WL-EMP-IN-127

Interaction Notes

Note 127

26 April 1972

Currents Induced on a Plane and Sphere Immersed in a Time-varying Conductive Medium

Roger C. Kennedy
The Boeing Company
Seattle, Washington

Abstract

Strong electromagnetic fields along with a time-varying air conductivity are generated from nearby nuclear surface bursts. This note describes an analysis of currents induced on a plane and a sphere in such an environment. The view point is taken that these scatterers act as the generators of the scattered field with an incident field as the source. The calculation falls naturally into two parts. One is where the air conductivity is low at early times. Here, Maxwell's equations are solved via an eigenfunction technique. For later times at higher conductivities, Maxwell's equations are simplified with the diffusion approximation and solved via Laplace transforms. From this analysis an important approximation at high conductivities is pointed out. Also emphasized is the great influence which the air conductivity has on currents and current rates. Solutions for the sphere have been coded for an IBM 360 and some example calculations are presented.

air conductivity, surface bursts, calculations

1. Introduction

This note is an attempt to assess with some accuracy the electric currents induced on above ground metallic bodies in the presence of a nearby nuclear surface burst. The electromagnetic environment from such a burst is generated by Compton currents and consists of intense electric and magnetic fields along with a gamma-ray induced time-varying air conductivity. The work here does not concentrate on specifying this environment any further. The emphasis is rather on the calculation of currents once the environment is given or assumed. A background discussion which places this problem in context has been given by Baum.[7]

Any conductors in this environment will necessarily experience large induced currents not only from the fields alone but also from the enhanced coupling due to the surrounding air conductivity. Specifically, the motiviation for this work grew out of the necessity to know the combined environmental effects on the above ground Minuteman UHF antenna. First attempts at this replaced the antenna with a simple equivalent circuit with the time-varying air conductivity replaced by a time-varying resistor incorporated into the circuit.[6] While this procedure has the obvious advantage of simplicity, the uncertainties involved were felt to justify a more firmly based approach. The approach taken here is to simulate rather complicated metallic objects with ones having simpler boundaries in order that the correct boundary conditions can easily be applied to solutions of Maxwell's equations. Some judgment, of course, must be exercised in applying the resulting calculated currents to any specific problem of interest.

It is the air conductivity and its temporal behavior which complicates the problem and necessitates the special mathematical techniques described below. The techniques are somewhat unconventional. The equations describing electromagnetic problems are usually Fourier transformed in the time domain and the analysis carried out in the frequency domain. However, this requires that the coefficients of the pertinent equations (e.g., the conductivity) be independent of time. The fact that the conductors are considered to be immersed in a medium with a time-varying conductivity forces a different attack on the problem.

The model conductors considered here represent the extremes in which the relevant wavelengths of the fields in question are much smaller than the metallic body of interest or larger or comparable to it. For simplicity, these are chosen to be a perfectly conducting plane and a perfectly conducting sphere, respectively. These are then considered to be subjected to an incident electric field in a conducting medium. An important point of interest is not only how the combined electric field and conductivity enhance the magnitude of the induced current but also how this current varies with time. To emphasize this, the following assumptions are made about the field and conductivity which retain their time histories but which also reduce the complexity of the solutions so that numerical estimates can more easily be carried out. First, the electric field is assumed to have a constant direction in space and to vary with time only (no spatial dependence). Second, the conductivity is a simple scalar function of time only, like the electric field. Just how these assumptions simplify the analysis will become evident from the explanation which follows in the next three sections.

The view point is taken[7] that the plane or sphere acts as the generator of the scattered fields with the incident electric field as the source. The only field that has to be calculated is the magnetic field intensity $\vec{H}$ since induced currents are obtained[1] from

$$\vec{K} = \hat{n} \times \vec{H} \qquad \text{(I-1)}$$

where $\hat{n}$ is a unit normal to the surface and $\vec{K}$ is the induced surface current.

