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QUASI-TEM TRANSMISSION LINE THEORY

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ABSTRACT

If a multi-wire cable or transmission line is embedded in a non-homogeneous or mixed dielectric, or if its conductor losses are taken into account, it turns out, generally, that more than one propagation parameter is required to describe the response of the line to an externally-applied field. With waves travelling generally in both directions on the cable, the series resistances representing the power loss in the various conductors become functions of position along the line, the nature of the functions themselves varying with the nature of the line terminations.

The difficulty can, in principle, be eliminated by resolving the line dynamic quantities into biorthogonal mode sets. Criteria for establishing such mode sets are presented.

For the homogeneous dielectric case, complexity of analysis can be reduced if one is willing to accept some error in the values of series impedance assigned to the various conductors and to make use of a reasonable approximation that becomes consequently possible. In view of the intended application of the analysis, namely, an evaluation of external interference effects, the resulting errors must be considered as of small significance. The solution thus obtained is only slightly more complicated than that previously obtained for lossless lines.

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1. INTRODUCTION

When a number of lossless cylindrical conductors constituting a multiwire transmission line, or cable, is embedded in a homogeneous, isotropic (but not necessarily lossless) dielectric, energy is transmitted in either direction with a single propagation constant. When the line is excited at one end and match-terminated at the other, the relative amplitudes of potentials and currents on the conductors are the same at every transverse plane, diminishing in absolute amplitude in the propagation direction in the case of a lossy dielectric. This simple model leads to relatively easy analytical procedures for determining the dynamic response of such lines.

If the conductors are in a non-homogeneous dielectric, or if they are lossy, transmission is no longer in terms of a single propagation constant. For a line of N conductors plus reference (or ground), i.e., an N-line, there are generally N modes of propagation in each direction. Each of the N conductors carries N modes, making N^2 components in all. However, for each mode, the ratios of all voltages are determined by the mode eigenvalues, so that only one constant has to be determined for each mode in each direction, i.e., 2N constants altogether, as in the strictly TEM case.

If we postulate a system in which the conductors are lossless, then no conceptual barriers arise, even in the presence of partial mode degeneracy. However, practical difficulties may limit the generality of the class of problems that can be solved with accuracy. On the other hand, if we are dealing with low-loss conductors in a homogeneous dielectric, practical analytical considerations give rise to certain questions which must be clarified to justify the procedures used. Thus, in such a system it is customary, except in the simplest cases, to find the field and boundary distributions in the limiting, conductor-lossless case, and then to determine the conductor losses by allowing the conductor current density to be independent of small values of conductor resistivity.

However, if we assume, even temporarily, that the conductors are lossless, we are back to the pure-TEM, single-propagation-mode case. It would appear that the multi-mode approach is not useable in this procedure. On the contrary, it can be useful to separate the line dynamic quantities into sets of bi-orthogonal modes in order to obtain a closed solution to the line excitation problem. Furthermore, subject to certain constraints, it is possible to construct a non-denumerable infinity of orthogonal mode sets for a given line configuration; in other words, the resolution into bi-orthogonal mode sets is not unique.

The background information for the bi-orthogonal formulation is reviewed in section 2, and the formulation itself discussed briefly in section 3.1 in connection with end-excited lines. When such mode sets can be found, the conductor losses (and therefore the attenuation of all line components) may be determined by finding the losses for the individual modes, and incorporating them into these modes as independent attenuation constants.

The problem that next presents itself is the actual determination of a set of modes for any particular case. While this is a relatively simple matter in the case of a 2-, 3-, or 4-line, it remains to be seen whether such a determination is practical in the case, for example, of a cable consisting of one hundred conductors of various sizes in a uniform dielectric.

An alternative is to abandon the multi-mode approach and to employ an approximation, or a succession of approximations, for determining the line attenuation. In view of the generally inexact nature of the phenomenological studies toward which the present study is directed, such an approach might well represent a suitable compromise between quality and cost of analysis. An approximate method which deliberately degenerates the multiple-mode status of the line to an infinite-series expansion, of completely degenerate modes of equal eigenvalues but differing multiplicities is discussed in section 3.2.

In either case, attenuation constants may be determined by minor adaptation of the same electrostatic analytical procedures that yield the lines coefficients of capacitance and inductance.

2. MULTI-MODE METHOD: EIGENVALUES AND EIGENVECTORS

Non-trivial solutions of the homogeneous line differentail equations imply a finite number, -at most, one less than the total number of conductors,- of modes of TEM or quasi-Tem propagation. In this section we outline procedures for determining the eigenvalues of such modes, under certain restricted circumstances, and necessary conditions that their components must satisfy. Special problems arising when mode degeneracy occurs are treated. Simplifying procedures for approximate solutions in the likely case that the eigenvalues are nearly equal are reserved for section 3.2.

2.1 DIFFERENTIAL EQUATIONS OF A QUASI-TEM LINE: PROPAGATION MODES

When a set of N cylindrical lossless conductors plus reference conductor are embedded in a homogeneous, isotropic, but not necessarily lossless, dielectric, the system can constitute a guide transmitting electromagnetic waves in a pure TEM mode. If the axis of such an N-line is parallel to the x-axis of a rectangular coordinate system, we have the following homogeneous differential equations for the system (Reference (9):

$$\left.\begin{aligned} \frac{d\underline{V}}{dx} + \underline{\zeta}\ \underline{I} &= \underline{0} \\ \frac{d\underline{I}}{dx} + \underline{\eta}\ \underline{V} &= \underline{0} \end{aligned}\right\} \quad (1)$$

where

$$\underline{V}^T = [V_1, \ldots, V_N] \quad (2a)$$

$$\underline{I}^T = [I_1, \ldots, I_N] \quad (2b)$$

$$\left.\begin{aligned} \underline{\zeta} &= [\zeta_{ij}],\ i,j = 1, \ldots, N \\ &= j\omega\ [L_{ij}] = j\omega\underline{L} \end{aligned}\right\} (2c)$$

$$\left.\begin{aligned} \underline{\eta} &= [\eta_{ij}],\ i,\ j = 1, \ldots, N \\ &= [G_{ij} + j\omega C_{ij}] = \underline{G} + j\omega\underline{C} \end{aligned}\right\} (2d)$$

$\underline{V}^T$ and $\underline{I}^T$ are transposes of $\underline{V}$ and $\underline{I}$ respectively. I_i is the total current in the i<u>th</u> conductor. V_i is the potential of the i<u>th</u> conductor, with respect to reference, in a transverse plane.

L_{ij} is the coefficient of inductance between the i<u>th</u> and j<u>th</u> conductor, Hy. (Reference 1). $L_{ij} = L_{ji}$

G_{ij} is the coefficient of conductance between the i<u>th</u> and j<u>th</u> conductor, mho. $G_{ij} = G_{ji}$

C_{ij} is the Maxwell coefficient of capacitance between the i<u>th</u> and j<u>th</u> conductor, Fd. (Reference 1,2).

$$C_{ij} = C_{ji}$$

For a homogeneous isotropic dielectric, G_{ij} and C_{ij} are identical with the electrostatic values, and, furthermore,

$$\frac{G_{ij}}{C_{ij}} = \frac{g_d}{\varepsilon_d} \tag{3}$$

where

g_d = dielectric conductivity, mho/meter

ε_d = dielectric permittivity, Fd./meter

When the conductors have a small resistivity, or when the dielectric is inhomogeneous or mixed, a component of electric field exists in the propagation direction; only the magnetic-intensity component of the field is transverse. We call it a quasi-TEM mode since it approaches a pure TEM mode in the limit as conductor resistivity and permittivity differences tend to zero. Inasmuch as the magnetic field is transverse, we can continue to characterize the transverse component of the electric field as the gradient of a potential, so that equations (1) continue to be applicable. However, Laplace's equation in the transverse plane is no longer satisfied, so that the use of electrostatic values for the C_{ij} is only an approximation. Similarly, the L_{ij}, which, in the strict TEM case, can be computed from a knowledge of the C_{ij} and the propagation velocity, deviate from their electrostatic limits. Thus, we continue to use equations (1) with the understanding that determination of the line parameters, $\underline{\zeta}$ and $\underline{\eta}$ constitute a separate problem.

In equations (1), eliminating $\underline{I}$ and $\underline{V}$ successively yields

$$\left.\begin{aligned} \frac{d^2\underline{V}}{dx^2} - \underline{\zeta}\ \underline{\eta}\ \underline{V} = \underline{0} \\ \frac{d^2\underline{I}}{dx^2} - \underline{\eta}\ \underline{\zeta}\ \underline{I} = \underline{0} \end{aligned}\right\} \tag{4}$$

Write

$$\underline{A} = \underline{\zeta}\ \underline{\eta} \tag{5}$$

Then

$$\underline{\eta}\ \underline{\zeta} = (\underline{\zeta}^T \underline{\eta}^T)^T = (\underline{\zeta}\ \underline{\eta})^T = \underline{A}^T \tag{6}$$

since $\underline{\zeta}$ and $\underline{\eta}$ are symmetric matrices.

For a forward wave, assume a solution of equations (4) of the form

$$\left.\begin{aligned} \underline{V} = \underline{V}_o \exp(-\gamma x) \\ \underline{I} = \underline{I}_o \exp(-\gamma x) \end{aligned}\right\} \quad Re(\gamma) \geq 0 \tag{7}$$

Then from equations (4) we get

$$\left.\begin{aligned} (\gamma^2\ \underline{I} - \underline{A})\ \underline{V} = \underline{0} \\ (\gamma^2\ \underline{I} - \underline{A}^T)\ \underline{I} = \underline{0} \end{aligned}\right\} \tag{8}$$

where

$$\begin{aligned} \underline{I} &= \text{unit } N \times N \text{ matrix} \\ &= [\delta_{ij}]\ ,\ i,j = 1,\ldots, N \end{aligned} \tag{9}$$

and δ_{ij} is the Kronecker delta.

Write $\lambda = \gamma^2$. Then non-trivial solutions of the homogeneous equations (8) exist only if (Reference 3)

$$\left.\begin{aligned} &\det.\ (\lambda\underline{I} - \underline{A}) = |\lambda\underline{I} - \underline{A}| = 0 \\ \text{and}\quad &\det.\ (\lambda\underline{I} - \underline{A}^T) = |\lambda\underline{I} - \underline{A}^T| = 0 \end{aligned}\right\} \tag{10}$$

Since $\underline{I}^T$ is the same as $\underline{I}$, and since

$$(\lambda \underline{I}^T - A^T) = (\lambda \underline{I} - \underline{A})^T$$

has the same determinant as $(\lambda \underline{I} - \underline{A})$, both equations (10) have the same solutions for λ. The left members of equations (10) are identical Nth order polynomials in λ.

2.1.1. Propagation Modes When Eigenvalues are all Different; Mode Orthogonality and Independence; Additional Constraints.

Assume that solution of equation (10) yields N distinct eigenvalues, λ_j, $(j = 1, \ldots, N)$. Each λ_j in turn yields a pair of values

$$\gamma_j = \pm \sqrt{\lambda_j} \tag{11}$$

where only the upper sign corresponds to a forward travelling wave. Each γ_j substituted in equations (8) yields a consistent set of equations from which the ratios of currents and voltages may be found. The solution corresponding to the jth eigenvalue, λ_j, yields a voltage vector

$$\underline{V}^{(j)^T} = [V_1^{(j)}, \ldots, V_N^{(j)}] \tag{12a}$$

and a current vector

$$\underline{I}^{(j)^T} = [I_1^{(j)}, \ldots, I_N^{(j)}] \tag{12b}$$

The N voltage eigenvectors are arranged as columns of a square matrix

$$\underline{\underline{V}} = [\underline{V}^{(j)}] = \begin{bmatrix} V_1^{(1)}, \ldots, & V_1^{(N)} \\ \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \\ V_N^{(1)}, \ldots, & V_N^{(N)} \end{bmatrix} = [V_i^{(j)}] \qquad (13a)$$

Similarly, the current eigenvectors are shown as

$$\underline{\underline{I}} = [\underline{I}^{(j)}] = \begin{bmatrix} I_1^{(1)}, \ldots, & I_1^{(N)} \\ \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \\ I_N^{(1)}, \ldots, & I_N^{(N)} \end{bmatrix} = [I_i^{(j)}] \qquad (13b)$$

A standard result in eigenvector theory yields (Reference 4)

$$\underline{\underline{V}}^T \underline{\underline{I}} = \underline{W} = [w_j \, \delta_{ij}] \qquad (14)$$

where w_j are undetermined constants. Equation (14) implies

$$\sum_{i=1}^{N} V_i^{(j)} \, I_i^{(j)} = w_j \,, \; j = 1, \ldots, N \qquad (14a)$$

$$\sum_{i=1}^{N} V_i^{(j)} \, I_i^{(k)} = 0, \; j \neq k \qquad (14b)$$

If the line is excited in the jth mode, the transmitted power is represented by the summation of the potential-current products on all the conductors, i.e., by the summation shown in the left member of equation (14a). Thus, w_j is identified as the power transmitted in the jth mode. If the line is excited generally by all modes, the total transmitted power is

$$W_T = \sum_{i=1}^{N} V_i \; I_i$$

$$= \sum_{i=1}^{N} \left(\sum_{j=1}^{N} V_i^{(j)} \right) \left(\sum_{k=1}^{N} I_i^{(k)} \right)$$

$$= \sum_{j=1}^{N} \sum_{k=1}^{N} \sum_{i=1}^{N} V_i^{(j)} \; I_i^{(k)}$$

$$= \sum_{j=1}^{N} \sum_{i=1}^{N} V_i^{(j)} \; I_i^{(j)}$$

in virtue of equation (14b)

$$= \sum_{j=1}^{N} W_j \tag{15}$$

in virtue of equation (14a).

Two sets of vectors, $\underline{V}$ and $\underline{\underline{I}}$, satisfying equation (14) are said to be bi-orthogonal. Thus, we have the result that when the propagation eigenvalues are distinct, the resulting current and voltage vectors necessarily obey the bi-orthogonality condition.

However, equation (14) is not a sufficient constraint on these modes. If one of the assumed solutions (equations(7)) is substituted in equations (1) we get, respectively,

$$\gamma_j \; \underline{V}^{(j)} = \underline{\zeta}^{(j)} \; \underline{I}^{(j)} \tag{16a}$$

$$\gamma_j \; \underline{I}^{(j)} = \underline{\eta}^{(j)} \; \underline{V}^{(j)} \tag{16b}$$

where we have indicated that $\underline{\zeta}$ and $\underline{\eta}$ may be functions of the mode configuration.

We can write

$$\left[\gamma_j \underline{V}^{(j)}\right] = \left[\gamma_j V_i^{(j)}\right] = \left[V_i^{(j)}\right] \left[\gamma_j \delta_{ij}\right] = \underline{\underline{V}} \, \underline{\gamma} \tag{17a}$$

where

$$\underline{\gamma} = \left[\gamma_j \delta_{ij}\right]$$

Similarly,

$$\left[\gamma_j \underline{I}^{(j)}\right] = \left[\gamma_j I_i^{(j)}\right] = \underline{\underline{I}} \, \underline{\gamma} \tag{17b}$$

For the right members of equations (16a,b) we have, respectively,

$$\underline{\zeta}^{(j)} \underline{I}^{(j)} = \left[\zeta_{ik}^{(j)}\right]_{(j)} \cdot \left[I_k^{(j)}\right]_{(j)} \tag{18a}$$

where the symbol (j) outside the brackets indicates that j is not a position index for its associated matrix. Continuing,

$$\underline{\zeta}^{(j)} \underline{I}^{(j)} = \left[\sum_{k=1}^{N} \zeta_{ik}^{(j)} I_k^{(j)}\right]_{(j)} = \left[E_i^{(j)}\right]_{(j)} = \underline{E}^{(j)} \tag{18b}$$

and

$$\underline{\eta}^{(j)} \underline{V}^{(j)} = \left[\sum_{k=1}^{N} \eta_{ik}^{(j)} V_k^{(j)}\right]_{(j)} = \left[H_i^{(j)}\right]_{(j)} = \underline{H}^{(j)} \tag{18c}$$

Then equations (16a,b) extended to all j become

$$\underline{\underline{V}} \, \underline{\gamma} = \underline{\underline{E}} = [\underline{E}^{(j)}] = [E_i^{(j)}]$$

$$\underline{\underline{I}} \, \underline{\gamma} = \underline{\underline{H}} = [\underline{H}^{(j)}] = [H_i^{(j)}]$$

whence

$$\left.\begin{aligned} \underline{\underline{V}} &= \underline{\underline{E}} \, \underline{\gamma}^{-1} \\ \underline{\underline{I}} &= \underline{\underline{H}} \, \underline{\gamma}^{-1} \end{aligned}\right\} \tag{19}$$

Using these successively in equations (14), and noting that

$$(\underline{Y}^{-1})^T = \underline{Y}^{-1}$$

we get

$$\underline{Y}^{-1}\,\underline{\underline{E}}^T\,\underline{\underline{I}} = \underline{W}$$

and

$$\underline{Y}^{-1}\,\underline{\underline{H}}^T\,\underline{\underline{V}} = \underline{W}^T = \underline{W}$$

In more explicit form, upon making the appropriate substitutions,

$$[\gamma_i^{-1} \sum_{k=1}^{N} \sum_{\ell=1}^{N} \zeta_{k\ell}^{(i)} I_\ell^{(i)} I_k^{(j)}] = [w_i\ \delta_{ij}] \tag{20a}$$

and

$$[\gamma_i^{-1} \sum_{k=1}^{N} \sum_{\ell=1}^{N} \eta_{k\ell}^{(i)} V_\ell^{(i)} V_k^{(j)}] = [w_i\ \delta_{ij}] \tag{20b}$$

These equations may be taken as necessary conditions that a set of modes satisfy the homogeneous differential equations of the line, and be consistent with the power condition equation (14a).

