}^{\infty} \frac{(p-p_i)^r}{r!} \tag{90}$$

Thus, equation (86) becomes

$$\mathrm{Res}(p_i) = e^{p_i x} \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{b_{s,i} \hat{\underline{Q}}^{(n)}(p_i)}{n!r!} \left\{ \frac{1}{j2\Pi} \int_{(p_i)} (p-p_i)^{n+r+s-r_i} dp \right\} \tag{91}$$

The residue is the coefficient of $(p - p_i)^{-1}$; that is, it is that part of the summation corresponding to

$$n + r + s = r_i - 1 \tag{91a}$$

It is clear that neither the sum, - nor any of the individual values, - of the summation indices can exceed $(r_i - 1)$. For a given value of n, r cannot exceed the value $(r_i - 1) - n$. Finally, s is constrained by the condition stipulated in equation (91a). Thus, equation (91) reduces to

$$\mathrm{Res}(p_i) = \sum_{n=0}^{r_i-1} \sum_{r=0}^{(r_i-1)-n} \frac{b_{s,i} \hat{\underline{Q}}^{(n)}(p_i)}{n!r!} e^{p_i x} \tag{92}$$

where

$$s = (r_i-1) - (n+r) \tag{93}$$

To compute the residue at $(-p_i)$, write

$$\phi_-(p;p_i) = (p-p_i)^{r_i} \prod_{\substack{k=1\\(i)}}^{m} (p^2-p_k{}^2)^{r_k} \tag{94}$$

We must now expand

$$\frac{\hat{\underline{Q}}\, e^{px}}{\phi_-(p;p_i)}$$

in a Taylor's series around $(-p_i)$. We have

$$\hat{\underline{Q}}(p) = \sum_{n=0}^{\infty} \hat{\underline{Q}}^{(n)}(-p_i)\, \frac{(p+p_i)^n}{n!} \tag{95}$$

where, recalling that $\hat{\underline{Q}}$ is an even function of p:

$$\hat{\underline{Q}}^{(n)}(-p_i) = (-1)^n \hat{\underline{Q}}^{(n)}(p_i) \tag{96}$$

Next,

$$e^{px} = e^{-p_i x} \sum_{r=0}^{\infty} \frac{(p+p_i)^r x^r}{r!} \tag{97}$$

Finally, consider

$$\begin{aligned} \phi_+(-p;p_i) &= (-p+p_i)^{r_i} \prod_{\substack{k=1\\(i)}}^{m} (p^2-p_k{}^2)^{r_k} \\ &= (-1)^{r_i} (p-p_i)^{r_i} \prod_{\substack{k=1\\(i)}}^{m} (p^2-p_k{}^2)^{r_k} \qquad (98) \\ &= (-1)^{r_i}\, \phi_-(p;p_i) \end{aligned}$$

But according to equation (88),

$$\left[\phi_+(-p;p_i)\right]^{-1} = \sum_{s=0}^{\infty} (-1)^s \, b_{s,i} (p+p_i)^s \;;$$

that is,

$$\left[\phi_-(p;p_i)\right]^{-1} = (-1)^{r_i} \sum_{s=0}^{\infty} (-1)^s \, b_{s,i} (p+p_i)^s \qquad (99)$$

Inspection of equations (95) to (99) reveals that, except for the exponential factors in equations (90) and (97), the residues at $\pm p_i$ differ only by the additional factor

$$(-1)^{r_i+s+n} = (-)^{2r_i-r-1} = (-1)^{r+1}$$

Thus we have

$$\mathrm{Res}(p_i) + \mathrm{Res}(-p_i) = \sum_{n=0}^{r_i-1} \sum_{r=0}^{(r_i-1)-n} \frac{b_{s,i}\hat{\underline{Q}}^{(n)}(p_i)}{n!r!} \left[e^{p_i x} + (-1)^{r+1} e^{-p_i x}\right]$$

and, finally,

$$\underline{T} = \sum_{i=1}^{m} \sum_{n=0}^{r_i-1} \sum_{r=0}^{(r_i-1)-n} \frac{b_{s,i}\hat{\underline{Q}}^{(n)}(p_i)}{n!r!} \left[e^{p_i x} + (-1)^{r+1} e^{-p_i x}\right] \qquad (100)$$

where

$$s = (r_i - 1) - (n+r)$$

This complicated result is the most general form for the fundamental quantity, $\underline{T}(x)$, characterizing the behavior of a uniform multiconductor transmission line. The reader is reminded that this function only constitutes a basis for the final determination of the response. From it are obtained its transpose, $\underline{T}^T(x)$, and the first derivatives, -- $\underline{T}'(x)$ and $\underline{T}^{T'}(x)$, -- of both of these. Finally, all four of these are convolved with the impressed equivalent series and shunt sources, $\underline{E}^e(x)$ and $\underline{H}^e(x)$, and the results used in equations (26) and the auxiliary equations (27) to determine line potentials and currents at any point, -- including, in particular, the line terminals.