Both the fields and the air conductivity from a close-in surface burst rise rapidly to their peak values and then fall off. In general, the conductivity lags somewhat behind the fields and does not fall off quite as rapidly. On physical grounds, then, it can be anticipated that the calculation of $\vec{H}$ as a function of time will fall naturally into two parts. The first is at earlier times when the conductivity is low and the wavelike properties of the solutions are emphasized. The second part is later, when the conductivity is larger, and the conduction currents dominate the displacement currents. At these late times the field equations may be approximated by ignoring the displacement current altogether and it is in this region where peak currents are to be expected. A feature of the latter approximation is that it requires initial time information from the first part for a complete and accurate time history. The appropriateness of the division of the general problem in this way must, of course, be determined for any particular case of interest.

Although the following discussion is limited to scattering from a plane and sphere, there are circumstances for which the induced current can be estimated <u>independently</u> of any geometry. If the conductivity becomes great enough, experience with calculations has shown that a good estimate can be made with Eq. (V-1) below. This is an important result owing to the relative simplicity of the integral in this equation, and we point it out here because of this. Further discussion of this approximation is appropriately deferred until Section V after the solutions for the plane and sphere have been obtained.

Each part of the problem requires a different technique to arrive at a solution. Briefly, the technique of the first part is essentially a generalization of the Fourier transform method in which expansions of the fields are made with a set of orthogonal functions defined by a Sturm-Liouville system with time as the independent variable. This set of functions depends directly on the conductivity and reduces to the usual sines and cosines when the conductivity vanishes. That is to say, for zero conductivity the expansion is just a Fourier expansion. The Sturm-Liouville system is chosen such that the coefficients of the field expansions, which depend on spatial coordinates, satisfy the Helmholtz equation which is subsequently solved for the two geometries already mentioned. For the second part, omitting the displacement current results in a vector diffusion equation. This is then solved with the appropriate initial values and boundary conditions by means of Laplace transforms. The answer for this part is expressed as a sum of integrals. Two features of the entire solution are that all analyses are carried out with fields rather than potentials and that no complex quantities are ever introduced.

Some examples are presented in Section V to illustrate numerically the influence and importance of the air conductivity.

II. General Analysis of the Problem

In this section an over-all analysis is made of the general problem in which advantage is taken of the special assumptions made about the surrounding air conductivity and the incident electric field (see the discussion in the Introduction). The details presented here emphasize the time-varying aspects of the problem, and are independent of any particular geometry in question. The discussion centers around a division of the problem as outlined in the Introduction. For the initial low conductivity region a review of the Toulios[2] eigenfunction approach is given starting from Maxwell's equations. For the diffusion approximation (high conductivity region) a simple transformation is made on the basic equations so that the Laplace transform method can be applied to arrive at a solution.

The pertinent Maxwell equations are

$$\nabla \times \vec{E} = -\mu_o \frac{\partial \vec{H}}{\partial t} \qquad \text{(II-1)}$$

and

$$\nabla \times \vec{H} = \vec{j} + \varepsilon_o \frac{\partial \vec{E}}{\partial t} = \sigma(t)\vec{E} + \varepsilon_o \frac{\partial \vec{E}}{\partial t} . \qquad \text{(II-2)}$$

Note that Compton currents have been omitted from Eq. (II-2) as a source term. These could be included easily with the same assumptions about the conductivity and the incident electric field, i.e. time variations but no spatial dependence. The effect would be to modify the calculations of the field expansion coefficients (see Sections II and III below). Both E and H are divergence - free

$$\nabla \cdot \vec{E} = \nabla \cdot \vec{H} = 0. \qquad \text{(II-3)}$$

A new function g(t) is now defined in anticipation of later use:

$$\frac{\sigma(t)}{\varepsilon_o} = \frac{1}{g}\frac{dg}{dt} = \frac{\dot{g}}{g} . \qquad \text{(II-4)}$$

A wave equation for $\vec{H}$ can now be easily derived from Eqs. (II-1), (II-2) and (II-4), it can be shown to be

$$\nabla^2\vec{H} - \frac{1}{c^2}\left[\frac{\dot{g}}{g}\frac{\partial\vec{H}}{\partial t} + \frac{\partial^2\vec{H}}{\partial t^2}\right] = 0. \qquad \text{(II-5)}$$

It is this equation which has to be solved in order to find surface currents according to Eq. (I-1).