The N vectors of either mate of a pair of bi-orthogonal modes constitute a linearly independent set (App. A). But any (N + 1)st vector is expressible as a linear combination of these. The linearly independent set, $\underline{V}^{(j)}$ (j = 1,..., N) constitutes a (generally non-orthogonal) coordinate system in N-space. The forward-wave voltage for the transmission line is

$$\underline{V} = \sum_{j=1}^{N} \underline{V}^{(j)}(x) = \sum_{j=1}^{N} \underline{V}^{(j)}(0)\ \exp(-\gamma_j x) \tag{21a}$$

$$= [V_i^{(j)}(0)]\ [\exp(-\gamma_j x)] \tag{21b}$$

where

$$[V_i^{(j)}(0)] = \begin{bmatrix} V_1^{(1)}(0),\ldots, & V_1^{(N)}(0) \\ \cdot \quad \cdot & \cdot \\ V_N^{(1)}(0),\ldots, & V_N^{(N)}(0) \end{bmatrix} = \underline{\underline{V}}(0) \qquad (22)$$

and

$$[\exp(-\gamma_j x)]^T = [\exp(-\gamma_1 x),\ldots, \exp(-\gamma_N x)] \qquad (23)$$

Similarly,

$$\underline{I} = \sum_{j=1}^{N} \underline{I}^{(j)}(x) = \sum_{j=1}^{N} \underline{I}^{(j)}(0) \exp(-\gamma_j x) \qquad (24a)$$

$$= [I_i^{(j)}(0)][\exp(-\gamma_j x)] \qquad (24b)$$

where

$$[I_i^{(j)}(0)] = \begin{bmatrix} I_1^{(1)}(0),\ldots, & I_1^{(N)}(0) \\ \cdot \quad \cdot & \cdot \\ I_N^{(1)}(0),\ldots, & I_N^{(N)}(0) \end{bmatrix} = \underline{\underline{I}}(0) \qquad (25)$$

2.1.2. Quasi-TEM Mode; Eigenvalues not all Distinct

Consider the case where (N - 2) of the λ_j are distinct, while the remaining two are coincident. Without loss of generality let λ_j (j = 1,..., N - 2) be distinct, while $\lambda_N = \lambda_{N-1} \neq \lambda_j$ (j = 1,..., N - 2). There are now only (N-1) different voltage vectors, $\underline{V}^{(j)}$. For every λ_j, (j < N-1), the matrix $\lambda \underline{I} - \underline{A}$ **(26)** is of rank (N - 1) (Ref. 3). Consequently, each of the vectors, $\underline{V}^{(j)}$, (j = 1,..., N - 2) is specified within a scalar constant, as in previous case for all vectors (sec. 2.1.1). On the other hand, corresponding to the double root, λ_{N-1}, the matrix (26) is of rank (N-2), (App. B), whence $\underline{V}^{(N-1)}$ is unspecified within two constants. Two components of $\underline{V}^{(N-1)}$ may be specified arbitrarily, and two equations removed from the set, provided the

the determinant of the diminished set is different from zero. The remaining components are then uniquely determined in terms of the two chosen arbitrarily. In geometric interpretation, we are free to choose both amplitude and direction of the projection of $\underline{V}^{(N-1)}$ on an appropriate 2-space, after which, amplitude and direction of all remaining components are fixed.

Similarly, if the first (N - 3) eigenvalues are distinct, while $\lambda_N = \lambda_{N-1} = \lambda_{N-2} \neq \lambda_j (j = 1, \ldots, N - 3)$, the matrix (26) is of rank (N - 3). Three components may be specified arbitrarily, whence the remainder are fixed. The projection of $\underline{V}^{(N-2)}$ on an appropriate 3-space may take any amplitude and direction in that space, whence the remaining components are fixed.

In general, if the first (N - r) eigenvalues are distinct, while

$$\lambda_N = \lambda_{N-1} = \ldots = \lambda_{N-r+1} \neq \lambda_j (j = 1, \ldots, N - r)$$

the matrix (26) is of rank (N - r), $(r \geq 2)$. The projection of $\underline{V}^{(j)}$ $(N - r < j \leq N)$ on an appropriate r-space may be specified arbitrarily in amplitude and direction, after which the remaining components are fixed.

Finally, if all eigenvalues are equal, there is only one eigenmode, and its amplitude and direction in N-space are completely arbitrary.

Translated into multiconductor line terminology, these results are interpreted as follows:

The component, $V_i^{(j)}$, is the value of the j<u>th</u> mode potential on the ith conductor. When the eigenvalues are all different, one of the conductor potentials may be specified arbitrarily to satisfy terminal conditions, whence the remaining components of the jth mode are fixed. Since this holds true for all N modes, we have a total of N arbitrary values available for terminal excitation conditions. An equal number of arbitrary values is available by virtue of a back-travelling wave, making 2N values in all.

When (N-r) values are distinct ($r \geq 2$), there are (N-r+1) different eigenmodes. For (N-r) of these, one arbitrary terminal assignment is available, while for the remaining one, r values may be assigned, again making N arbitrary terminal values in all for a forward wave. Finally, when r = N, corresponding to a pure TEM mode, there is a single eigenmode associated with N arbitrary terminal conditions for a wave in one direction.

Since, in the general case, the rank of the characteristic matrix is (N-r), it follows from the argument in part (b) of Appendix A that the (N-r+1) eigenvectors are linearly independent.

The whole foregoing discussion applies, with appropriate adjustment in terminology, to the current modes, $\underline{I}^{(j)}$. Furthermore, equations (14b) continue to apply to the modes corresponding to simple zeros of the characteristic equation, i.e., to the first (N-r) modes in the foregoing discussion. As for the modes corresponding to multiple zeros, we have shown, at least in the case of a single multiple root of multiplicity, M, that the projection of the vector on an appropriate M-space may be assigned any set of arbitrarily chosen components. Now, provided that it is practical to do so, it turns out to be sometimes useful, as discussed in section 2.2.1, to choose these M components of potential and of current in such a way that potential and current vectors constitute a bi-orthogonal set satisfying equations (20a, b). If $\underline{V}^{(j)}$ and $\underline{I}^{(j)}$ $(j = 1, \ldots, N)$ are, in fact, bi-orthogonal, they must satisfy equation (14). In addition, any solution must satisfy equations (19). But these two sets of equations were used to obtain equations (20a, b), which therefore, remain valid in the degenerate case. (Note that $\underline{\gamma}$ remains non-singular under degeneracy conditions).

If the characteristic equation has more than one multiple root, the previous arguments are extended to each of the degenerate eigenmodes in turn.

In equation (20a), for $i \neq j$, we have

$$\gamma_i^{-1} \sum_{k=1}^{N} \sum_{\ell=1}^{N} \zeta_{k\ell}^{(i)} I_\ell^{(i)} I_k^{(j)} = 0 \ , \ i \neq j$$

or, since $\gamma_i \neq 0$ for $\omega \neq 0$

$$\sum_{k=1}^{N} \sum_{\ell=1}^{N} \zeta_{k\ell}^{(i)} I_\ell^{(i)} I_k^{(j)} = 0 \ , \ i \neq j$$

The true number of independent parameters involved in this set of equations is exhibited more clearly by normalizing the various parameters in some sense.

Write

$$\left.\begin{aligned} s_{k\ell}^{(i)} &= \frac{\zeta_{k\ell}^{(i)}}{\zeta_{mm}^{(i)}} \; ; \; s_{mm}^{(i)} = 1 \\ x_{\ell}^{(i)} &= \frac{I_{\ell}^{(i)}}{I_{m}^{(i)}} \; , \text{ etc; } x_{m}^{(i)} = 1 \end{aligned}\right\} i,k,\ell = 1,\ldots, N \tag{27a}$$

where m is chosen such that $I_m^{(i)} \neq 0$. Then we get

$$\sum_{k=1}^{N} \sum_{\ell=1}^{N} s_{k\ell}^{(i)} x_{\ell}^{(i)} x_{k}^{(j)} = 0, \; i \neq j \tag{27b}$$

Similarly, from equation (20b)

$$\sum_{k=1}^{N} \sum_{\ell=1}^{N} t_{k\ell}^{(i)} y_{\ell}^{(i)} y_{k}^{(j)} = 0, \; i \neq j \tag{27c}$$

where

$$\left.\begin{aligned} t_{k\ell}^{(i)} &= \frac{\eta_{k\ell}^{(i)}}{\eta_{mm}^{(i)}} \; ; \; t_{mm}^{(i)} = 1 \\ y_{\ell}^{(i)} &= \frac{V_{\ell}^{(i)}}{V_{m}^{(i)}} \; , \text{ etc.; } y_{m}^{(i)} = 1 \end{aligned}\right\} i,k,\ell = 1,\ldots, N \tag{27d}$$

The proof, in appendix A, that the vectors of a bi-orthogonal set are linearly independent is not a function of any conditions on the eigenvalues. It follows that any bi-orthogonal set of current or voltage vectors is linearly independent, and therefore adequate for supplying the number of degrees of freedom required to satisfy line terminal conditions.

Application of the multimode formulation to the behavior of a 2-line is illustrated in appendix C.

2.2. Line of Lossy Conductors in a Homogeneous, Isotropic Dielectric.

We review, first, the lossless conductor case. Propagation is in the pure TEM mode, and equations (2c, d) apply. We have

$$\underline{A} = \underline{\zeta}\ \underline{\eta} = j\omega\ \underline{L}\ (\underline{G} + j\omega\ \underline{C}) \tag{28}$$

where

$$\underline{L} = [L_{ij}],\ \underline{G} = [G_{ij}],\ \underline{C} = [C_{ij}] \tag{29}$$

Using equation (3) in equation (28),

$$\underline{A} = j\omega \left(\frac{g_d}{\varepsilon_d} + j\omega \right) \underline{L}\ \underline{C} \tag{30}$$

Furthermore, for the pure TEM case,

$$\underline{L}\ \underline{C} = \mu_d\ \varepsilon_d\ \underline{I} = \frac{1}{v_d^2}\ \underline{I} \tag{31}$$

whence

$$\underline{A} = \left(-\frac{\omega^2}{v_d^2} + j\omega\mu_d g_d \right) \underline{I}$$

$$= (- k_o^2 + j\omega\mu_d g_d)\ \underline{I} \tag{32}$$

where $k_o = \frac{\omega}{v_d}$ (33)

The characteristic equation is

$$\text{det. } [(\lambda + k_o^2 - j\omega\mu_d g_d)\ \delta_{ij}] = (\lambda + k_o^2 - j\omega\mu_d g_d)^N = 0 \tag{34}$$

yielding the single root of multiplicity N:

$$\lambda = \gamma^2 = - k_o^2 + j\omega\mu_d g_d$$

or

$$\gamma = \pm jk_o \left[1 + \frac{g_d}{j\omega\varepsilon_d} \right]^{1/2}$$

$$= \pm jk_o \; (1 - j \tan \delta_d)^{1/2} \tag{35}$$

where

$$\tan \delta_d = \frac{g_d}{\omega\varepsilon_d} \tag{36}$$

is the loss tangent of the dielectric. δ_d is usually small enough so that for practical purposes,

$$(1 - j \tan \delta_d)^{1/2} \approx (1 - j \delta_d)^{1/2} \approx (1 + \tfrac{1}{8} \delta_d^2) - j(1/2)\delta_d \approx 1 - j(1/2)\delta_d \tag{37}$$

If we write

$$\gamma = \alpha + j\beta$$

we have, for a forward wave

$$\gamma \approx jk_o \; [1 - j \tfrac{1}{2} \delta_d]$$

so that

$$\alpha \approx \tfrac{1}{2} \delta_d \, k_o = \tfrac{1}{2} g_d \eta_d \tag{38}$$

where

$$\eta_d = \sqrt{\frac{\mu_d}{\varepsilon_d}} \tag{39}$$

= dielectric wave impedance in the absence of losses.

Other details for this case are available in reference 5. Although there is a small modification required to the $\underline{Z}$ and $\underline{Y}$ matrices of the lossless case, it is, for practical purposes, ordinarily permissible to ignore the change in these parameters, modifying the lossless case only to the extent that

$$\gamma = j\beta$$

is replaced by

$$\gamma = \alpha + j\beta$$

When conductor conductivity is finite, this simple picture is, in principle, drastically modified. Whereas, in the previous case, the relative conductor potentials had no effect on the attenuation constant (which measures the relative loss on the line), this is not true of conductor losses, which are a function of current distribution around the conductor peripheries. These distributions, in turn, depend on the nature of the line excitations and terminations.

It is customary, ordinarily, to determine the conductor current distributions by first assuming the conductors to be lossless, finding the peripheral and axial distributions, and then, assuming that these distributions are modified negligibly when the conductivities are made finite, calculating the loss rate at each point along the line. This approach generally introduces a number of difficulties. For example, if we are dealing with the problem of finding the currents and potentials on the exteriors of a number of closely-packed cable shields, with terminations which are either short-circuits to ground or open circuits, the stated procedure yields infinite resonances at isolated frequencies which happen to be most important for solution of the physical problem. Furthermore, relative potentials vary along the line, generally by virtue both of differing losses on the various conductors and of terminal reflections, so that behavior must be determined point by point along the line, with points separated by distances much smaller than the smallest wavelength of interest. In that case, much of the advantage of distributed line theory is lost.

If great accuracy is not of importance, this problem may be circumvented by assuming uniform current distributions around conductor peripheries, regardless of relative conductor potentials. For typical conductor proximity in multi-wire cable, this writer has estimated that this yields an estimate of conductor resistance too low probably by something of the order of fifteen percent, at most. Calculations of interference based on such an assumption would therefore yield results on the conservative side, but considering the very approximate nature of the assumed excitation forces, conservatism of that magnitude can hardly be considered excessive. Analysis of line response based on pre-assumed values of conductor resistance and internal inductance is dealt with in section 3.2. In the section immediately following this one, we discuss the use of bi-orthogonal modes for finding the line response without the need for pre-assumed current distributions.

2.2.1. Conductor Losses Produced by Bi-orthogonal Mode Currents

Once again we note that equations (27b, c) are independent of the associated propagation constants. A priori, the possibility that two modes may be bi-orthogonal, is not excluded merely because they have identical propagation constants.

Assume, again, that the line voltage and current vectors have been resolved into bi-orthogonal mode sets in accordance with equations (21a) and (24a), where, however, the γ_j are not required to be all different. It is clear by inspection that the total voltage and current are linear superpositions of the individual modes, independently of the nature of the propagation constants.

2.2.2. Conductor-Loss Attenuation Constant

Given a specific bi-orthogonal mode ($\underline{V}^{(j)}$, $\underline{I}^{(j)}$) determined on the lossless conductor basis, conductivities may then be assigned to the conductors, and, on the assumption that the current distributions remain unchanged, the a.c. resistances of the various conductors may be determined. The attenuation constant for the mode is then obtained on a power-loss basis:

$$\alpha_c = - \frac{1}{2W} \frac{dW}{dx} \qquad (40)$$

where W is the transmitted power at some convenient cross-section of the line, and -dW/dx is its rate of power loss at the same cross-section.

2.2.3. Modifications Resulting from the Presence of Mixed Dielectrics

In the case of lossless conductors in a homogeneous dielectric, equations (3) and (31) lead to a single propagation mode. When mixed dielectrics are present, it is again convenient to think initially in terms of lossless conductors, introducing modification for finite conductivity later. Transmission is in the TM (nearly TEM) mode. The electric field has a component, E_x, in the direction of propagation; the two-dimensional Laplace equation no longer holds exactly. In fact, we have

$$\overline{E} = - \overline{\nabla} V - j\omega\mu_d \overline{A}$$

and since we have a TM mode, $A_y = A_z = 0$. Thus,

$$\begin{aligned} E_x &= - \frac{\partial V}{\partial x} - j\omega\mu_d A_x \\ E_y &= - \frac{\partial V}{\partial y} \\ E_z &= - \frac{\partial V}{\partial z} \end{aligned} \qquad (41)$$

and

$$\text{div } \overline{E} = 0 = - \frac{\partial^2 V}{\partial x^2} - \frac{\partial^2 V}{\partial y^2} + \frac{\partial E_x}{\partial x}$$

that is,

$$\nabla_t^2 V = \frac{\partial E_x}{\partial x} = - \gamma E_x \neq 0 \qquad (42)$$

The transverse field is no longer quasi-static, so, strictly speaking, the concept of electrostatic capacitance is invalid. Nevertheless, it has become customary to apply the concept to the present situation, even ignoring the likely possibility that effective capacitances will vary with frequency (Ref. 7,8). More recent literature has taken a closer look at this problem for some special cases (e.g., reference 13, which contains additional recent references). Inductances have been determined by assuming that magnetic flux distribution is independent of the dielectric constant, uniform or not.