Equation (100) reduces easily to the special case of non-degeneracy (equation (62)).

3. Discussion

The procedure developed in Section 2 affords a general means for evaluating the response of a uniform multi-wire transmission line to an impressed field, regardless of the nature of the line eigenvalues, provided the line parameters are known or can be estimated. The general result is very complicated; even in the simplest case of a completely degenerate system, it is well-understood that, practically, numerical methods must be used if the number of conductors (exclusive of reference) exceeds two or three.

When the eigenvalues are all nearly equal in magnitude, considerable simplification is afforded by application of the "average pole" method of Section 2.3 to obtain an approximate result. The result is in the form of an infinite series whose convergence rate depends on the spread in eigenvalue magnitudes. When applicable, the prescribed procedure is not only simpler than the general form; provided a sufficient number of terms in the series is used, it should prove more accurate as well, inasmuch as the characteristic matrix of such a system is nearly singular, and approaches singularity as two or more of the eigenvalues approach equality.

Although the subject has not been discussed in this report, it should be clear that if eigenvalues are grouped around more than one value, the method can be extended by assuming average pole-pairs for each group.

Although the multiple-eigenvalue result for $\underline{T}(x)$, equation (100), is applicable generally, it may be worth while to develop separate programs for computing line response, depending on the nature of the eigenvalues. Thus, in a given numerical problem, an early step in computation consists of determining the eigenvalues. A judgment should then be made to fit the eigenvalue set into one of the following categories:

(1) eigenvalues all equal

(2) eigenvalues all different

(3) eigenvalues all nearly equal

(4) one or more groups of equal eigenvalues

(5) one or more groups of nearly equal eigenvalues.

Based on this judgment, an appropriate computation program may then be selected.

4. Conclusion

A general procedure for evaluating the single-frequency response of a uniform multi-wire line to distributed excitation has been described. The method is applicable regardless of the nature of the line eigenvalues. Mode degeneracy is handled in a routine (but complicated) manner. However, excessive complexity and computational error associated with nearly degenerate modes is avoided by expanding the Laplace transform of the response in Laurent series around a pair of average poles.

5. Acknowledgment

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APPENDIX

A. Development of Formal General Solution.

In equations (21) and (24) write

$$\underline{T}'(x) = \frac{d\underline{T}}{dx}$$

$$\underline{T}^{T'}(x) = \frac{d\underline{T}^T}{dx}$$

and use the revised notation for terminal voltage and current:

$$\left.\begin{aligned} \underline{V}(x) &= \underline{T}'(x)\underline{V}^i - \underline{T}(x)\underline{\zeta}\underline{I}^i + \underline{T}'(x) * \underline{E}^e(x) - \underline{T}(x) * \underline{\zeta}\underline{H}^e(x) \\ \underline{I}(x) &= \underline{T}^{T'}(x)\underline{I}^i - \underline{T}^T(x)\underline{\eta}\underline{V}^i + \underline{T}^{T'}(x) * \underline{H}^e(x) - \underline{T}^T(x) * \underline{\eta}\underline{E}^e(x) \end{aligned}\right\} \quad \text{(A-1)}$$

Then for $x = \ell$ we have

$$\left.\begin{aligned} \underline{V}^0 &= \underline{T}'(\ell)\underline{V}^i - \underline{T}(\ell)\underline{\zeta}\underline{I}^i + \underline{T}'(\ell) * \underline{E}^e(\ell) - \underline{T}(\ell) * \underline{\zeta}\underline{H}^e(\ell) \\ \underline{I}^0 &= \underline{T}^{T'}(\ell)\underline{I}^i - \underline{T}^T(\ell)\underline{\eta}\underline{V}^i + \underline{T}^{T'}(\ell) * \underline{H}^e(\ell) - \underline{T}^T(\ell) * \underline{\eta}\underline{E}^e(\ell) \end{aligned}\right\} \quad \text{(A-2)}$$