For the low conductivity region the equation is attacked by the classic method of separation of variables which is discussed at length in many text-books[1,3,4]. Briefly what happens is that the partial differential equation (II-5) separates into a set of ordinary differential equations which with appropriate boundary conditions form classic Sturm-Liouville systems,[3,4]. The result is that the solution is expressed as an expansion over one or more sets of eigenfunctions. This is the procedure followed here with regard to the space and time variables and we anticipate therefore an expansion of the form

$$\vec{H}(\vec{r},t) = \sum_n \vec{H}_n(\vec{r})\, y_n(t) \qquad \text{(II-6)}$$

A summation is written here because the time domain is chosen to be a finite interval $(-t_o,t_o)$. We now proceed to show that the set $\{y_n(t)\}$ can indeed be specified as an orthogonal eigenfunction set generated by a Sturm-Liouville system.

Insertion of Eq. (II-6) into Eq. (II-5) gives

$$\sum\left[\nabla^2\vec{H}_n\, y_n - \frac{\vec{H}_n}{c^2}\left(\frac{\dot{g}}{g}\dot{y}_n + \ddot{y}_n\right)\right] = 0. \qquad \text{(II-7)}$$

Now we choose

$$\frac{\dot{g}}{g}\dot{y}_n + \ddot{y}_n = -\lambda_n y_n \qquad \text{(II-8)}$$

or

$$\frac{d}{dt}(g\,\dot{y}_n) + \lambda_n g\, y_n = 0. \qquad \text{(II-9)}$$

After suitable boundary conditions are imposed on the solutions of this equation, the set $\{y_n\}$ is shown below to be a complete orthogonal set. Then each mode of Eq. (II-7) is simply

$$\nabla^2 \vec{H}_n + \frac{\lambda_n}{c^2} \vec{H}_n = 0 \qquad \text{(II-10)}$$

which is the familiar vector Helmholtz equation. This is the equation which contains all reference to geometry and the solutions of which form the content of the two succeeding sections on the plane and sphere. The attainment of this much studied equation is one of the reasons for the choice of Eq. (II-8).

Assume with Toulios[2] the artificiality

$$g(t) = g(-t). \qquad \text{(II-11)}$$

This makes no difference to the discussion at hand since only t>o is of interest. Then Eq. (II-9) has even and odd function solutions on the fundamental interval $(-t_o, t_o)$. Boundary conditions are now imposed on the solutions as follows:

$$y_n^o \; (-t_o) = y_n^o \; (t_o) = 0 \qquad \text{(II-12)}$$

and

$$\dot{y}_n^e \; (-t_o) = \dot{y}_n^e \; (t_o) = 0 \qquad \text{(II-13)}$$

where the additional (e,o) refer to even or odd functions respectively. These boundary conditions are chosen for the particular reason that in the limit of zero conductivity the eigenfunctions reduce to sines and cosines with the <u>same</u> eigenvalues

$$\lambda_n^{e,o} = \left(\frac{n\pi}{t_o}\right)^2. \qquad \text{(II-14)}$$

There are, of course, many ways to impose boundary conditions but this particular choice facilitates the identification of properly scattered outgoing waves for $\sigma = 0$. The question of identifying outgoing waves for $\sigma \neq 0$ is an important one and is dealt with a little later.