The restriction to quasi-static approximations may be avoided by finding the bi-orthogonal sets and their eigenvalues without recourse to the static differential equations (1). This problem has not received our attention in the present study.

In addition to these distortions to the TEM model, we still have the effect of finite conductor conductivity, which modifies the mode propagation constants and the impedance parameters. Again, the details of these distortions have not been covered in the present study.

No further consideration is given in this report to the discussion of mixed dielectrics as a separate subject.

3. FORMAL SOLUTIONS

The eigenmodes found in the previous section, along with appropriate terminal conditions, are sufficient to define the response of the line to terminal excitation only. We have yet to consider solution for the continuously excited line (non-homogeneous equations). Inasmuch as we believe that this may have been documented elsewhere in terms of eigenmode solutions (e.g., Ref. 14), and since, furthermore, we are, for the present, concerned with homogeneous dielectrics and lossy conductors, we investigate the response of the continuously-excited line only by the approximate average-pole method of section 3.2.

We summarize, first, the solution for the end-excited line in terms of bi-orthogonal modes.

3.1. <u>End-Excited Line</u>

Let

$_f\underline{V}^{(j)}$, $_f\underline{I}^{(j)}$, be a bi-orthogonal set of forward-wave voltage and current vectors, respectively, $j = 1, \ldots, N$

$_b\underline{V}^{(j)}$, $_b\underline{I}^{(j)}$, be a bi-orthogonal set of back-wave voltage and current vectors, respectively.

Then we have, from the second of equations (19),

$$\left.\begin{aligned} {}_f\underline{\underline{I}} &= {}_f\underline{\underline{H}}\ \underline{\Upsilon}^{-1} \\ {}_b\underline{\underline{I}} &= -\ {}_b\underline{\underline{H}}\ \underline{\Upsilon}^{-1} \end{aligned}\right\} \tag{43}$$

where the minus sign in the second equation results from changing γ_j to $-\gamma_j$ in the derivation of equations (19).

The total forward wave of current is

$$ {}_f\underline{I} = \sum_{j=1}^{N} {}_f\underline{I}^{(j)} = \left[{}_f\underline{I}^{(j)} \right]_{\text{row}} \underline{I}_c = {}_f\underline{\underline{I}}\ \underline{I}_c \tag{44}$$

where $\underline{I}_c$ is a unit column matrix and we have written "row" under the bracketed quantity as a reminder that it is a row matrix.

Similarly,

$$ {}_b\underline{I} = {}_b\underline{\underline{I}}\ \underline{I}_c \tag{45}$$

The total line current vector is

$$\underline{I} = {}_f\underline{I} + {}_b\underline{I} = ({}_f\underline{\underline{I}} + {}_b\underline{\underline{I}})\ \underline{\mathcal{I}}_c$$

$$= ({}_f\underline{\underline{H}} - {}_b\underline{\underline{H}})\ \underline{\gamma}^{-1}\ \underline{\mathcal{I}}_c \tag{46}$$

From equation (18c)

$${}_f\underline{\underline{H}} = \begin{bmatrix} {}_f\underline{H}^{(j)} \\ \text{row} \end{bmatrix} = \begin{bmatrix} \eta^{(j)}\ {}_f\underline{V}^{(j)} \\ \text{row} \end{bmatrix} \tag{47}$$

Write

$${}_f\underline{V}^{(j)}(x) = {}_f\underline{V}^{(j)}(0)\ \exp(-\gamma_j x) = {}_f\underline{V}^{(j)}(0)\, s_j^{-1} \tag{48}$$

where

$$s_j = \exp(\gamma_j x) \tag{49}$$

Then equation (47) becomes

$${}_f\underline{\underline{H}} = \begin{bmatrix} \underline{\eta}^{(j)}\ {}_f\underline{V}^{(j)}(0)\, s_j^{-1} \\ \text{row} \end{bmatrix} = [\underline{\eta}^{(j)}\ {}_f\underline{V}^{(j)}(0)]_{\text{row}}\ \underline{s}^{-1}$$

$$= {}_f\underline{\underline{H}}(0)\ \underline{s}^{-1} \tag{50}$$

where $\underline{s}$ is the diagonal matrix

$$\underline{s} = \begin{bmatrix} s_1 & & \\ & s_2 & \\ & & \ddots \\ & & & s_N \end{bmatrix} = \begin{bmatrix} \exp(\gamma_1 x) & & \\ & \ddots & \\ & & \exp(\gamma_N x) \end{bmatrix} \tag{51}$$

and

$${}_f\underline{\underline{H}}(0) = [\underline{\eta}^{(j)}\ {}_f\underline{V}^{(j)}(0)]_{\text{row}} \tag{52}$$

Similarly

$${}_b\underline{\underline{H}} = {}_b\underline{\underline{H}}(0)\ \underline{s} \tag{53}$$

where

$$ {}_b\underline{\underline{H}}(0) = [\ \underbrace{\underline{\eta}^{(j)}\ {}_b\underline{V}^{(j)}(0)}_{\text{row}}\ ] \tag{54} $$

Then equations (50,53) in equation (46) yields

$$ \underline{I} = ({}_f\underline{\underline{H}}(0)\underline{s}^{-1} - {}_b\underline{\underline{H}}(0)\ \underline{s})\ \underline{\gamma}^{-1}\ \underline{I}_c $$

$$ = \underbrace{[\ \underline{\eta}^{(j)}\ ({}_f\underline{V}^{(j)}(0)\cdot\ \underline{s}^{-1} - {}_b\underline{V}^{(j)}(0)\cdot\ \underline{s})\ ]}_{\text{row}}\underline{\gamma}^{-1}\underline{I}_c \tag{55} $$

and, for the line voltage vector,

$$ \underline{V} = {}_f\underline{V} + {}_b\underline{V} = \sum_{j=1}^{N}\ {}_f\underline{V}^{(j)} + \sum_{j=1}^{N}\ {}_bV^{(j)} $$

$$ = ({}_f\underline{\underline{V}} + {}_b\underline{\underline{V}})\ \underline{I}_c $$

$$ = ({}_f\underline{\underline{V}}(0)\cdot\ \underline{s}^{-1} + {}_b\underline{\underline{V}}(0)\cdot\underline{s})\ \underline{I}_c \tag{56} $$

For the pure TEM mode

$$ \underline{\eta}^{(j)} = \underline{\eta} = [\eta_{ij}] = [G_{ij} + j\omega\ C_{ij}] $$

$$ = \underline{G} + j\omega\ \underline{C} $$

$$ \underline{s} = \exp(\gamma x)\cdot\ \underline{I} = s\ \underline{I};\ s = \exp(\gamma x) $$

$$ \underline{\gamma} = \gamma\ \underline{I} $$

and γ is defined in equations (35-39)

Then equations (56) and (55) become, respectively,

$$ \underline{V} = s^{-1}\cdot{}_f\underline{\underline{V}}(0)\ \underline{I}_c + s\cdot{}_b\underline{\underline{V}}(0)\ \underline{I}_c $$

$$ = s^{-1}\cdot{}_f\underline{V}(0) + s\cdot{}_b\underline{V}(0) \tag{57} $$

$$\underline{I} = \gamma^{-1}\ \underline{\eta}\ (s^{-1}\cdot {}_{f}\underline{V}(0) - s\cdot {}_{b}\underline{V}(0))$$

$$= \underline{Y}'\ (s^{-1}\cdot {}_{f}\underline{V}(0) - s\cdot {}_{b}\underline{V}(0)) \qquad (58)$$

where (Ref. 5, Chapter 5),

$$\underline{Y}' = (1 - j\ \tan\ \delta_d)^{1/2}\ v\ \underline{C} \qquad (59)$$

in which $\underline{C}$ is the line capacitance matrix, and

$$v = (\mu_d \varepsilon_d)^{-1/2} \qquad (60)$$

Equations (57,58) constitute one form of canonical equations for a multi-wire TEM line with end excitation only (Ref. 1 and 5) while equations (56) and (55) are their respective generalizations for a quasi-TEM line.

Some additional development of the theory of multimode end excited lines may be found in reference 12.

3.2. Continuously-Excited Line; Approximate Average-Pole Method.

In the following development we assume that the resistance and internal inductance of the various conductors are known in advance. This, of course, is not true, but we can approximate the true values by assuming for this purpose that current distribution on the conductors is uniform. The estimated error in resistance for typical situations is expected to be no greater than about fifteen percent on the low side, thus yielding a conservative result for externally induced, undesired signals.

When an external field is impressed on a line, forcing functions must be added to the homogeneous equations (1); thus (Ref. 9, App. B)

$$\left.\begin{aligned} \frac{d\underline{V}}{dx} + \underline{\zeta}\,\underline{I} &= \underline{E}^e(x) \\ \frac{d\underline{I}}{dx} + \underline{\eta}\,\underline{V} &= \underline{H}^e(x) \end{aligned}\right. \qquad (61)$$

where $\underline{E}^e(x)$ is an equivalent series voltage source distribution resulting from the impressed transverse magnetic field, and $\underline{H}^e(x)$ is an equivalent shunt current source distribution resulting from the impressed transverse electric field.

3.2.1. Laplace Transforms for Line Potentials and Currents.

The generalization of Laplace transform methods to vectors is straightforward (loc. cit.). Use the general designation

$\underline{\tilde{F}}(p)$ = Laplace transform of $\underline{F}(x)$

$$\underline{\tilde{F}}(p) = \int_0^{\infty} \underline{F}(\lambda)\exp(-p\lambda)\,d\lambda;\ p = c + j\eta,\ c > 0 \qquad (62)$$

and the inverse transform

$$\underline{F}(x) = \frac{1}{j2\pi}\int_{c-j\infty}^{c+j\infty} \underline{\tilde{F}}(p)\exp(px)\,dp \qquad (63)$$

where $\underline{F}(x)$ is a column matrix (vector).

Taking transforms in equations (61) leads to

$$\left.\begin{aligned} p\,\underline{\tilde{V}} + \underline{\zeta}\,\underline{\tilde{I}} &= \underline{V}(0) + \underline{\tilde{E}}^e \\ \eta\,\underline{\tilde{V}} + p\,\underline{\tilde{I}} &= \underline{I}(0) + \underline{\tilde{H}}^e \end{aligned}\right\} \qquad (64)$$

Multiply the first of these by p, the second by $\underline{\zeta}$, and subtract

$$(p^2 \underline{I} - \underline{\zeta}\ \underline{\eta})\ \underline{\tilde{V}} = p\underline{V}(0) - \underline{\zeta}\ \underline{I}(0) + p\ \underline{\tilde{E}}^e - \underline{\zeta}\ \underline{\tilde{H}}^e \qquad (65)$$

In the pure TEM case we have

$$\underline{\zeta}\ \underline{\eta} = \underline{\eta}\ \underline{\zeta} = -\beta^2 \underline{I};\ \beta = \frac{\omega}{v} \qquad (66)$$

and equation (65) is easily solved explicitly for $\underline{V}$:

$$\underline{\tilde{V}} = \frac{p}{p^2+\beta^2} \underline{V}(0) - \frac{1}{p^2+\beta^2} \underline{\zeta}\ \underline{I}(0) + \frac{p}{p^2+\beta^2} \underline{\tilde{E}}^e(p) - \frac{\underline{\zeta}\ \underline{\tilde{H}}^e(p)}{p^2+\beta^2} \qquad (67)$$

whence the inverse transform, $\underline{V}$, is obtained in a straightforward manner (loc. cit.).

More generally, equation (65) is solved for $\underline{\tilde{V}}$ by multiplying both sides on the left by $(p^2 \underline{I} - \underline{\zeta}\ \underline{\eta})^{-1}$:

$$\underline{\tilde{V}} = (p^2 \underline{I} - \underline{\zeta}\ \underline{\eta})^{-1} \{p\underline{V}(0) - \underline{\zeta}\ \underline{I}(0) + p\underline{\tilde{E}}^e - \underline{\zeta}\ \underline{\tilde{H}}^e\} \qquad (68)$$

whence the formal solution for $\underline{V}$ follows by way of equation (63).

Similarly, elimination of $\underline{\tilde{V}}$ in equations (64) yields

$$\underline{\tilde{I}} = (p^2 \underline{I} - \underline{\eta}\ \underline{\zeta})^{-1} \{p\underline{I}(0) - \underline{\eta}\ \underline{V}(0) + p\underline{\tilde{H}}^e - \underline{\eta}\ \underline{\tilde{E}}^e\} \qquad (69)$$

3.2.2. <u>Inverse Transforms</u>

Write

$$\underline{Q} = (p^2 \underline{I} - \underline{\zeta}\ \underline{\eta}) = [q_{ij}] \qquad (70)$$

$$\underline{Q}^{-1} = \frac{\underline{\hat{Q}}}{|\underline{Q}|} \qquad (71)$$

where $\hat{\underline{Q}}$ is the adjoint of $\underline{Q}$:

$$\hat{\underline{Q}} = [Q_{ji}] = \begin{bmatrix} Q_{11}, \ldots, & Q_{N1} \\ \cdot \quad \cdot & \cdot \\ Q_{1N}, \ldots, & Q_{NN} \end{bmatrix} = [Q_{ij}]^T \tag{72}$$

and Q_{ji} is the cofactor of q_{ji} in $|\underline{Q}|$;

$|\underline{Q}|$ is the determinant of $\underline{Q}$.

Thus the formal solution for $\underline{V}$ is (Ref. 10, 11).

$$\underline{V} = \frac{1}{j2\pi} \int_{c-j\infty}^{c+j\infty} \frac{\hat{\underline{Q}}(p)\{p\underline{V}(0) - \underline{\zeta}\underline{I}(0) + p\tilde{\underline{E}}^e(p) - \underline{\zeta}\tilde{\underline{H}}^e(p)\}\exp(px)\,dp}{|\underline{Q}|} \tag{73}$$

This may be written

$$\underline{V} = \underline{T}^{(1)}\underline{V}(0) - \underline{T}^{(2)}\underline{\zeta}\,\underline{I}(0) + \underline{T}^{(3)} - \underline{T}^{(4)} \tag{74}$$

where

$$\underline{T}^{(1)}(x) = \frac{1}{j2\pi} \int_{c-j\infty}^{c+j\infty} \frac{p\hat{\underline{Q}}(p)\exp(px)\,dp}{|\underline{Q}|} = \frac{1}{j2\pi} \int_{c-j\infty}^{c+j\infty} p\tilde{\underline{T}}^{(2)}(p)\exp(px)\,dp$$

$$= \frac{d\underline{T}^{(2)}}{dx} \tag{75}$$

$$\underline{T}^{(2)}(x) = \frac{1}{j2\pi} \int_{c-j\infty}^{c+j\infty} \frac{\hat{\underline{Q}}(p)\exp(px)\,dp}{|\underline{Q}|} = \frac{1}{j2\pi} \int_{c-j\infty}^{c+j\infty} \tilde{\underline{T}}^{(2)}(p)\exp(px)\,dp \tag{76}$$

$$\underline{T}^{(3)}(x) = \frac{1}{j2\pi} \int_{c-j\infty}^{c+j\infty} \frac{p\hat{\underline{Q}}(p)\tilde{\underline{E}}^e(p)\exp(px)\,dp}{|\underline{Q}|}$$

$$= \frac{1}{j2\pi} \int_{c-j\infty}^{c+j\infty} \tilde{\underline{T}}^{(1)}(p)\tilde{\underline{E}}^e(p)\exp(px)\,dp \tag{77}$$

$$T^{(4)}(x) = \frac{1}{j2\pi} \int_{c-j\infty}^{c+j\infty} \frac{\hat{\underline{Q}}(p)\,\underline{\zeta \tilde{H}^e}(p)\exp(px)\,dp}{|\underline{Q}|}$$

$$= \frac{1}{j2\pi} \int_{c-j\infty}^{c+j\infty} \tilde{\underline{T}}^{(2)}(p)\,\underline{\zeta \tilde{H}^e}(p)\exp(px)\,dp \qquad (78)$$

Inspection of equations (75-78) shows that the basic solution needed is $\underline{T}^{(2)}(x)$, equation (76). $\underline{T}^{(1)}(x)$ is then obtained from $\underline{T}^{(2)}$ by differentiation. $\underline{T}^{(3)}$ is determined by convolution of $\underline{T}^{(1)}$ with $\underline{E}^e$, while $\underline{T}^{(4)}$ is obtained by convolution of $\underline{T}^{(2)}$ with $\underline{\zeta H}^e$. (See App. D).