Substitute the first of equations (25) for $\underline{I}^i$:

$$\left.\begin{aligned} \underline{V}^0 &= \underline{T}'(\ell)\underline{V}^i + \underline{T}(\ell)\underline{\zeta}\underline{Y}^i\underline{V}^i + \underline{T}'(\ell) * \underline{E}^e(\ell) - \underline{T}(\ell) * \underline{\zeta}\underline{H}^e(\ell) \\ \underline{I}^0 &= -\underline{T}^{T'}(\ell)\underline{Y}^i\underline{V}^i - \underline{T}^T(\ell)\underline{\eta}\underline{V}^i + \underline{T}^{T'}(\ell) * \underline{H}^e(\ell) - \underline{T}^T(\ell) * \underline{\eta}\underline{E}^e(\ell) \end{aligned}\right\} \quad \text{(A-3)}$$

Combine these in the second of equations (25) and reduce:

$$\underline{S}_0\underline{V}^i - \underline{K}_0(\ell) = 0 \quad \text{(A-4)}$$

where

$$\underline{S}_0 = \left[\underline{T}^{T'}(\ell)\underline{Y}^i + \underline{T}^T(\ell)\underline{\eta}\right] + \underline{Y}^0\left[\underline{T}'(\ell) + \underline{T}(\ell)\underline{\zeta}\underline{Y}^i\right] \quad \text{(A-5)}$$

$$\underline{K}_0(\ell) = \left[\underline{T}^{T'}(\ell) * \underline{H}^e(\ell) - \underline{T}^T(\ell) * \underline{\eta}\underline{E}^e(\ell)\right]$$

$$- \underline{Y}^0\left[\underline{T}'(\ell) * \underline{E}^e(\ell) - \underline{T}(\ell) * \underline{\zeta}\underline{H}^e(\ell)\right] \tag{A-6}$$

From equation (A-4),

$$\underline{V}^i = \underline{S}_0^{-1}\underline{K}_0(\ell) \tag{A-7}$$

while, with the help of the first of equations (25), we get

$$\underline{I}^i = -\underline{Y}^i\underline{S}_0^{-1}\underline{K}_0(\ell) \tag{A-8}$$

Using equations (A-7,A-8) in equations (21) and (24) of the main text,

$$\left.\begin{aligned} \underline{V}(x) &= \left[\underline{T}'(x) + \underline{T}(x)\underline{\zeta}\underline{Y}^i\right] \underline{S}_0^{-1}\underline{K}_0(\ell) + \underline{U}(x) \\ \underline{I}(x) &= -\left[\underline{T}^{T'}(x)\underline{Y}^i + \underline{T}^T(x)\underline{\eta}\right]\underline{S}_0^{-1}\underline{K}_0(\ell) + \underline{W}(x) \end{aligned}\right\} \tag{A-9}$$

where

$$\left.\begin{aligned} \underline{U}(x) &= \underline{T}'(x) * \underline{E}^e(x) - \underline{T}(x) * \zeta \underline{H}^e(x) \\ \underline{W}(x) &= \underline{T}^{T'}(x) * \underline{H}^e(x) - \underline{T}^T(x) * \eta \underline{E}^e(x) \end{aligned}\right\} \tag{A-10}$$

B. Recursion Formula for S_m.

According to the development leading to equation (66), the S_m, ($m = 0,1,2,\ldots$) are related to the δ_i ($i = 1,\ldots,N$) through

$$\left\{ \prod_{i=1}^{N} \left(1 - \frac{\delta_i}{p-p_a} \right) \right\}^{-1} = \sum_{m=0}^{\infty} \frac{S_m}{(p-p_a)^m} \tag{B-1}$$

Write

$$u = (p - p_a)^{-1}$$

and multiply both sides of equation (B-1) by the multiple product:

$$1 = \left[\prod_{i=1}^{N} (1 - \delta_i u) \right] \left[\sum_{m=0}^{\infty} S_m u^m \right] \tag{B-2}$$

The multiple product may be expanded into a polynomial in u:

$$\prod_{i=1}^{N} (1 - \delta_i u) = \sum_{r=0}^{N} (-1)^r a_r u^r; \quad a_0 = 1 \tag{B-3}$$

where

$$a_r = \sum_{i_1=1}^{(N-r+1)} \sum_{i_2=i_1+1}^{(N-r+2)} \cdots \sum_{i_r=i_{r+1}+1}^{N} \left(\prod_{j=1}^{r} \delta_{i_j} \right), \quad r=1,2,3,\ldots \tag{B-4}$$