The general conditions for the set to be a complete orthogonal set are

(1) they satisfy a Sturm-Liouville equation

(2) the lowest eigenvalue is zero (y_o^e = constant in this case)

(3) $g\, y_m \dot{y}_n \Big]_{-t_o}^{t_o} = 0.$

Since the system in question meets these conditions, it is complete and orthogonal and expansions in this set can be safely performed. In addition, the set can be normalized by merely adjusting multiplicative factors. With the weight function g(t),

$$\int_{-t_o}^{t_o} g(t)\, y_n(t)\, y_m(t)\, dt = \delta_{nm} \qquad \text{(II-15)}$$

where for the moment all designations of the functions (e,o included) have been incorporated into the single indices.

We emphasize again that the generation of the set y_m depends only on the time-varying conductivity. This means that once the set has been obtained along with its eigenvalues, it can be applied to systems of any geometry and in particular to the plane and sphere considered below.

In general, the calculation of the eigenvalues and eigenfunctions will have to be done numerically. There are, however, two simple cases in which the eigenvalues and eigenfunctions can be found explicitly. First, there is the case of $\sigma = 0$ which has already been mentioned. To repeat, the eigenvalues are

$$\lambda_o^e = 0, \quad \lambda_n^{e,o} = \left(\frac{n\pi}{t_o}\right)^2 \qquad \text{(II-16)}$$

with

$$y_o^e = \frac{1}{\sqrt{2\, t_o}}, \quad y_n^{\substack{e\\o}} = \frac{1}{\sqrt{t_o}} \begin{cases} \cos\sqrt{\lambda_n}\, t \\ \sin\sqrt{\lambda_n}\, t \end{cases} . \qquad \text{(II-17)}$$

The second case is for σ = constant. Equation (II-8) is then

$$\ddot{y}_n + \frac{\sigma}{\varepsilon_o} \dot{y}_n + \lambda_n y_n = 0. \quad \text{(II-18)}$$

The eigenvalues are

$$\lambda_o^e = 0, \; \lambda_n^{e,o} = \left(\frac{n\pi}{t_o}\right)^2 + \left(\frac{\sigma}{2\varepsilon_o}\right)^2 \quad \text{(II-19)}$$

with the eigenfunctions

$$y_o^e = \left(\frac{2\sigma}{\varepsilon_o}\right)^{\frac{1}{2}} \left[e^{\sigma t_o/\varepsilon_o} - 1\right]^{-\frac{1}{2}}, \quad \text{(II-20)}$$

$$y_n^o = \frac{1}{\sqrt{t_o}} \, e^{\frac{-\sigma t}{2t_o}} \sin \frac{n\pi t}{t_o}, \; t>0 \quad \text{(II-21)}$$

and

$$y_n^e = \frac{1}{\sqrt{t_o \lambda_n}} \, e^{\frac{-\sigma t}{2t_o}} \left[\frac{n\pi}{t_o} \cos \frac{n\pi t}{t_o} + \frac{\sigma}{2\varepsilon_o} \sin \frac{n\pi t}{t_o}\right], \; t>0. \quad \text{(II-22)}$$

Note that these functions explicitly display damped behavior and that in the limit $\sigma \to 0$ they degenerate into sines and cosines as expected. These functions have proven very useful and have had practical application in examining and checking out the computer code which was written for the spherical solution.

In the expansion given by Eq. (II-6) there will be in general expansion coefficients, which in this case are incorporated in the factor $\vec{H}_n(\vec{r})$, and which have to be calculated from physical boundary conditions. For a perfect conductor one condition is that the total tangential electric field vanish at the surface.[1] This condition is not difficult to apply if the object is simple enough geometrically (which is the case for the solutions for the plane and sphere below). There is one other condition sometimes called the boundary condition "at infinity". In the case of zero conductivity this is taken in a scattering problem to be an outgoing wave in the asymptotic limit from the scattering object. A relationship between particular coefficients can usually be established from a direct examination of the expansions involved and these together with the above conditions are usually sufficient for a solution of the problem at hand. On the other hand, when $\sigma \neq 0$ such identifications are not obvious and the calculational procedure is not clear. However, the fact remains that the wave front of the scattered magnetic field intensity can travel outward no faster than the speed of light.