Thus, the solution for the potential may be written

$$\underline{V} = \frac{d\underline{T}^{(2)}}{dx}\underline{V}(0) - \underline{T}^{(2)}\ \underline{\zeta}\ \underline{I}(0) + \int_0^x \frac{d\underline{T}^{(2)}(\xi)}{d\xi}\,\underline{E}^e(x-\xi)\,d\xi$$

$$- \int_0^x \underline{T}^{(2)}(\xi)\ \underline{\zeta}\ \underline{H}^e(x-\xi)\,d\xi \qquad (79)$$

For the current, $\underline{I}$, we note in equation (69) that

$$(p^2\underline{I} - \underline{\eta\zeta}) = [p^2\underline{I} - (\underline{\zeta}\ \underline{\eta})^T] = [p^2\underline{I} - \underline{\zeta}\ \underline{\eta}]^T$$

$$= \underline{Q}^T = [q_{ji}] \qquad (80)$$

and

$$(p^2\underline{I} - \underline{\eta}\ \underline{\zeta})^{-1} = (\underline{Q}^T)^{-1} = \frac{(\hat{\underline{Q}}^T)}{|\underline{Q}^T|}$$

But $|\underline{Q}^T| = |\underline{Q}|$

and $(\hat{\underline{Q}}^T)$ = matrix of transposed cofactors of q_{ji}

$= [Q_{ij}]$

The formal solution for the current is, from equation (69)

$$\underline{I} = \underline{R}^{(1)}\, \underline{I}(0) - \underline{R}^{(2)}\, \underline{\eta}\, \underline{V}(0) + \underline{R}^{(3)} - \underline{R}^{(4)} \tag{81}$$

where

$$\underline{R}^{(1)}(x) = \frac{1}{j2\pi} \int_{c-j\infty}^{c+j\infty} \frac{p\hat{\underline{Q}}^T \exp(px)\,dp}{|\underline{Q}|}$$

$$= \frac{1}{j2\pi} \int_{c-j\infty}^{c+j\infty} p\tilde{\underline{R}}^{(2)}(p) \exp(px)\,dp = \frac{d\underline{R}^{(2)}}{dx} \tag{82}$$

$$\underline{R}^{(2)}(x) = \frac{1}{j2\pi} \int_{c-j\infty}^{c+j\infty} \frac{\hat{\underline{Q}}^T \exp(px)\,dp}{|\underline{Q}|}$$

$$= \frac{1}{j2\pi} \int_{c-j\infty}^{c+j\infty} \tilde{\underline{R}}^{(2)}(p) \exp(px)\,dp \tag{83}$$

$$\underline{R}^{(3)}(x) = \frac{1}{j2\pi} \int_{c-j\infty}^{c+j\infty} \frac{p\hat{\underline{Q}}^T \tilde{\underline{H}}^e \exp(px)\,dp}{|\underline{Q}|}$$

$$= \frac{1}{j2\pi} \int_{c-j\infty}^{c+j\infty} \tilde{\underline{R}}^{(1)}(p)\, \tilde{\underline{H}}^e(p) \exp(px)\,dp \tag{84}$$

$$\underline{R}^{(4)}(x) = \frac{1}{j2\pi} \int_{c-j\infty}^{c+j\infty} \frac{\hat{\underline{Q}}^T \underline{\eta} \tilde{\underline{E}}^e \exp(px)\,dp}{|\underline{Q}|}$$

$$= \frac{1}{j2\pi} \int_{c-j\infty}^{c+j\infty} \tilde{\underline{R}}^{(2)}(p)\, \underline{\eta} \tilde{\underline{E}}^e(p) \exp(px)\,dp \tag{85}$$

In this case, equation (83) for $\underline{R}^{(2)}(x)$ is the basic solution, the others being derived from it in the same way as in the potential case. Equation (81) may be written

$$\underline{I} = \frac{d\underline{R}^{(2)}}{dx} \underline{I}(0) - \underline{R}^{(2)}\underline{\eta V}(0) + \int_0^x \frac{d\underline{R}^{(2)}(\xi)}{d\xi} \underline{H}^e(x-\xi)d\xi$$

$$- \int_0^x \underline{R}^{(2)}(\xi)\ \underline{\eta}\ \underline{E}^e(x-\xi)d\xi \qquad (86)$$

Returning to equation (76), we note from equation (70) that $|\underline{Q}|$ is a polynomial of the Nth degree in p^2, and may therefore be written

$$|\underline{Q}| = \prod_{i=1}^{N} (p^2 - p_i^2) = \sum_{r=0}^{N} d_r p^{2(N-r)};\ d_o = 1 \qquad (87)$$

where the p_i are the roots of

$$|\underline{Q}| = 0$$

In the numerator of equation (76) the elements of $\underline{\hat{Q}}$ are rational integral functions of p and therefore contain no singularities. In fact they are polynomials in p^2 of degree no greater than (N-1). Thus the solution for $\underline{T}^{(2)}(x)$ is just the sum of the residues at the poles, $\pm p_i$ $(i = 1, \ldots, N)$ of the integrand.

3.2.3. Approximate Solution: First Order Effects.

With the conductor resistance assumed small, the deviation of the propagation modes from pure TEM is also small. The roots, $\pm p_i$ are contained in regions R_+ and R_-, respectively, bounded by small circles of radius, ρ, such that

$$\left|\frac{\rho}{p_i}\right| \ll 1, \quad i = 1, \ldots, N$$

We make use of this fact by expanding the integrand of equation (76) in Laurent series around the points $\pm\, p_a$ in R_+ and R_- respectively, where

$$p_i = p_a + \delta_i, \quad \left|\frac{\delta_i}{p_a}\right| \ll 1 \tag{88}$$

and

$$\left.\begin{aligned} p_a &= \frac{1}{N}\sum_{i=1}^{N} p_i \\ &\equiv \text{average value of the } p_i \end{aligned}\right\} \tag{89}$$

It follows that

$$\sum_{i=1}^{N} \delta_i = 0 \tag{90}$$

The value of $T_{ij}^{(2)}$ is then just the sum of the coefficients of $(p - p_a)^{-1}$ and $(p + p_a)^{-1}$ in the respective expansions. Details are given in Appendix E.

The result is

$$T_{ij}^{(2)} = \sum_{q=1}^{N} \frac{a_{ji}^{(q)}}{(4p_a{}^2)^{q-1}} F(x;p_a;q) \tag{91}$$

where

$$F = \sum_{r=0}^{q+s-1} \sum_{s=0}^{\infty} \frac{(-1)^{q+s-r-1}}{(2p_a)^{s-r+1}} \, S_s\!\left(\frac{x^r}{r!}\right) \binom{2q+s-r-2}{q-1} \times \left[e^{p_a x} - (-1)^r e^{-p_a x}\right] \tag{92}$$

The $a_{ji}^{(q)}$ are coefficients in an expansion of Q_{ji} (c.f. equation (E-3)):

$$Q_{ji} = \sum_{r=1}^{N} a_{ji}^{(r)} (p^2 - p_a^{\,2})^{n-r} \tag{93}$$

The first few values of the S_s are

$$\begin{aligned} S_0 &= 1 \\ S_1 &= 0 \\ S_2 &= \sum_{i=1}^{N} \sum_{j=1}^{i} \delta_i \, \delta_j \end{aligned} \tag{94}$$

In particular, for q = 1,2:

$$\begin{aligned} F(x;p_a;1) = \frac{1}{p_a} \{ &\sinh p_a x \\ &+ \frac{S_2}{(2p_a)^2} \left[(1+2p_a^{\,2}x^2) \sinh p_a x - 2p_a x \cdot \cosh p_a x \right] \\ &+ \ldots \} \end{aligned} \tag{95}$$

$$\begin{aligned} F(x;p_a;2) = -\frac{2}{p_a} \{ &[\sinh p_a x - p_a x \cdot \cosh p_a x] \\ &+ \frac{S_2}{2p_a^{\,2}} [\sinh p_a x - 3p_a x \cosh p_a x \\ &+ p_a^{\,2}x^2 \sinh p_a x - \tfrac{1}{3} p_a^{\,3} x^3 \cosh p_a x] \\ &+ \ldots \} \end{aligned} \tag{96}$$

At this writing, we have not taken the time to explore carefully the convergence properties of the double series, equation (92). Some preliminary analysis suggests that (1) the series converges more slowly for larger values of $(p_a x)$; (2) terms begin to get smaller when s is of the order of $\delta_m x$, where δ_m is the largest of the $|\delta_i|$. Actually, for low-loss conductors in a homogeneous isotropic dielectric, we expect applications of these results to show that the dispersion effects are accounted for adequately by the $a_{ji}^{(q)}$, $(q>1)$, and that their multiplication by factors containing S_s, $(s>1)$, introduces only second-order corrections to the dispersion calculations. In that case, the determination of F becomes simply

$$F = F_0(x;p_a;q) = \sum_{r=0}^{q-1} \frac{(-1)^{q-r-1}}{(2p_a)^{1-r}} \left(\frac{x^r}{r!}\right)\binom{2q-r-2}{q-1}\left[e^{p_a x} - (-1)^r e^{-p_a x}\right] \quad (97)$$

In particular,

$$\left.\begin{aligned} F_0\ (x;p_a;1) &= \frac{1}{p_a} \sinh p_a x \\ F_0\ (x;p_a;2) &= -\frac{2}{p_a}(\sinh p_a x - p_a x \cosh p_a x) \end{aligned}\right\} \quad (97a)$$

From equation (91) we have immediately

$$\underline{T}^{(2)} = \sum_{q=1}^{N} \frac{\underline{A}^{(q)^T}}{(4p_a{}^2)^{q-1}} F(x;p_a;q) \quad (98a)$$

where

$$\underline{A}^{(q)} = \left[a_{ij}{}^{(q)}\right] \quad (99)$$

Furthermore, comparison of equations (76) and (83) yields

$$\underline{R}^{(2)} = \sum_{q=1}^{N} \frac{\underline{A}^{(q)}}{(4p_a{}^2)^{q-1}} F(x;p_a;q) \quad (98b)$$

To determine the response of the line as a function of its terminations, we proceed essentially as in reference 9, appendix B. In conformity with previously established notation (loc.cit.) we introduce the following symbol changes:

$$\underline{V}(0) \rightarrow \underline{V}^i$$
$$\underline{V}(\ell) \rightarrow \underline{V}^o$$
$$\underline{I}(0) \rightarrow \underline{I}^i$$
$$\underline{I}(\ell) \rightarrow \underline{I}^o$$

where ℓ is the length of the line.

In addition we have the terminal conditions

$$\left.\begin{aligned} \underline{I}^i + \underline{Y}^i\,\underline{V}^i &= \underline{0} \\ \underline{I}^o - \underline{Y}^o\,\underline{V}^o &= \underline{0} \end{aligned}\right\} \tag{100}$$

where $\underline{Y}^i$, $\underline{Y}^o$ are the line termination matrices at $x = 0, \ell$, respectively (Ref. 15).

In Appendix F we use these conditions to obtain the following results:

$$\left.\begin{aligned} \underline{V}(x) &= [\underline{T}^{(1)}(x) + \underline{T}^{(2)}(x)\,\underline{\zeta}\,\underline{Y}^i]\,\underline{S}^{-1}\underline{K}(\ell) + \underline{U}(x) \\ \underline{I}(x) &= -\,[\underline{R}^{(1)}(x)\underline{Y}^i + \underline{R}^{(2)}(x)\underline{\eta}]\underline{S}^{-1}\underline{K}(\ell) + \underline{W}(x) \end{aligned}\right\} \tag{101}$$

where

$$\underline{S} = p_a^{-1}\,\underline{\zeta}\{[\underline{R}^{(1)}(\ell)\underline{Y}^i + \underline{R}^{(2)}(\ell)\underline{\eta}] + \underline{Y}^o[\underline{T}^{(1)}(\ell) + \underline{T}^{(2)}(\ell)\underline{\zeta}\underline{Y}^i]\}$$

$$\underline{K}(\ell) = p_a^{-1}\,\underline{\zeta}\,\{[\underline{R}^{(3)}(\ell) - \underline{R}^{(4)}(\ell)] - \underline{Y}^o[\underline{T}^{(3)}(\ell) - \underline{T}^{(4)}(\ell)]\}$$

$$\underline{U}(x) = \underline{T}^{(3)}(x) - \underline{T}^{(4)}(x) \tag{102}$$

$$\underline{W}(x) = \underline{R}^{(3)}(x) - R^{(4)}(x)$$

Although small conductor losses can affect the line propagation constant significantly, the effect on the line impedance, or admittance, matrix is usually considered unimportant. In accord with this view we have, from equations (16a, b),

$$\left.\begin{aligned} \underline{V} &= p_a^{-1}\ \underline{\zeta}\ \underline{I} = \underline{Z}\ \underline{I} \\ \underline{I} &= p_a^{-1}\ \underline{\eta V} = \underline{Y}\ \underline{V} \end{aligned}\right\} \qquad (103)$$

where

$$\left.\begin{aligned} \underline{Z} &= p_a^{-1}\ \underline{\zeta} \approx j\omega p_a^{-1}\underline{L} = (1 - j\tan\delta_d)^{-1/2}\ v\ \underline{L} \\ \underline{Y} &= p_a^{-1}\ \underline{\eta} \approx j\omega p_a^{-1}\ \ \underline{C} = (1 - j\tan\delta_d)^{1/2}\ v\ \underline{C} \end{aligned}\right\} \qquad (104)$$

and we have used the average propagation constant, p_a, for γ.

$\underline{Z}$ and $\underline{Y}$ are the line impedance and admittance matrices, respectively (Ref. 15):

$$\underline{ZY} \approx v^2\underline{LC} \approx \underline{I} \qquad (105)$$

Again, using previously established notation we write the normalized load admittance matrices:

$$\left.\begin{aligned} \underline{P}^i &= \underline{ZY}^i \\ \underline{P}^o &= \underline{ZY}^o \end{aligned}\right\} \qquad (106)$$

whence equations (101) become

$$\begin{aligned} \underline{V}(x) &= [\underline{T}^{(1)}(x) + \underline{T}^{(2)}(x)(p_a\underline{Z}\ \underline{Y}^i)]\underline{S}^{-1}\ \underline{K}(\ell) + \underline{U}(x) \\ &= [\underline{T}^{(1)}(x) + p_a\ \underline{T}^{(2)}(x)\ \underline{P}^i]\ \underline{S}^{-1}\ \underline{K}(\ell) + \underline{U}(x) \qquad (107) \\ \underline{I}(x) &= -\ [\underline{R}^{(1)}(x)\ \underline{Y}\ \underline{P}^i + \underline{R}^{(2)}(x)\ \ (p_a\underline{Y})]\underline{S}^{-1}\underline{K}(\ell) + \underline{W}(x) \\ &= -\ [\underline{R}^{(1)}(x)\ \underline{Y}\ \underline{P}^i + p_a\underline{R}^{(2)}(x)\underline{Y}]\underline{S}^{-1}\underline{K}(\ell) + \underline{W}(x) \end{aligned}$$

where

$$\left.\begin{aligned} \underline{S} &= \underline{Z}\{[\underline{R}^{(1)}(\ell)\underline{YP}^i + p_a\underline{R}^{(2)}(\ell)\underline{Y}] + \underline{YP}^o[\underline{T}^{(1)}(\ell) + p_a\underline{T}^{(2)}(\ell)\underline{P}^i]\} \\ \underline{K}(\ell) &= \underline{Z}\{[\underline{R}^{(3)}(\ell) - \underline{R}^{(4)}(\ell)] - \underline{YP}^o[\underline{T}^{(3)}(\ell) - \underline{T}^{(4)}(\ell)]\} \quad (108) \end{aligned}\right\} (108)$$

4. DISCUSSION OF RESULTS

We will discuss, first, the approximate method of the preceding section, and then review, briefly, the method of bi-orthogonal modes.