In particular,

$$
\left.
\begin{aligned}
a_1 &= \sum_{i=1}^{N} \delta_i \\
a_2 &= \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \delta_i \delta_j \\
a_3 &= \sum_{i=1}^{N-2} \sum_{j=i+1}^{N-1} \sum_{k=j+1}^{N} \delta_i \delta_j \delta_k \\
&\cdots \\
a_N &= \sum_{i_1=1}^{1} \sum_{i_2=2}^{2} \cdots \sum_{i_N=N}^{N} \left(\prod_{j=1}^{N} \delta_{i_j} \right) = \prod_{i=1}^{N} \delta_i
\end{aligned}
\right\} \quad \text{(B-5)}
$$

Then equation (B-3) in (B-2) yields

$$\sum_{r=0}^{N} (-1) a_r u^r \sum_{m=0}^{\infty} S_m u^m = \sum_{r=0}^{N} \sum_{m=0}^{\infty} (-1)^r a_r S_m u^{m+r} = 1 \qquad \text{(B-6)}$$

Collecting coefficients of like powers of u and solving for the successive S_m:

$$S_0 = 1$$

$$S_1 = a_1 = \sum_{i=1}^{N} \delta_i = 0$$

as used in the main text. Continuing,

$$S_2 = a_1 S_1 - S_0 a_2,$$

etc. In general,

$$\left. \begin{aligned} S_m &= \sum_{k=1}^{m} (-1)^{k-1} a_k S_{m-k}, \quad m \leqq N \\ &= \sum_{k=1}^{N} (-1)^{k-1} a_k S_{m-k}, \quad m \geqq N \end{aligned} \right\} \qquad \text{(B-7)}$$

REFERENCES

1. S.A. Schelkunoff, Electromagnetic Waves, Van Nostrand, New York, 1943, p. 235.

2. J.R. Carson and R.S. Hoyt, Propagation of Periodic Currents Over a System of Parallel Wires, Bell System Technical Journal, V. 6, pp. 495-545, 1927.

3. S. Frankel, Response of a Multiconductor Transmission Line to Excitation by an Arbitrary Monochromatic Impressed Field Along the Line, Sandia Laboratories Contract Report SC-CR-71 5076, April 1971. Also AFWL Interaction Note No. 80, April 1971.

4. C.D. Taylor, R.S. Satterwhite, and C.W. Harrison, Jr., The Response of a Terminated Two-Wire Transmission Line Excited by a Non-Uniform Electromagnetic Field, Sandia Laboratories Report SC-R-65-978A, November, 1965. Interaction Note 66.

5. S. Frankel, Externally Excited Transmission Line: Definition of Procedures for Determining Coupling Parameters, Harry Diamond Laboratories Report No. HDL-TM-72-11, May 1972. Interaction Note 121.

6. S. Frankel, Field Coupling Parameters for a Single Round Wire Close to a Ground Plane or Two Large Round Wires in Free Space, Harry Diamond Laboratories Report No. HDL-TM-72-14, April 1972. Interaction Note 120.

7. Akio Matsumoto, Editor, Microwave Filters and Circuits, Academic Press, New York, 1970.

8. D.F. Strawe, Analysis of Uniform Multiwire Transmission Lines, Boeing Document D2-26088-1, October 1972.

9. D.F. Strawe, Analysis of Uniform Symmetric Transmission Lines, Boeing Document D2-19734-1, August 1971.

10. S. Frankel, Response of a Multiconductor Cable to Excitation by an Arbitrary Single-Frequency Constant-Impedance Source, Sandia Laboratories Contract Report SC-CR-71 5058, April 1971. Also AFWL Interaction Note No. 79, April 1971.

11. S. Frankel, Quasi-TEM Transmission Line Theory, Sidney Frankel & Associates Report No. FA-196. Interaction Note 135.

REFERENCES (cont'd)

12. S. Frankel, TEM Response of a Multiwire Transmission Line (Cable) to an Externally Impressed Electromagnetic Field: Recipe for Analysis, Harry Diamond Laboratories Technical Memorandum For File No. FA-181, February 1972. Also AFWL Interaction Note No. 130, February 1972.

13. H. Amemiya, Time-Domain Analysis of Multiple Parallel Transmission Lines, RCA Review, V. 28, pp. 241-276, June 1967.

14. V.N. Faadeeva, Computational Methods of Linear Algebra, Dover Publications, New York, 1959, pp.105ff.

15. H. Guckel, and Y.Y.Sun, Uniform Multimode Transmission Lines, IEEE Transactions on Microwave Theory and Techniques, Vol. MTT-20, No. 6, June 1972, pp. 412-413.

16. M. Abramowitz and I.A. Stegun, Editors, Handbook of Mathematical Functions, National Bureau of Standards Applied Mathematics Series 55, June 1964, Ch. 29.