There is no apparent way to apply this condition directly to the calculation of the expansion coefficients so instead the following procedure has been followed. Call $\vec{H}_F$ the value of the field at the wave front. Then

$$\vec{H}_F = 0. \tag{II-23}$$

The condition which in fact is used, is to minimize the integral

$$I = \int_0^{t_o} |\vec{H}_F|^2 \, dt \tag{II-24}$$

with respect to the uncalculated coefficients. This results in a linear system which can then be solved with the usual techniques of linear algebra. This procedure has given satisfactory results for many calculations, and the examples presented in Section V were all calculated in this manner. It should be emphasized that Eq. (II-24) has no strict theortical basis, but is merely an attempt to force the correct outgoing character on the wave front.

Attention is now turned to the diffusion approximation for which the calculations are more direct. The foregoing analysis was made with no approximations to the basic Eqs. (II-1) and (II-2). When the conductivity becomes large enough and the electric field is not changing too rapidly,

$$|\sigma(t)\vec{E}| >> |\epsilon_o\dot{\vec{E}}| \tag{II-25}$$

and Eq. (II-2) can be approximated

$$\nabla \times \vec{H} \simeq \sigma \vec{E}. \tag{II-26}$$

It is not difficult to show that the equation for the magnetic field intensity now becomes

$$\nabla^2 \vec{H} = \frac{\sigma(t)}{\varepsilon_o c^2} \frac{\partial \vec{H}}{\partial t} \tag{II-27}$$

which is a vector diffusion equation. This equation can be put into a simpler form with the transformation

$$\xi = \varepsilon_o c^2 \int_{t_o}^{t} \frac{dt}{\sigma(t)} \tag{II-28}$$

The result is

$$\nabla^2 \vec{H} = \frac{\partial \vec{H}}{\partial \xi} \tag{II-29}$$

which is easier to solve in terms of the variable ξ than was the original Eq. (II-27) in terms of t.

Calculations with the Toulios eigenfunctions require the weight function

$$g(t) = \exp\left[\frac{1}{\varepsilon_o} \int_{0}^{t} \sigma(t)\,dt\right]. \tag{II-30}$$

This function can become very large, depending on the problem under study, for the values of σ in the range of interest. Thus for regions of high conductivity the eigenfunction expansions can become impractical. The exponent in Eq. (II-30) is large when

$$\frac{\bar{\sigma} t_o}{\varepsilon_o} \gg 1 \tag{II-31}$$

where $\bar{\sigma}$ is some mean conductivity. Since t_o is usually of the order of magnitude of a fundamental period in the problem of interest, Eq. (II-31) also suggests the condition (II-25). The diffusion approximation is not only convenient but also necessary in some cases for the estimation of currents at later times.

III. Scattering from a Plane

This is the first of the two geometries to be considered in this note. Scattering from a plane can approximate the situation in which the relevant wavelengths of the incident electric field are much smaller than the dimensions of the scatterer. Thus, in a sense, the solutions presented here can be considered complementary to those of the sphere. Comparisons of the solutions for both cases are of some interest and make evident the importance of certain features in the diffusion approximation, a discussion of which is deferred until Section V. The solutions for the plane also have the edifying feature of being much simpler so that the essentials of this type of calculation stand out better and are not obscured by too much detail.

The situation to be considered is as follows. Consider a perfectly conducting plane perpendicular to the y-axis at the origin as illustrated in Fig. 1. Assume an incident electric field

$$\vec{E}^i(t) = \hat{z}\, E^i(t),\ t>0 \tag{III-1}$$

is scattered from this plane. From these imposed symmetries it is evident that the magnetic field intensity has only an x-component, the magnitude of which depends only on y and t,

$$\vec{H} = \hat{x}\, H(y,t). \tag{III-2}$$

In the low conductivity, early time region H is expanded in the Toulios eigenfunctions according to Eq. (II-6).