4.1. Discussion of the Approximate Solution

The results obtained in the previous section may be summarized as follows: The TEM potentials and currents induced along a multiwire line of low-loss conductors by an external electromagnetic field may be approximated by the equations

$$\left.\begin{aligned} \underline{V}(x) &= [\underline{T}^{(1)}(x) + p_a\underline{T}^{(2)}(x)\underline{p}^i]\underline{S}^{-1}\underline{K}(\ell) + \underline{U}(x) \\ \underline{I}(x) &= -[\underline{R}^{(1)}(x)\ \underline{YP}^i + p_a\underline{R}^{(2)}(x)\underline{Y}]\underline{S}^{-1}\underline{K}(\ell) + \underline{W}(x) \end{aligned}\right\} \quad (109)$$

where $\underline{S}$ is the non-singular matrix given by the first of equations (108)

$\underline{K}(\ell)$ is given by the second of equations (108)

$$\underline{U}(x) = T^{(3)}(x) - \underline{T}^{(4)}(x)$$

$$W(x) = \underline{R}^{(3)}(x) - \underline{R}^{(4)}(x)$$

$$\underline{T}^{(1)}(x) = d\underline{T}^{(2)}(x)/dx$$

$$\underline{T}^{(3)}(x) = \underline{T}^{(1)}(x) * \underline{E}^e(x) \qquad \text{(convolution)}$$

$$\underline{T}^{(4)}(x) = \underline{T}^{(2)}(x) * \underline{\zeta}\ \underline{H}^e(x)$$

$$\underline{T}^{(2)}(x) = \frac{\underline{I}}{p_a}\sinh p_a x - \frac{\underline{A}^{(2)^T}}{2p_a^{\,3}}(\sinh p_a x - p_a x\cosh p_a x)$$

$$\underline{R}^{(1)}(x) = d\,\underline{R}^{(2)}(x)/dx$$

$$\underline{R}^{(3)}(x) = \underline{R}^{(1)}(x) * \underline{\mathcal{H}}^{e}(x) \qquad \text{(convolution)}$$

$$\underline{R}^{(4)}(x) = \underline{R}^{(2)}(x) * \underline{\eta}\,\underline{\mathcal{E}}^{e}(x) \qquad \text{"}$$

$$\underline{R}^{(2)}(x) = \frac{\underline{I}}{p_a} \sinh p_a x - \frac{\underline{A}^{(2)}}{2\,p_a^{3}} (\sinh p_a x - p_a x \cosh p_a x)$$

$\underline{\zeta}$ = line series impedance matrix, ohm/meter

$= \underline{R} + j\omega \underline{L}$

$\underline{R} = [R_i \delta_{ij}]$; $\underline{L} = [L_{ij}]$; $R_i << L_{ii}$

$\underline{\eta}$ = line shunt admittance matrix, mho/meter

$= \underline{G} + j\omega\,\underline{C}$

$\underline{G} = [G_{ij}]$; $\underline{C} = [C_{ij}]$; $G_{ij} << C_{ij}$

$$\underline{\zeta}\,\underline{\eta} = (\underline{R} + j\omega\,\underline{L})(\underline{G} + j\omega\underline{C})$$

$$= [\underline{R} + j\omega(\underline{L}_e + \underline{L}_i)]\ (\underline{G} + j\omega\underline{C})$$

where

$\underline{L}_e$ = external inductance matrix

= inductance matrix in the absence of conductor losses

$\underline{L}_i$ = internal inductance matrix

$\underline{L}_i << \underline{L}_e$

Continuing, we can write

$$\underline{\zeta}\,\underline{\eta} = j\omega\underline{L}_e\ (\underline{G} + j\omega\underline{C}) + (\underline{R} + j\omega\underline{L}_i)(\underline{G} + j\omega\underline{C})$$

$$= -\,k_0^2\,\underline{I} + \underline{\varepsilon} \qquad (110)$$

where

$$k_o^2 = \frac{\omega^2}{v^2}\left(1 + \frac{g_d}{j\omega\varepsilon_d} \right) \tag{111a}$$

and

$$\underline{\varepsilon} = [\varepsilon_{ij}] = (\underline{R} + j\omega\underline{L}_i)(\underline{G} + j\omega\underline{C})$$

$$\varepsilon_{ij} << |k_o^2| \tag{111b}$$

jk_o is the line propagation constant in the absence of conductor losses. The polynomial

$$\det.\ \underline{Q} = \det.\ (p^2\ \underline{I} - \underline{\zeta}\ \underline{\eta}) = \det.((p^2 + k_o^2)\ \underline{I} - \underline{\varepsilon})$$

has zeros, $\pm\ p_i$, given by

$$p_i = p_a + \delta_i\ ,\ i = 1,\ldots, N$$

where p_a is the arithmetic average of the p_i:

$$p_a = \frac{1}{N}\sum_{i=1}^{N} p_i$$

and

$$\left|\frac{\delta_i}{p_a}\right| << 1\quad ,\ i = 1,\ldots, N$$

The zeros may probably be determined most accurately by first writing

$$u = p^2 + k_o^2\ ;\ \det.\ \underline{Q} = \det.\ (u\underline{I} - \underline{\varepsilon}) \tag{112}$$

and finding the zeros, u_i, of the resulting polynomial. The p_i are then given by

$$p_i = (u_i - k_o^2)^{1/2}$$

$$= j(k_o^2 - u_i)^{1/2} \quad (113a)$$

$$\approx jk_o - j\frac{u_i}{2k_o}, \; i = 1,\ldots, N \quad (113b)$$

Knowing the δ_i we can compute S_2, the coefficient of $(p - p_a)^{-2}$ in the power series expansion of

$$P = \left[\prod_{i=1}^{N} \left(1 - \frac{\delta_i}{p - p_a} \right)\right]^{-1}$$

The desired coefficient is readily obtained by initiating the indicated operations. Write

$$b_i = \frac{\delta_i}{p-p_a}$$

$$P^{-1} = \prod_{i=1}^{N} (1 - b_i)$$

$$= 1 - a_1 + a_2 - a_3 + \ldots.$$

where

$$a_1 = \sum_{i=1}^{N} b_i$$

$$a_2 = \sum_{i=1}^{N} \sum_{j=i+1}^{N} b_i b_j$$

$$a_3 = \sum_{i=1}^{N} \sum_{j=i+1}^{N} \sum_{k=j+1}^{N} b_i b_j b_k \qquad \text{etc.}$$

Then inverting by long division one obtains

$$p = 1 + a_1 + (a_1^2 - a_2) + \ldots\ldots$$

$$= 1 + (p - p_a)^{-1} \sum_{i=1}^{N} \delta_i$$

$$+ (p-p_a)^{-2} \left[\sum_{i=1}^{N} \sum_{j=1}^{N} \delta_i \delta_j - \sum_{i=1}^{N} \sum_{j=i+1}^{N} \delta_i \delta_j \right] + \ldots$$

Thus,

$$S_2 = \sum_{i=1}^{N} \delta_i \left[\sum_{j=1}^{N} \delta_j - \sum_{j=i+1}^{N} \delta_j \right] = \sum_{i=1}^{N} \sum_{j=1}^{i} \delta_i \delta_j \qquad (114)$$

Finally, we have the matrix, $\underline{A}^{(2)}$, of the coefficients $a_{ij}^{(2)}$ in the expansion of the cofactor Q_{ij}:

$$Q_{ij} = \sum_{r=1}^{N} a_{ij}^{(r)} (p^2 - p_a^2)^{N-r}, \; i,j = 1,\ldots, N \qquad (115)$$

Normally, Q_{ij}, being a cofactor of det. $\underline{Q}$, will be available initially in the form

$$\left. \begin{aligned} Q_{ij} &= \sum_{r=1}^{N} b_{ij}^{(r)} u^{N-r} \\ \text{where } u &= p^2 + k_o^2 \\ b_{ij} &= \delta_{ij}; \; \left| b_{ij}^{(r)} \right| \ll 1, \; r > 1 \end{aligned} \right\} \qquad (116)$$

The coefficients, $a_{ij}^{(2)}$ are obtained by initiating the binomial expansions of the first two terms of equation (116) in powers of $(p^2 - p_a^2)$; thus, for $r = 1$,

$$u^{N-1} = [(p^2 - p_a^2) + (p_a^2 + k_o^2)]^{N-1}$$

$$= (p^2 - p_a^2)^{N-1} + (N - 1)(p^2 - p_a^2)^{N-2}(p_a^2 + k_o^2) + \ldots$$

for r = 2,

$$u^{N-2} = (p^2 - p_a^2)^{N-2} + \ldots$$

Collecting coefficients of $(p^2 - p_a^2)^{N-2}$ in the two expansions (eq. (115) and (116)), and equating

$$a_{ij}^{(2)} = b_{ij}^{(2)} + (N - 1)(p_a^2 + k_o^2)\ \delta_{ij} \tag{117}$$

The remaining parameters ($\underline{Y}$, $\underline{E}^e$, $\underline{H}^e$, $\underline{P}^i$, $\underline{P}^o$) have been fully defined previously for the case of lossless lines (Ref 15).

The dominant terms in $\underline{T}^{(2)}(x)$, $\underline{R}^{(2)}(x)$ have the identical form

$$\underline{M}^{(2)}(x) = \frac{\underline{I}}{p_a} \sinh p_a x \tag{118}$$

Thus, for the lossless-conductor case we get

$$\underline{T}^{(2)}(x) = \underline{R}^{(2)}(x) = \frac{\sinh p_a x}{p_a}\ \underline{I}$$

$$= \frac{\sin k_o x}{k_o}\ \underline{I} \tag{119a}$$

in which, for lossy dielectrics, k_o is complex. In addition we get

$$\left.\begin{aligned} \underline{T}^{(1)}(x) &= \underline{R}^{(1)}(x) = \cos k_o x \cdot \underline{I} \\ \underline{T}^{(3)}(x) &= \underline{E}^e(x) * \cos k_o x \\ \underline{T}^{(4)}(x) &= k_o^{-1}\ \underline{\zeta}\ \underline{H}^e(x) * \sin k_o x \\ &= j\ \underline{Z}\ \underline{H}^e(x) * \sin k_o x \end{aligned}\right\} \tag{119b}$$

by equations (104). Furthermore,

$$\underline{R}^{(3)}(x) = \underline{H}^{e}(x) * \cos k_o x$$

$$\underline{R}^{(4)}(x) = k_o^{-1}\, \underline{\eta}\, \underline{E}^{e}(x) * \sin k_o x \qquad (119c)$$

$$= j\, \underline{Y}\, \underline{E}^{e}(x) * \sin k_o x$$

by equations (104).

Equations (108) become

$$\underline{S} = \underline{Z}\{[\underline{Y}\,\underline{P}^{i} \cos k_o\ell + j\underline{Y} \sin k_o\ell] + \underline{Y}\,\underline{P}^{o}[\cos k_o\ell\cdot\underline{I}$$

$$+ j\,\underline{P}^{i} \sin k_o\ell]\} = (\underline{P}^{i}+\underline{P}^{o})\cos k_o\ell + j(\underline{I}+\underline{P}^{o}\underline{P}^{i}) \sin k_o\ell \qquad (120a)$$

$$\underline{K}(\ell) = \underline{Z}\{[\underline{H}^{e}(\ell) * \cos k_o\ell - j\underline{Y}\,\underline{E}^{e}(\ell) * \sin k_o\ell]$$

$$- \underline{Y}\,\underline{P}^{o}[\underline{E}^{e}(\ell) * \cos k_o\ell - j\underline{Z}\,\underline{H}^{e}(\ell) * \sin k_o\ell]\}$$

$$= (\underline{I} \cos k_o\ell + j\underline{P}^{o} \sin k_o\ell) * \underline{Z}\,\underline{H}^{e}(\ell)$$

$$- (\underline{P}^{o} \cos k_o\ell + j\,\underline{I} \sin k_o\ell) * \underline{E}^{e}(\ell) \qquad (120b)$$

From equations (102) we get

$$\underline{U}(x) = \underline{E}^{e}(x) * \cos k_o x - j\,\underline{Z}\,\underline{H}^{e}(x) * \sin k_o x \qquad (121a)$$

$$\underline{W}(x) = \underline{H}^{e}(x) * \cos k_o x - j\,\underline{Y}\,\underline{E}^{e}(x) * \sin k_o x \qquad (121b)$$

and, finally, equations (101) become

$$\left.\begin{aligned} \underline{V}(x) &= [\underline{I} \cos_o k\, x + j\underline{P}^{i} \sin k_o x]\, \underline{S}^{-1}\, \underline{K}(\ell) + \underline{U}(x) \\ \underline{I}(x) &= -\,[\underline{Y}^{i} \cos k_o x + j\underline{Y} \sin k_o x]\underline{S}^{-1}\underline{K}(\ell) + \underline{W}(x) \\ &= -\,\underline{Y}[\underline{P}^{i} \cos k_o x + j \sin k_o x]\underline{S}^{-1}\underline{K}(\ell) + \underline{W}(x) \end{aligned}\right\}(122)$$

where $\underline{S}$, $\underline{K}(\ell)$, $\underline{U}(x)$, and $\underline{W}(x)$ are given by equations (120a,b) and (121a,b).

These results conform with those previously obtained for the lossless case (App. B, Ref. 9) on setting

$$k_o = \beta = \frac{\omega}{v}$$

4.1.1. Series Impedance Approximation

We have to re-emphasize the fact that the method developed here is based on the assumption of values of conductor series resistance and internal inductance which generally must be only approximate. On the assumption that the current distribution on each conductor periphery is uniform, the attenuation estimate will generally be too small. For a typical cable situation we have estimated the worst-condition error to be of the order of 15%. Furthermore, insofar as the study of interference phenomena is concerned, the line attenuation is important chiefly when one or more conductors are terminated in lossless loads, as for example, a group of cable sheaths grounded or open-circuited at the cable ends.

The error in estimating the internal inductance has not been studied. We guess it to be of the same order as that of the resistance, and, probably, of even less importance, since, in any case, the internal inductance is small compared with the ever-present external self-inductances of the various conductors.

4.2. Discussion of the Bi-orthogonal Mode Solution

Expansion of the line response in a set of bi-orthogonal modes has the advantage that the contribution of the conductor losses can be formulated independently of line terminations.

In at least one special case, that of perfect circular symmetry for all conductors within a circular sheath, a set of independent bi-orthogonal modes is readily postulated. (Ref. 1, Sec. 4.10, p. 172). Figure 1 illustrates this configuration class for N = 6. A suitable set of potential modes is

$$\underline{v}^{(k)} = [1, v^{k-1}, v^{2(k-1)}, \ldots, v^{5(k-1)}]^T, \quad k = 1, \ldots, 6 \quad (123)$$

where

$$v = \exp(j2\pi/6) \quad (124)$$

More generally, for N symmetrically-disposed conductors in a circular shield we have

$$\underline{v}^{(k)} = [1, v^{k-1}, v^{2(k-1)}, \ldots, v^{(N-1)(k-1)}]^T, \quad k = 1, \ldots, N \quad (125)$$

where

$$v = \exp(j2\pi/N) \quad (126)$$

Furthermore, the vectors so established, form a linearly independent set (App. G).

We presume that this subject is well-covered in reference 14.

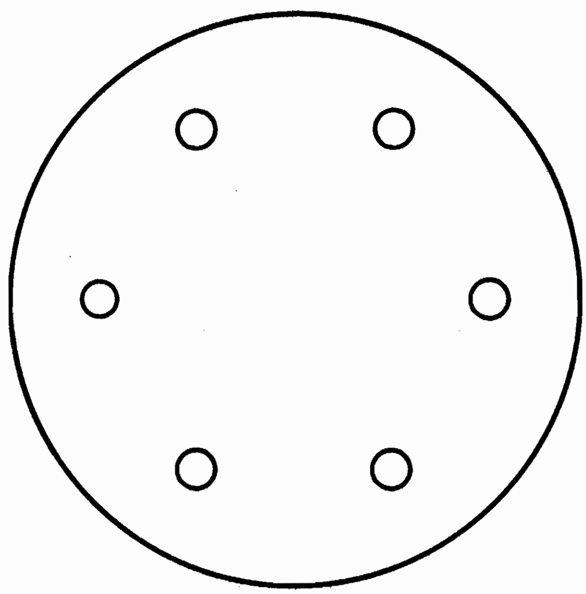

Fig. 1 Six Conductors Symmetrically Disposed in a Circular Shield.

In the meantime it must be stated that certain configurations of importance (e.g., Fig. 2) do not fit in this category. For such arrangements, it would appear necessary to accept the complexities inherent in the multi-mode solution for large N or to accept the errors inherent in the average-pole-residue solution, or else to become involved in a much more complicated step-by-step, iterative procedure that hardly seems justified by the overall improvement to be expected.

5. CONCLUSIONS

This report studies and contrasts two analytical methods for investigating the response of multiconductor lines to external fields. We have identified these methods as the bi-orthogonal multi-mode and the average-pole-residue method.

From open literature available to us[1,12], and from what we expect to find subsequently[14], we conclude that the multimode method is well-discussed elsewhere. Consequently, we have studied and sketched this approach to the problem somewhat superficially. We are satisfied that, at least for a line or cable with a moderate number of conductors (less than, say, twenty), this method is a powerful one.

The average-pole method requires that the resistance and internal inductance of the conductors be known in order that the system determinant, which contains the needed poles, may

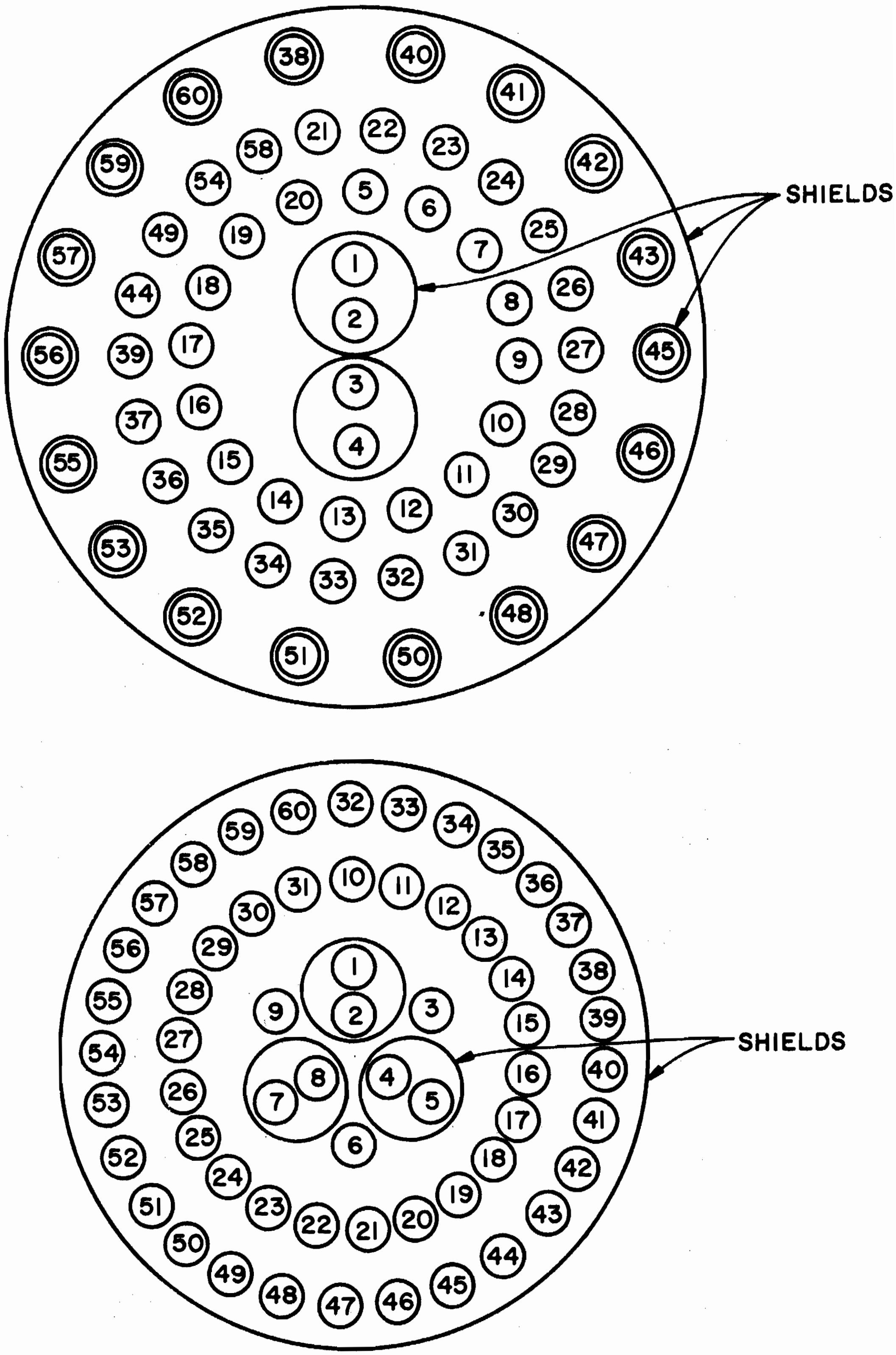

Fig. 2 Typical Large-N Multiconductor Cables.

be written. On the other hand, logically, not only can these quantities not be known until the solution of the problem is known, but they further complicate the problem by being functions of position along the line. However, they may be simply approximated by assuming uniform current distributions on the conductors. For the purpose of studying interference effects, the suggested approximation should be adequate, and, in any case, should overstate, slightly, the degree of interference expected. The method should be particularly advantageous for a cable with a large number (fifty or more) of conductors. The relatively small complexity it adds to the lossless conductor formulation appears to be commensurate with the gain in information accruing from inclusion of conductor losses in the model.

The chief pre-occupation in this report has been with the effect of conductor losses. The effect of mixed dielectrics has received passing mention because the analytical methods, particularly that of bi-orthogonal modes, appear to be suitable for that class of problems as well. The attractiveness of the average-pole method for this problem should vary, roughly, inversely with the spread in permittivities of the dielectrics involved.

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APPENDIX

A. LINEAR INDEPENDENCE OF THE VECTORS OF A BI-ORTHOGONAL PAIR.

For a bi-orthogonal pair we have [Eq. (14)]

$$\underline{\underline{V}}^T \underline{\underline{I}} = \underline{W} = [w_j \ \delta_{ij}] \qquad \text{(A-1)}$$

Assume, if possible, that the $\underline{V}^{(j)}$ are linearly dependent. In that case a constant vector

$$\underline{c} = [c_1, \ldots, c_N]^T$$

different from $\underline{0}$ exists, such that

$$\underline{c}^T \underline{\underline{V}}^T = \underline{0}$$

This is possible only if the determinant of $\underline{\underline{V}}$ is zero:

$$|\underline{\underline{V}}| = 0.$$

But, from equation (A-1),

$$|\underline{W}| = |\underline{\underline{V}}^T| \cdot |\underline{\underline{I}}| = |\underline{\underline{V}}| \cdot |\underline{\underline{I}}| = 0$$

which is contrary to hypothesis, since, by equation (A-1),

$$|\underline{W}| = |w_j \ \delta_{ij}| = \prod_{j=1}^{N} w_j \neq 0.$$

Therefore, the $\underline{V}^{(j)}$ are linearly independent. Proof that the $\underline{I}^{(j)}$ are linearly independent follows in an obvious manner.

B. CORRESPONDENCE BETWEEN EIGENVALUE MULTIPLICITY AND THE RANK OF THE CHARACTERISTIC MATRIX

The following discussion is based on the development in Reference 6, sections 10.13, pp. 303 ff. (q.v.).

Let $(\lambda \underline{I} - \underline{A})$ have eigenvalues, $\lambda_1, \lambda_2, \ldots \lambda_p$, where λ_k is of multiplicity, M_k. By means of a similarity transformation [loc.cit.]

$$\underline{B} = \underline{Q}^{-1} \underline{AQ} \tag{B-1}$$

the matrix, $\underline{A}$, equation (5), may be transformed to a matrix, B, with the same characteristic equation, i.e. the same characteristic determinant set equal to zero, and therefore the same eigenvalues, such that

$$\underline{B} = \text{diag.}\ (\underline{A}_1, \underline{A}_2, \ldots, \underline{A}_p) \tag{B-2}$$

where, by virtue of the symmetry of $\underline{A}$, $\underline{A}_k$ $(k = 1, \ldots, p)$ is reducible to a diagonal matrix of M_k rows and columns:

$$\underline{A}_k = \begin{bmatrix} \lambda_k, & 0, & 0, \ldots, & 0 \\ 0, & \lambda_k, & 0, \ldots, & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0, & 0, & 0, & ,\lambda_k \end{bmatrix}, \quad k = 1, \ldots, p \tag{B-3}$$

The characteristic equation is

$$\det.(\lambda \underline{I} - \underline{B}) = \begin{vmatrix} \lambda \underline{I} - \underline{A}_1, & 0 & , 0, \ldots, & 0 \\ 0 & , \lambda \underline{I} - \underline{A}_2, & 0, \ldots, & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & , 0 & , 0, \ldots, & (\lambda \underline{I} - \underline{A}_p) \end{vmatrix} = 0 \tag{B-4}$$

When we substitute $\lambda = \lambda_k$, it is clear that equation (B-4) will contain M_k rows and columns of zeros, whence its rank cannot exceed $(N - M_k)$. In fact, if we form the $(N - M_k)$-rowed determinant, $D_{(k)}$, obtained by excluding the M_k rows and columns of zeros, we get by Laplace's expansion

$$D_{(k)} = \prod_{\substack{i=1 \\ (k)}}^{N} \{\det.\ (\lambda_k \underline{I} - \underline{A}_i)\} \neq 0$$

since, by hypothesis, det. $(\lambda_k \underline{I} - \underline{A}_i) \neq 0$, $i \neq k$. Thus, det. $(\lambda_k \underline{I} - \underline{B})$, and therefore, det.$(\lambda_k \underline{I} - A)$, are of rank $(N - M_k)$.

C. BI-ORTHOGONAL MODES FOR A 2-LINE

From equations (27b, c),

$$\left.\begin{aligned} \sum_{k=1}^{N} \sum_{\ell=1}^{N} s_{k\ell}^{(i)}\ x_{\ell}^{(i)}\ x_{k}^{(j)} &= 0 \\ \sum_{k=1}^{N} \sum_{\ell=1}^{N} t_{k\ell}^{(i)}\ y_{\ell}^{(i)}\ y_{k}^{(j)} &= 0 \end{aligned}\right\} i \neq j$$

For homogeneous dielectrics and lossless conductors, the superscripts on $s_{k\ell}^{(i)}$ and $t_{k\ell}^{(i)}$ may be dropped:

$$\sum_{k=1}^{N} \sum_{\ell=1}^{N} s_{k\ell}\ x_{\ell}^{(i)}\ x_{k}^{(j)} = 0 \qquad \text{(C-1a)}$$

$$i \neq j$$

$$\sum_{k=1}^{N} \sum_{\ell=1}^{N} t_{k\ell} y_{\ell}^{(i)} y_{k}^{(j)} = 0 \qquad \text{(C-1b)}$$

Note that interchanging i and j in equations (C-1a,b) adds no new result. Thus, the condition $i \neq j$ changes to

$$i > j,\ j = 1, \ldots, (N - 1) \qquad \text{(C-2)}$$

We can use either of equations (C-1a,b) to discover sets of bi-orthogonal modes for a 2-line. Considering equation (C-1a) for a 2-line we have

$$1 + s_{12} \left(x_2^{(1)} + x_2^{(2)}\right) + s_{22}\, x_2^{(1)}\, x_2^{(2)} = 0 \qquad \text{(C-3)}$$

We have two unknown ratios and one equation. An infinite number of solutions is possible, subject to physical realizability constraints on the ζ's, including

$$0 \leq \left|\frac{\zeta_{12}^2}{\zeta_{11}\,\zeta_{22}}\right| < 1$$

$$\frac{\zeta_{22}}{\zeta_{11}} > 0$$

We can obtain another equation by imposing the condition that the mode set be independent of s_{12}. In that case,

$$x_2^{(1)} + x_2^{(1)} = 0 \qquad \text{(C-4a)}$$

and, consequently,

$$x_2^{(1)}\, x_2^{(2)} = -\, s_{22}^{-1} = -\frac{\zeta_{11}}{\zeta_{22}} \qquad \text{(C-4b)}$$

Solving these simultaneously yields

$$\frac{I_2^{(1)}}{I_1^{(1)}} = \pm \sqrt{\frac{\zeta_{11}}{\zeta_{22}}} = -\frac{I_2^{(2)}}{I_1^{(2)}} \qquad \text{(C-5)}$$

In the case of a symmetric line, $\zeta_{11} = \zeta_{22}$, and we get

$$\frac{I_2^{(1)}}{I_1^{(1)}} = \pm 1 = -\frac{I_2^{(2)}}{I_1^{(2)}} \qquad \text{(C-6)}$$

If we choose the upper sign, $\underline{I}^{(1)}$ is the so-called even, or common, or unbalanced-line mode, while $\underline{I}^{(2)}$ is the odd, or balanced-line mode.

Still, restricting discussion to the symmetric line, if we remove the restriction that the mode solution be independent of ζ_{12}, we have to satisfy

$$1 + s_{12} \left(x_2^{(1)} + x_2^{(2)}\right) + x_2^{(1)} x_2^{(2)} = 0 \qquad \text{(C-7)}$$

$x_2^{(2)}$ is plotted as a function of $x_2^{(1)}$ in Figure C-1. Values of $x_2^{(2)} = \pm \infty$ are best interpreted as $I_1^{(2)} = 0$.

The voltage vectors are then readily obtained from equations (14b):

$$V_1^{(1)} I_1^{(2)} + V_2^{(1)} I_2^{(2)} = 0$$
$$V_1^{(2)} I_1^{(1)} + V_2^{(2)} I_2^{(1)} = 0$$

that is,

$$\left.\begin{aligned} 1 + y_2^{(1)} x_2^{(2)} &= 0 \\ 1 + y_2^{(2)} x_2^{(1)} &= 0 \end{aligned}\right\} \qquad \text{(C-8)}$$

whence

$$\frac{V_2^{(1)}}{V_1^{(1)}} = - \frac{I_1^{(2)}}{I_2^{(2)}}$$

$$\frac{V_2^{(2)}}{V_1^{(2)}} = - \frac{I_1^{(1)}}{I_2^{(1)}} \qquad \text{(C-9)}$$

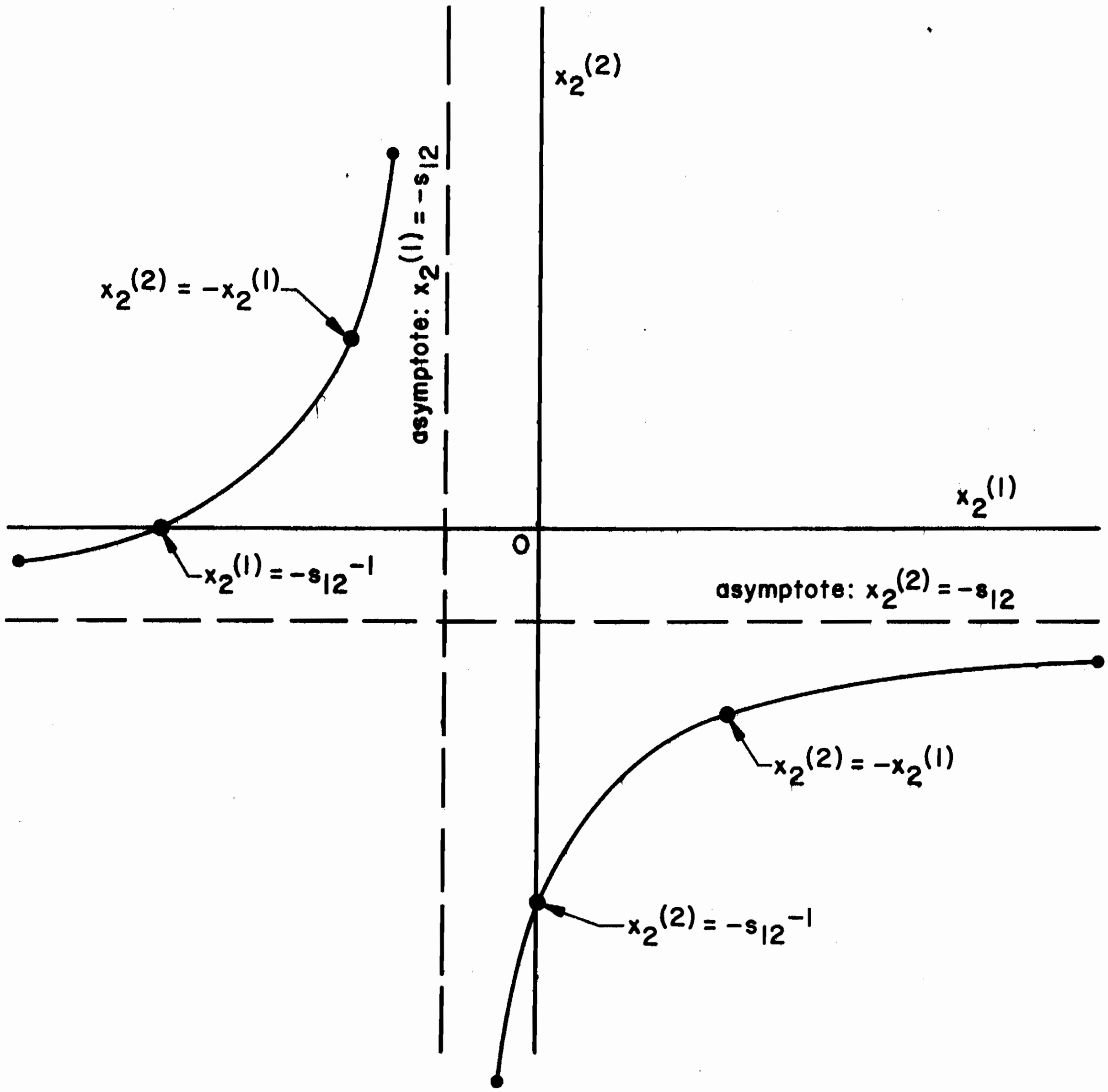

Fig. C-1 Relationship Between Mode-current Ratios for a 2-line.