$$H(y,t) = \sum H_n(y)\, y_n(t) \tag{III-3}$$

where the Helmholtz Eq. (II-10) for H_n is

$$\frac{d^2 H_n}{dy^2} + \frac{\lambda_n H_n}{c^2} = 0. \tag{III-4}$$

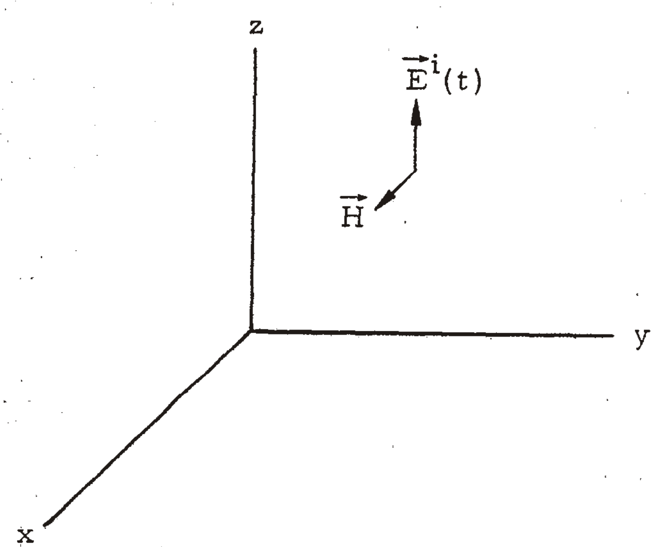

Fig. 1 Cartesian Co-ordinate System

The lowest eigenvalue is, of course, zero and the corresponding solution of Eq. (III-4) is

$$H_o = a_o\, y + b_o \tag{III-5}$$

where a_o and b_o are constants. Both parts of this part of the solution must be retained. Usually the first part would be ignored on the grounds that the solution is unbounded. However, it must be remembered that a finite time interval $(0,t_o)$ is the domain of calculation and that this limits the extent of the fields from the plane to $y < ct_o$. Thus for the calculations here (and later for the sphere) decisions based upon asymptotic arguments are not pertinent. The values of the fields beyond ct_o are known: they must be zero. For the remainder of the eigenvalues it is not difficult to see that

$$H_n^{e,o} = a_n^{e,o} \sin k_n^{e,o}\, y + b_n^{e,o} \cos k_n^{e,o}\, y \tag{III-6}$$

where the even-odd designations have now been introduced and

$$k_n^{e,o} \equiv \frac{\sqrt{\lambda_n^{e,o}}}{c} . \tag{III-7}$$

The total solution is then

$$\vec{H} = \hat{x}\left[a_o\, y + b_o + \sum \left\{ y_n^e \left(a_n^e \sin k_n^e\, y + b_n^e \cos k_n^e\, y\right) + y_n^o \left(a_n^o \sin k_n^o\, y + b_n^o \cos k_n^o\right)\right\}\right] . \tag{III-8}$$

For the solution to be complete it is necessary to specify the coefficients $a_n^{e,o}$ and $b_n^{e,o}$. The technique for doing this is outlined in the Introduction and the condition that the tangential electric field be zero on the plane

$$E + E^i = 0 \tag{III-9}$$

is applied first. From Maxwell's Eq. (II-2)

$$-\frac{1}{\varepsilon_o}\frac{\partial H}{\partial y} = \frac{\sigma}{\varepsilon_o} E + \dot{E} \tag{III-10}$$

and Eq. (III-9) we have

$$\frac{1}{\varepsilon_o} \frac{\partial H}{\partial y}\bigg|_{y=o} = \frac{\sigma}{\varepsilon_o} E^i + \dot{E}^i \qquad \text{(III-11)}$$

or

$$a_o + \sum \left[k_n^e \, y_n^e \, a_n^e + k_n^o \, y_n^o \, a_n^o \right] =$$

$$\text{(III-12)}$$

$$\frac{\sigma}{\varepsilon_o} E^i + \dot{E}^i = \frac{\dot{g}}{g} E^i + \dot{E}^i .$$

Now the orthonormality of the Toulios eigenfunctions (Eq. (II-15)) can be employed directly to give

$$k_n^e \, a_n^e = g(t_o) \, y_n^e(t_o) \, E^i(t_o) - \int_o^{t_o} g \, E^i \, \dot{y}_n^e \, dt, \qquad \text{(III-13)}$$

$$k_n^o \, a_n^o = - \int_o^{t_o} g \, E^i \, \dot{y}_n^o \, dt \qquad \text{(III-14)}$$

and

$$a_o = g(t_o) \, E^i(t_o) \Big/ \left[2 \int_o^{t_o} g \, dt \right] . \qquad \text{(III-15)}$$

When these are substituted into the expression for the magnetic field intensity, Eq. (III-8), the coefficients b_o, b_n^e, b_n^o remain to be found. For the finite conductivity case these will have to be calculated according to some scheme based on the outgoing wave conditions, (Eq. (II-23)). For this, the method discussed in association with Eq. (II-24) could be tried.

For the case of $\sigma = 0$, $t_o \to \infty$ there should exist expressions for b_o, b_n^e and b_n^o which can be determined directly from the solution itself. The scattered field must be an outgoing wave with the conventional generic solution of the wave equation, i.e. some function of the argument y-ct. To see this note that g = 1 and

$$a_o = \frac{E^i(t_o)}{2t_o}. \tag{III-16}$$

If we assume that E^i remains bounded then

$$a_o \xrightarrow[t_o \to \infty]{} 0. \tag{III-17}$$

Since the wave must have an outgoing character it is not hard to ascertain that

$$b_o = 0 \tag{III-18}$$

$$b_n^o = - a_n^e \tag{III-19}$$

and

$$b_n^e = a_n^o \tag{III-20}$$

fulfill the condition for the choice of Toulios functions given by Eq. (II-17).

We now turn to a discussion of the situation in which the conduction current dominates the displacement current as already anticipated in Sections I and II. From Eq (II-29), we obtain

$$\frac{\partial^2 H}{\partial y^2} = \frac{\partial H}{\partial \xi} . \tag{III-21}$$

A unique solution to this equation is insured[3] given boundary conditions at y = 0 and initial conditions at $t = t_o$ ($\xi = 0$) for all y. These are

$$\nabla x\vec{H} = - \sigma\vec{E}^i, \ y = 0 \tag{III-22}$$

or

$$\frac{\partial H}{\partial y} = \sigma E^i \equiv \Phi\ (\xi) \tag{III-23}$$

and

$$H_o(y) = H(y,t_o),\ t = t_o,\ \xi = 0. \tag{III-24}$$

A common technique for the solution of diffusion equations such as Eq. (III-21) is the Laplace transform method and it is used here in the case of the plane and later for the sphere. The Laplace transform of a function (ξ) will be denoted

$$\bar{f}(s) = \int_0^\infty e^{-s\xi} f(\xi)\, d\xi \tag{III-25}$$

and of its derivative $f'(\xi)$

$$s\bar{f}(s) - f(o) = \int_0^\infty e^{-s\xi} f'(\xi)\, d\xi . \tag{III-26}$$

With these, Eq. (III-21) transforms to

$$\frac{d^2\bar{H}}{dy^2} = s\bar{H} - H_o . \tag{III-27}$$

The transform of the boundary condition, Eq. (III-23) is

$$\left.\frac{d\bar{H}}{dy}\right|_{y=0} = \bar{\phi}(s). \tag{III-28}$$

Now the homogeneous solution of Eq. (III-26) which remains bounded as $y \to \infty$ is

$$\bar{H}_n = e^{-y\sqrt{s}} \tag{III-29}$$

A particular solution can be found by assuming

$$\bar{H}_p = \bar{H}_n F(y) \tag{III-30}$$

where $F(y)$ is yet to be determined. The advantage of this device is that it results in an easily solved equation for $F(y)$,

$$\frac{d}{dy}\left(F' e^{-2\sqrt{s}\, y}\right) = - H_o e^{-\sqrt{s}\, y}. \tag{III-31}$$