In the generally unsymmetric case, if modes are limited to those independent of ζ_{12},

$$\frac{V_2^{(1)}}{V_1^{(1)}} = \pm \quad \frac{\zeta_{22}}{\zeta_{11}} = - \frac{V_2^{(2)}}{V_1^{(2)}} \tag{C-10}$$

and for the special case of symmetric lines,

$$\frac{V_2^{(1)}}{V_1^{(1)}} = \pm 1 = - \frac{V_2^{(2)}}{V_1^{(2)}} \tag{C-11}$$

Sometimes (see, for instance, Reference 1, Chapter 3), for unsymmetrical 2-lines, it is assumed that

$$x_2^{(1)} = - 1 = \frac{I_2^{(1)}}{I_1^{(1)}} \tag{C-12a}$$

In that case,

$$x_2^{(2)} = \frac{1 - s_{12}}{s_{22} - s_{12}} = \frac{\zeta_{11} - \zeta_{12}}{\zeta_{22} - \zeta_{12}} = \frac{I_2^{(2)}}{I_1^{(2)}} \tag{C-12b}$$

and

$$y_2^{(1)} = - \frac{1}{x_2^{(2)}} = - \frac{\zeta_{11} - \zeta_{12}}{\zeta_{22} - \zeta_{12}} = \frac{V_2^{(1)}}{V_1^{(1)}} \tag{C-13a}$$

$$y_2^{(2)} = - \frac{1}{x_2^{(1)}} = 1 = \frac{V_2^{(2)}}{V_1^{(1)}} \tag{C-13b}$$

Thus, for the general unsymmetric 2-line, if one mode is chosen as an odd-current mode, the other mode is an even-voltage mode. The fact that an even-current mode, chosen for the first mode, leads to an odd-voltage mode for the second, is clear on reversing the sign of $x_2^{(1)}$ in equation (C-12a).

To continue illustrating the results obtained in section 2, assume we are dealing with a symmetrical 2-line above ground (Fig. C-2). The dielectric conductivity and permittivity are g_d and ε_d respectively. The conductors all have conductivity, g_c. For simplicity we assume h/a, D/a >> 1.

As we have just seen, a set of bi-orthogonal modes independent of line parameters is

$$\begin{aligned} \underline{I}^{(1)} &= [I_1^{(1)}, \; I_1^{(1)}]^T \\ \underline{I}^{(2)} &= [I_1^{(2)}, \; - I_1^{(2)}]^T \\ \underline{V}^{(1)} &= [V_1^{(1)}, \; V_1^{(1)}]^T \\ \underline{V}^{(2)} &= [V_1^{(2)}, \; - V_1^{(2)}]^T \end{aligned} \qquad \text{(C-14)}$$

Furthermore, assuming g_c to be infinite, initially, we have (Ref. 1)

$$\underline{V}^{(j)} = \underline{ZI}^{(j)}, \; j = 1,\ldots, N \qquad \text{(C-15)}$$

where $\underline{Z}$ is the line impedance matrix

$$\underline{Z} = [Z_{ij}] = \begin{bmatrix} Z_{11}, & Z_{12} \\ Z_{21}, & Z_{22} \end{bmatrix}; \; Z_{11} = Z_{22}; \; Z_{21} = Z_{12} \qquad \text{(C-16)}$$

Equation (C-15) applies to any combination of the modes, as well as total potentials and currents on the conductors.

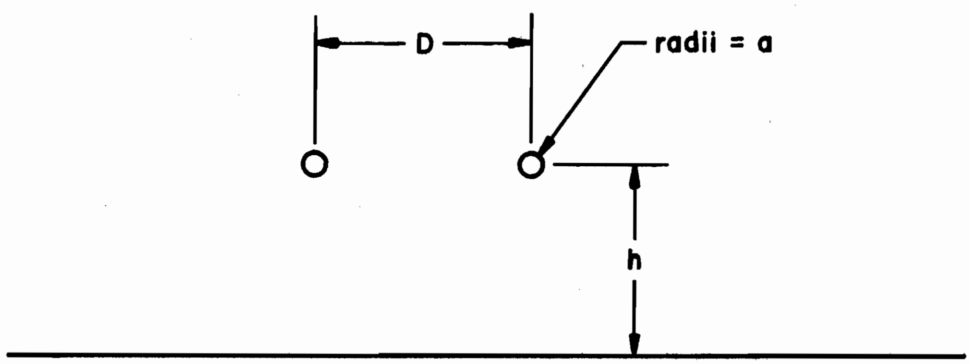

Fig. C-2 Symmetric 2-line above ground.

For the dielectric we have (eq. (38a)),

$$\alpha_d = \frac{1}{2}\,\delta_d\,k_o$$

$$\delta_d = \frac{g_d}{\omega\varepsilon_d}$$

$$k_o = \omega(\mu_d\varepsilon_d)^{1/2} = \frac{2\pi}{\lambda_d}$$

where λ_d is the propagation wavelength in the dielectric.

For the conductors we have (Ref. 5, Chap. 5)

$$\alpha_c^{(1)} = \frac{R_o}{8\pi a Z_{0u}}\left\{1+2\left(\frac{a}{D}\right)\left[\xi\ \frac{2+\xi^2}{1+\xi^2}\right]+2\left(\frac{a}{D}\right)^2\left[\frac{1+3\xi^2+\xi^4}{(1+\xi^2)}\right]\right\}$$

$$\alpha_c^{(2)} = \frac{R_0}{2\pi a Z_{0b}}\left\{1+2\left(\frac{a}{D}\right)\left[\frac{\xi^3}{1+\xi^2}\right]+2\left(\frac{a}{D}\right)^2\left[\frac{\xi^6}{(1+\xi^2)^2}\right]\right\} \qquad \text{(C-17)}$$

where

$$R_0 = \left(\frac{\pi f \mu_c}{g_c}\right)^{1/2} \quad \text{ohms/square} \qquad \text{(C-18)}$$

$$\left.\begin{aligned} Z_{0u} &= \frac{1}{2}\,(Z_{11} + Z_{12}) \\ Z_{0b} &= 2\,(Z_{11} - Z_{12}) \end{aligned}\right\} \qquad \text{(C-19)}$$

$$\xi = \frac{D}{2h} \qquad \text{(C-20)}$$

f = frequency, Hz.

= $4\pi \times 10^{-7}$ Hy/m for non-ferrous metals

g_c = 5.8×10^{7} mho/m for copper

Therefore

$$R_0 = 2.61 \times 10^{-7}\, f^{1/2} \text{ ohms/square} \qquad \text{(C-21)}$$

Furthermore (Ref. 5, p. 4-17), ignoring conductor internal inductance,

$$\left.\begin{aligned} Z_{11} &= \frac{\eta_d}{2\pi} \ln \frac{2h}{a} \\ Z_{12} &= \frac{\eta_d}{2\pi} \ln \frac{S}{D} \end{aligned}\right\} \qquad \text{(C-22)}$$

where

$$S = (D^2 + 4h^2)^{1/2}; \; S/D = (1 + \frac{1}{\xi^2})^{1/2}$$

For a non-ferrous dielectric,

$$\eta_d = \frac{120\pi}{\sqrt{\varepsilon_r}} \qquad \text{(C-23)}$$

where ε_r is the dielectric constant relative to free space. Then equations (C-21 are

$$\left.\begin{aligned} Z_{11} &= \frac{60}{\sqrt{\varepsilon_r}} \ln \frac{2h}{a} \\ Z_{12} &= \frac{60}{\sqrt{\varepsilon_r}} \ln \frac{S}{D} \end{aligned}\right\} \qquad \text{(C-24)}$$

Choose constants as follows:

$$\varepsilon_r = 2.3$$

$$\frac{h}{a} = 5$$

$$\xi = \frac{D}{2h} = 1; \text{ therefore } \left(\frac{a}{D} \right) = \frac{1}{10}$$

Then

$$Z_{11} = Z_{22} = \frac{60}{\sqrt{2.3}} \ln(10) = 91 \text{ ohms}$$

$$Z_{12} = Z_{21} = \frac{60}{\sqrt{2.3}} \ln(1+1)^{1/2} = 27.4 \text{ ohms}$$

Therefore

$$Z_{0u} = \frac{1}{2} (91 + 27.4) = 59.2 \text{ ohms}$$

$$Z_{0b} = 2 (91 - 27.4) = 127.2 \text{ ohms}$$

Take $a = 2.5 \times 10^{-3}$ m.

$f = 10^{8}$ Hz.

Then

$$R_0 = 2.67 \times 10^{-7} \times 10^{4} = 2.67 \times 10^{-3} \text{ ohms/square}$$

Substituting the foregoing results in equations (C-17) yields

$$\alpha_c^{(1)} = 0.968 \times 10^{-3} \text{ Np./m.} = 8.4 \times 10^{-3} \text{ db/m.}$$

$$\alpha_c^{(2)} = 1.48 \times 10^{-3} \text{ Np/m.} = 1.28 \times 10^{-2} \text{ db/m.}$$

To compute α_d, take

$$\delta_d = 2 \times 10^{-4}$$

$$\lambda_d = \frac{v}{f} = \frac{3 \times 10^{8}}{10^{8} \times \sqrt{2.3}} = 1.982$$

whence

$$k_o = \frac{2\pi}{1.982} = 3.17/\text{m.}$$

and

$$\alpha_d = \frac{1}{2} (2 \times 10^{-4})(3.17) = 3.17 \times 10^{-4}$$

The total attenuation constants are

$$\alpha^{(1)} = 8.7 \times 10^{-3} \text{ /m.}$$

$$\alpha^{(2)} = 1.31 \times 10^{-3} \text{ /m.}$$

For a semi-infinite line excited at one end,

$$I_1 = I_1^{(1)} + I_1^{(2)}$$
$$= I_1^{(1)}(0)\ \exp\{-[8.7 \times 10^{-3} + j3.17]x\}$$
$$+ I_1^{(2)}(0)\exp\{-[1.31 \times 10^{-3} + j3.17]x\}$$
$$I_2 = I_2^{(1)} + I_2^{(2)}$$
$$= I_1^{(1)}(0)\exp\{-[8.7 \times 10^{-3} + j3.17]x\}$$
$$- I_1^{(2)}(0)\exp\{-[1.31 \times 10^{-3} + j3.17]x\}$$
$$V_1 = V_1^{(1)} + V_1^{(2)}$$
$$= (Z_{11}I_1^{(1)} + Z_{12}I_2^{(1)}) + (Z_{11}I_1^{(2)} + Z_{12}I_2^{(2)})$$
$$= (Z_{11} + Z_{12})\ I_1^{(1)} + (Z_{11} - Z_{12})\ I_1^{(2)}$$
$$= 2\ Z_{0u}I_1^{(1)} + \frac{1}{2}\ Z_{0b}\ I_1^{(2)}$$
$$= 118.4\ I_1^{(1)}(0)\exp\{-[8.7 \times 10^{-3} + j3.17]x\}$$
$$+ 63.6\ I_1^{(2)}(0)\exp\{-[1.31 \times 10^{-3} + j3.17]x\}$$
$$V_2 = V_2^{(1)} + V_2^{(2)}$$
$$= (Z_{12}I_1^{(1)} + Z_{22}I_2^{(1)}) + (Z_{12}I_1^{(2)} + Z_{22}I_2^{(2)})$$
$$= 2\ Z_{0u}I_1^{(1)} - \frac{1}{2}\ Z_{0b}I_1^{(2)}$$
$$= 118.4\ I_1^{(1)}(0)\exp\{-[8.7 \times 10^{-3} + j3.17]\}$$
$$- 63.6\ I_1^{(2)}(0)\exp\{-[1.31 \times 10^{-3} + j3.17]x\}$$

The two unknowns, $I_1^{(1)}(0)$ and $I_2^{(2)}(0)$ must be determined from the terminal conditions. For instance, if the initial currents are given, write

$$I_1(0) = I_1^{(1)}(0) + I_1^{(2)}(0)$$
$$I_2(0) = I_2^{(1)}(0) + I_2^{(2)}(0) = I_1^{(1)}(0) - I_1^{(2)}(0)$$

whence

$$I_1^{(1)}(0) = \frac{1}{2}[I_1(0) + I_2(0)]$$

$$I_1^{(2)}(0) = \frac{1}{2}[I_1(0) - I_2(0)]$$

thus completing the solution.

D. CONVOLUTION INTEGRAL FOR THE TRANSFORM OF THE PRODUCT OF A SQUARE MATRIX AND A VECTOR.

Let $\underline{M}(x)$ be an N x N square matrix function of x, and let $\underline{P}(x)$ be an N-rowed vector. Let the corresponding Laplace transforms be $\tilde{\underline{M}}(p)$ and $\tilde{\underline{P}}(p)$ respectively.

Given

$$\tilde{\underline{G}}(p) = \tilde{\underline{M}}(p)\ \tilde{\underline{P}}(p) \qquad \text{(D-1)}$$

an expression for $\underline{G}(x)$ is derived as follows:

$$\tilde{\underline{G}}(p) = [\tilde{M}_{ij}(p)][\tilde{P}_i(p)]$$

$$= \left[\sum_{j=1}^{N} \tilde{M}_{ij}\ \tilde{P}_j\right] = [\tilde{G}_i(p)] \qquad \text{(D-2)}$$

From scalar transform theory (Ref. 11), if

$$\tilde{G}_i(p) = \sum_{j=1}^{N} \tilde{M}_{ij}(p)\ \tilde{P}_j(p) \qquad \text{(D-3)}$$

then

$$G_i(x) = L^{-1}\{G_i(p)\} = L^{-1}\ \{\sum_{j=1}^{N} \tilde{M}_{ij}(p)\ \tilde{P}_j(p)\}$$

$$= \sum_{j=1}^{N} \{L^{-1}(\tilde{M}_{ij}(p)\ \tilde{P}_j(p))\}$$

$$= \sum_{j=1}^{N} (M_{ij}(x) * P_j(x))$$

and

$$\underline{G}(x) = [G_i(x)] = \left[\sum_{j=1}^{N} (M_{ij}(x) * P_j(x))\right]$$

$$= [M_{ij}(x)] * [P_j(x)] = \underline{M}(x) * \underline{P}(x)$$

$$= \int_0^x \underline{M}(x-\xi)\ \underline{P}(\xi)d\xi = \int_0^x \underline{M}(\xi)\underline{P}(x-\xi)d\xi \qquad \text{(D-4)}$$

E. EVALUATION OF A RESIDUE AT A MULTIPLE POLE

From equation (87) write

$$|\underline{Q}| = \Pi_+(p) \cdot \Pi_-(p)$$

where

$$\Pi_\pm(p) = \prod_{i=1}^{N} (p \pm p_i)$$

To evaluate the residues at the poles in R_+, note that, for this purpose, p is confined to R_+ and its boundary, so that

$$\Pi_+(p) = \prod_{i=1}^{N} (p-p_i) = \prod_{i=1}^{N} (p-p_a-\delta_i)$$

$$= (p-p_a)^N \prod_{i=1}^{N} \left(1 - \frac{\delta_i}{p-p_a}\right)$$

and

$$\Pi_-(p) = \prod_{i=1}^{N} (p+p_i) = \prod_{i=1}^{N} (p+p_a+\delta_i)$$

$$\approx \prod_{i=1}^{N} (p+p_a) = (p+p_a)^N$$

thus,

$$|\underline{Q}|^{-1} \approx \left[(p^2 - p_a{}^2)^N \prod_{i=1}^{N} (1 - \frac{\delta_i}{p - p_a}) \right]^{-1}$$

$$= (p - p_a)^{-N} \sum_{s=0}^{\infty} \frac{S_s}{(p-p_a)^s} \qquad \text{(E-1)}$$

where

$$\left. \begin{aligned} S_0 &= 1 \\ S_1 &= \sum_{i=1}^{N} \delta_i = 0 \\ S_2 &= \sum_{i=1}^{N} \sum_{j=1}^{i} \delta_i \, \delta_j \end{aligned} \right\} \qquad \text{(E-2)}$$

and, in general, S_s is of the order of $\delta_{i_1} \delta_{i_2} \ldots, \delta_{i_s}$, where each $i_1, \ldots, i_s$ is one of the integers from 1 to N.

Next, turning attention to the numerator of equation (76) we have noted previously that the cofactors, Q_{ij}, of $\underline{Q}$ are polynomials of order (N-1) in p^2, at most. In fact, a little study shows that the cofactors of the diagonal elements are of order (N-1) in p^2, while all off-diagonal elements are of order (N-2) in p^2, at most. If jk_o is the propagation constant of the line in the absence of conductor losses (k_o generally complex), the Q_{ij} are available in the first instance from $\underline{Q}$ in the form

$$Q_{ij} = \sum_{r=1}^{N} b_{ij}^{(r)} (p^2 + k_o^2)^{N-r}$$

where the coefficient of the leading term is δ_{ij}, while, for $r > 1$,

$$b_{ij}^{(r)} = \prod_{k=1}^{r-1} \varepsilon_{i_k j_k}, \quad i_k, j_k = 1,\ldots, N; \quad \left|\frac{\varepsilon_{i_k j_k}}{k_o^2}\right| \ll 1$$

Thus

$$b_{ij}^{(1)} = \delta_{ij}, \quad i, j, = 1,\ldots, N$$

$$\left|\frac{b_{ij}^{(r+t)}}{k_o^{2(r+t-1)}}\right| \ll \left|\frac{b_{ij}^{(r)}}{k_o^{2(r-1)}}\right| \ll 1, \quad r > 1, \ t \text{ pos. int.}$$

These may be re-expressed in powers of $(p^2 - p_a^2)$ by use of Taylor's theorem:

$$Q_{ij} = \sum_{s=0}^{N-1} \left[\frac{\partial Q_{ij}(p^2)}{\partial p^2}\right]_{p=p_a} \frac{(p^2-p_a^2)^s}{s!}$$

or, by changing the summation index, thus:

$$s = N - r; \quad r = N - s$$

$$Q_{ij} = \sum_{r=1}^{N} a_{ij}^{(r)} (p^2 - p_a^2)^{N-r}, \quad i, j = 1,\ldots, N \tag{E-3}$$

where

$$\left.\begin{aligned} a_{ij}^{(1)} &= \delta_{ij} \\ a_{ij}^{(r)} &= \left[\frac{\partial^{N-r} Q_{ij}(p^2)}{\partial (p^2)^{N-r}}\right]_{p=p_a} \bigg/ [(N-r)!] \\ \left|\frac{a_{ij}^{(r+t)}}{p_a^{2(r+t-1)}}\right| &\ll \left|\frac{a_{ij}^{(r)}}{p_a^{2(r-1)}}\right| \ll 1, \quad r > 1, \ t \text{ pos. int.} \end{aligned}\right\} \tag{E-4}$$

Finally, expand the exponential of equation (76) into a series in $(p - p_a)$:

$$\begin{aligned} \exp(px) &= \exp(p_a x)\exp(p - p_a x) \\ &= \exp(p_a x) \end{aligned} \qquad \text{(E-5)}$$

Substitution of equations (E-1), (E-3), and (E-5) for the element of the $\underline{i}$th row, $\underline{j}$th column of $\underline{T}^{(2)}(x)$ in equation (76) yields, for the poles around p_a:

$$\text{Res}(p_a) = \frac{1}{j2\pi}\int_{(p_a)} \frac{\left[\sum_{q=1}^{N} a_{ji}^{(q)} (p^2-p_a{}^2)^{N-q}\right]\left[e^{p_a x} \sum_{r=0}^{\infty} \frac{(p-p_a)^r x^r}{r!}\right]\left[\sum_{s=0}^{\infty} \frac{S_s}{(p-p_a)^s}\right]}{(p^2-p_a{}^2)^N} dp$$

$$= e^{p_a x} \sum_{q=1}^{N} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} a_{ji}^{(q)} S_s \left(\frac{x^r}{r!}\right)\left\{\frac{1}{j2\pi}\int_{(p_a)} \frac{(p-p_a)^{r-q-s}}{(p+p_a)^q} dp\right\} \qquad \text{(E-6)}$$

Next, expand $(p + p_a)^{-q}$ in powers of $(p - p_a)$:

$$(p+p_a)^{-q} = (2\ p_a)^{-q} \sum_{m=0}^{\infty} (-1)^m \binom{q+m-1}{m} \left(\frac{p-p_a}{2p_a}\right)^m$$

where

$$\binom{q+m-1}{m} = \text{binomial coefficient}$$

$$= \frac{(q+m-1)(q+m-2)\ldots,(q+1)q}{m!}$$

$$= \binom{q+m-1}{q-1}$$

Equation (E-6) becomes

$$\mathrm{Res}(p_a) = e^{p_a x} \sum_{q=1}^{N} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \sum_{m=0}^{\infty} (-1)^m \binom{q+m-1}{q-1} \frac{a_{ji}^{(q)}}{(2p_a)^{q+m}} S_s\left(\frac{x^r}{r!}\right) C_\alpha(p_a) \qquad \text{(E-7)}$$

where

$$\left.\begin{aligned} C_\alpha &= \frac{1}{j2\pi} \int_{(p_a)} (p-p_a)^\alpha \, dp \\ \alpha &= r + m - q - s \end{aligned}\right\} \qquad \text{(E-7a)}$$

The residue at $p = p_a$ is the coefficient of C_{-1}; that is, it is the coefficient corresponding to the constraint

$$\alpha = r + m - q - s = -1 \qquad \text{(E-8)}$$

We are interested in terms corresponding to the first few values of q and s. For specific values of these parameters, equation (E-8) implies

$$r + m = q + s - 1 \geqq 0$$

Furthermore,

$$m = (q + s - 1) - r \geqq 0$$

which implies that

$$r \leqq q + s - 1$$

Making use of these facts we have, for the residue at p_a

$$\mathrm{Res}(p_a) = e^{p_a x} \sum_{q=1}^{N} \sum_{r=0}^{q+s-1} \sum_{s=0}^{\infty} \frac{a_{ji}^{(q)}}{(2p_a)^q} S_s\left(\frac{x^r}{r!}\right) \frac{(-1)^{q+s-r-1}}{(2p_a)^{q+s-r-1}} \binom{2q+s-r-2}{q-1} \quad \text{(E-9)}$$

Following the same procedure in the region R_- we get

$$\mathrm{Res}(-p_a) = e^{-p_a x} \sum_{q=1}^{N} \sum_{r=0}^{q+s-1} \sum_{s=0}^{\infty} \frac{a_{ji}^{(q)}}{(2p_a)^q} S_s\left(\frac{x^r}{r!}\right) \frac{(-1)^{q+s}}{(2p_a)^{q+s-r-1}} \binom{2q+s-r-2}{q-1} \quad \text{(E-10)}$$

The solution for $T_{ij}^{(2)}$ is

$$T_{ij}^{(2)}(x) = \mathrm{Res}(p_a) + \mathrm{Res}(-p_a)$$

$$= \sum_{q=1}^{N} \frac{a_{ji}^{(q)}}{(4p_a^2)^{q-1}} F(x;p_a;q) \quad \text{(E-11)}$$

where

$$F = \sum_{r=0}^{q+s-1} \sum_{s=0}^{\infty} \frac{(-1)^{q+s-r-1}}{(2p_a)^{s-r+1}} S_s\left(\frac{x^r}{r!}\right)\binom{2q+s-r-2}{q-1}\left[e^{p_a x} - (-1)^r e^{-p_a x}\right] \quad \text{(E-12)}$$

We are particularly interested in values of F for q = 1,2; with the help of equations (E-2) we get

$$F(x;p_a;1) = \frac{1}{p_a}\left\{\sinh p_a x + \frac{S_2}{(2p_a)^2}\left[(1+2p_a^2x^2)\sinh p_a x - 2\,p_a x\cdot\cosh p_a x\right] + \dots\right\} \quad \text{(E-13)}$$

$$F(x;p_a;2) = -\frac{2}{p_a}\left\{\left[\sinh p_a x - p_a x \cosh p_a x\right] - \frac{S_2}{p_a^3}\left[(1+p_a^2x^2)\sinh p_a x - 3p_a x(1+\tfrac{1}{9}p_a^2x^2)\cosh p_a x\right] + \dots\right\} \quad \text{(E-14)}$$

F. ADDITIONAL DETAILS - FINAL APPROXIMATE SOLUTION

Starting with equations (74) and (81) and using the revised notation for terminal voltages and currents, write

$$\underline{V}^{o} = \underline{T}^{(1)}(\ell)\underline{V}^{i} - \underline{T}^{(2)}(\ell)\underline{\zeta}\ \underline{I}^{i} + \underline{T}^{(3)}(\ell) - \underline{T}^{(4)}(\ell)$$

$$\underline{I}^{o} = \underline{R}^{(1)}(\ell)\underline{I}^{i} - \underline{R}^{(2)}(\ell)\underline{\eta}\ \underline{V}^{i} + \underline{R}^{(3)}(\ell) - \underline{R}^{(4)}(\ell)$$

Substitute the first of equations (100) for $\underline{I}^{i}$:

$$\underline{V}^{o} = \underline{T}^{(1)}(\ell)\underline{V}^{i} + \underline{T}^{(2)}(\ell)\underline{\zeta}\ \underline{Y}^{i}\underline{V}^{i} + \underline{T}^{(3)}(\ell) - \underline{T}^{(4)}(\ell)$$

$$\underline{I}^{o} = -\ \underline{R}^{(1)}(\ell)\underline{Y}^{i}\underline{V}^{i} - \underline{R}^{(2)}(\ell)\underline{\eta}\ \underline{V}^{i} + \underline{R}^{(3)}(\ell) - \underline{R}^{(4)}(\ell)$$

Combine these in the second of equations (100):

$$-\ \underline{R}^{(1)}(\ell)\underline{Y}^{i}\underline{V}^{i} - \underline{R}^{(2)}(\ell)\underline{\eta}\ \underline{V}^{i} + \underline{R}^{(3)}(\ell) - \underline{R}^{(4)}(\ell) =$$

$$\underline{Y}^{o}[\underline{T}^{(1)}(\ell)\underline{V}^{i} + \underline{T}^{(2)}(\ell)\underline{\zeta}\ \underline{Y}^{i}\underline{V}^{i} + \underline{T}^{(3)}(\ell) - \underline{T}^{(4)}(\ell)]$$

which reduces to

$$\{[\underline{R}^{(1)}(\ell)\underline{Y}^{i} + \underline{R}^{(2)}(\ell)\underline{\eta}] + \underline{Y}^{o}[\underline{T}^{(1)}(\ell) + \underline{T}^{(2)}(\ell)\underline{\zeta}\ \underline{Y}^{i}]\}\underline{V}^{i}$$

$$= \{[\underline{R}^{(3)}(\ell) - \underline{R}^{(4)}(\ell)] - \underline{Y}^{o}[\underline{T}^{(3)}(\ell) - \underline{T}^{(4)}(\ell)]\} \qquad \text{(F-1)}$$

Write

$$\underline{S} = \frac{\underline{\zeta}}{p_a}\{[\underline{R}^{(1)}(\ell)Y^{i} + \underline{R}^{(2)}(\ell)\underline{\eta}] + \underline{Y}^{o}[\underline{T}^{(1)}(\ell) + \underline{T}^{(2)}(\ell)\underline{\zeta Y}^{i}]\}$$

$$\underline{K}(\ell) = \frac{\underline{\zeta}}{p_a}\{[\underline{R}^{(3)}(\ell) - \underline{R}^{(4)}(\ell)] - \underline{Y}^{o}[\underline{T}^{(3)}(\ell) - T^{(4)}(\ell)]\} \qquad \text{(F-2)}$$

It will be noted that $\underline{S}$ and $\underline{K}(\ell)$ are the bracketed quantities in the left and right members of equation (F-1) respectively, except that each has been multiplied by the factor, $\underline{\zeta}/p_a$. This has been done in order that the terminology will conform with that of reference 9 when the results are reduced to the lossless-conductor case.

Substituting equations (F-2) in (F-1) and solving for $\underline{V}^i$,

$$\underline{V}^i = \underline{S}^{-1}\ \underline{K}(\ell) \tag{F-3a}$$

while, with the help of the first of equations (100), we get

$$\underline{I}^i = -\ \underline{Y}^i \underline{S}^{-1}\ \underline{K}(\ell) \tag{F-3b}$$

Finally, using these results in equations (74) and (81) and re-arranging terms yields equations (101) and (102) of the main text.

G. PROOF THAT THE VOLTAGE VECTORS OF A CERTAIN POLYPHASE SET ARE LINEARLY INDEPENDENT.

Start with equations (125) and (126) defining the vector set

$$\underline{v}^{(k)} = [1,\ v^{k-1},\ v^{2(k-1)}, \ldots,\ v^{(N-1)(k-1)}]^T$$

where

$$v = \exp(j2\pi/N)$$

The proof consists in showing that the Gram determinant

$$|\Gamma| = \begin{vmatrix} \underline{v}^{(1)T}\ \underline{v}^{(1)}, & \underline{v}^{(1)T}\ \underline{v}^{(2)}, \ldots, & \underline{v}^{(1)T}\ \underline{v}^{(N)} \\ \underline{v}^{(2)T}\ \underline{v}^{(1)}, & \underline{v}^{(2)T}\ \underline{v}^{(2)}, \ldots, & \underline{v}^{(2)T}\ \underline{v}^{(N)} \\ \cdot & \cdot \quad \cdot & \cdot \\ \underline{v}^{(N)^T}\ \underline{v}^{(1)}, & \underline{v}^{(N)^T}\ \underline{v}^{(2)}, \ldots, & \underline{v}^{(N)^T}\ \underline{v}^{(N)} \end{vmatrix} = [v^{(i)^T}\ \underline{v}^{(j)}] \tag{G-1}$$

is different from zero. (Ref. 6, Sec. 10.8, p. 297). By equation (125) we have

$$\underline{v}^{(i)^T} \underline{v}^{(j)} = 1 + v^{i+j-2} + (v^{i+j-2})^2 + (v^{i+j-2})^{N-1} \quad \text{(G-2)}$$

By equation (126) we have

$$v^{kN} = 1, \text{ k any integer, including zero;}$$

i.e.,

$$(v^k)^N - 1 = 0$$

which factors into

$$(v^k - 1)[(v^k)^{N-1} + (v^k)^{N-2} + \ldots + (v^k)^2 + v^k + 1] = 0 \quad \text{(G-3)}$$

If we set $k = i + j - 2$, then k ranges from $k = 0$, corresponding to $i = j = 1$, to $k = 2N - 2$, corresponding to $i = j = N$. In the permitted range,

$$v^k - 1 = 0$$

only for $k = 0, N$. For all other values in

$$0 \leqq k = i + j - 2 \leqq 2N - 2,$$

the expression, (G-3), is zero only if the right member of equation (G-2) is zero. That is,

$$\underline{v}^{(i)^T} \underline{v}^{(j)} = 0, \quad 0 \leqq i + j - 2 \leqq 2N - 2$$

provided that

$$i + j - 2 \neq 0, N$$

Consider the values of i, j, for which $\underline{v}^{(i)^T} \underline{v}^{(j)} \neq 0$.

We have,

$$\text{for } i = 1,\ j = 1$$
$$i = 2,\ j = N$$
$$i = 3,\ j = N - 1$$
$$\cdots \quad \cdots$$
$$i = N,\ j = 2$$

Therefore,

$$|\Gamma| = \begin{vmatrix} \underline{V}^{(1)T}\,\underline{V}^{(1)}, & 0 & ,0,\ldots,0, & 0 & , & 0 \\ 0 & , & 0 & ,0,\ldots,0, & 0 & ,\underline{V}^{(2)T}\,\underline{V}^{(N)} \\ 0 & , & 0 & ,0,\ldots,0,\underline{V}^{(3)T}\,\underline{V}^{(N-1)}, & & 0 \\ \cdot & & \cdot & \cdot\ \cdot\ \cdot & \cdot & \cdot \\ 0 & ,\underline{V}^{(N)T}\,\underline{V}^{(2)} & & ,0,\ldots,0, & 0 & , & 0 \end{vmatrix}$$

$$= (-1)^{N-1} \begin{vmatrix} 0 & ,0,0,\ldots, & 0 & ,\underline{V}^{(1)T}\,\underline{V}^{(1)} \\ 0 & ,0,0,\ldots,\underline{V}^{(2)T}\,\underline{V}^{(N)}, & & 0 \\ \cdot & \cdot\ \cdot\ \cdot & \cdot & \cdot \\ \underline{V}^{(N)T}\,\underline{V}^{(2)} & ,0,-,\ldots, & 0 & , & 0 \end{vmatrix}$$

$$= (-1)^{N-1}(-1)^{\frac{1}{2}N(N-1)}(\underline{V}^{(1)T}\,\underline{V}^{(1)})(\underline{V}^{(2)T}\,\underline{V}^{(N)})(V^{(3)T}\,\underline{V}^{(N-1)})\ldots$$
$$(\underline{V}^{(N)T}\,\underline{V}^{(2)})$$

The first product is

$$\underline{V}^{(1)T}\,\underline{V}^{(1)} = [1,\ldots,1]^T\,[1,\ldots,1] = N$$

For the remaining factors we have the product

$$\prod_{r=1}^{N=1} (\underline{V}^{(r+1)^T} \underline{V}^{(N-r+1)})$$

$$= \prod_{r-1}^{N-1} [1, v^r, v^{2r}, \ldots, v^{(N-1)r}]^T [1, v^{N-r}, v^{2(N-r)}, \ldots, v^{(N-1)(N-r)}]$$

$$= \prod_{r=1}^{N-1} [1 + v^N + v^{2N} + \ldots + v^{(N-1)N}] = \prod_{r=1}^{N-1} (N) = N^{N-1}$$

whence

$|\Gamma| = (-1)^{\frac{1}{2}(N-1)(N+2)} N^N \neq 0$. Therefore, the vectors of the polyphase set are linearly independent.

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