The total solution of Eq. (III-26) is then

$$\bar{H} = \bar{H}_n \,(A + F\,(y)) \tag{III-32}$$

where A is some constant. From the boundary condition Eq. (III-27), it can be seen that

$$\bar{\phi}(s) = F'(o) - \sqrt{s}\,F(o) - \sqrt{s}\,A. \tag{III-33}$$

To obtain a simple expression for A, we place a condition on the unspecified function F. Let

$$\sqrt{s}\,F(o) = F'(o). \tag{III-34}$$

Then immediately

$$A = -\frac{\bar{\phi}(s)}{\sqrt{s}}. \tag{III-35}$$

A function F(y) which satisfies both Eqs. (III-30) and (III-33) such that $\bar{H}_p$ remains bounded as $y \to \infty$ is

$$F(y) = \int_o^y e^{2\sqrt{s}\,y''}\,dy'' \int_{y''}^{\infty} H_o(y')\,e^{-\sqrt{s}\,y'}\,dy' + \frac{1}{\sqrt{s}}\int_o^{\infty} H_o(y')\,e^{-\sqrt{s}y'}\,dy' \tag{III-36}$$

with

$$F'(y) = e^{2\sqrt{s}\,y}\int_y^{\infty} H_o(y')\,e^{-\sqrt{s}\,y'}\,dy'. \tag{III-37}$$

That F(y) is indeed a solution of Eq. (III-30) can be determined by simple substitution and Eq. (III-33) is obviously satisfied. The boundedness of the particular solution is evident from the limit

$$\bar{H}_p \xrightarrow[y \to \infty]{} \frac{H_o(y)}{s}. \tag{III-38}$$

Since the current density on the plane y = 0 is the quantity of interest, the first term of Eq. (III-35) disappears, resulting in the simpler expression

$$F(o) = \frac{1}{\sqrt{s}}\int_o^{\infty} H_o(y')\,e^{-\sqrt{s}\,y'}\,dy'. \tag{III-39}$$

Now the total solution for the Laplace transform of the magnetic field intensity on the plane is

$$\bar{H} = -\frac{\bar{\phi}(s)}{\sqrt{s}} + \frac{1}{\sqrt{s}} \int_0^\infty H_o(y') \, e^{-\sqrt{s}\, y'} \, dy'. \qquad \text{(III-40)}$$

To identify the inverse transform of the first term, the convolution theorem for Laplace transforms,

$$\bar{f}(s) \, \bar{g}(s) = \int_0^\infty e^{-st} \, dt \int_0^t f(\tau) \, g(t-\tau) \, d\tau \qquad \text{(III-41)}$$

is employed. The inverse transform of the second term is readily available from existing tables.[5] Then

$$H = -\int_0^\xi \frac{\phi(\xi') \, d\xi'}{[\pi(\xi - \xi')]^{1/2}} + \int_0^\infty H_o(y') \frac{e^{-y'^2/4\xi}}{(\pi\xi)^{1/2}} \, dy'. \qquad \text{(III-42)}$$

The second integral can be simplified by letting

$$x^2 = \frac{y'^2}{4\xi} . \qquad \text{(III-43)}$$

Then

$$H = -\frac{1}{\sqrt{\pi}} \int_0^\xi \frac{\phi(\xi') d\xi'}{(\xi - \xi')^{1/2}} + \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-x^2} \, H_o \, (2x\sqrt{\xi}\,) \, dx, \qquad \text{(III-44)}$$

from which it is evident that in the limit $t \to t_o$ $(\xi = 0)$, this reduces to

$$H = H_o(o) \qquad \text{(III-45)}$$

as it should.

At this point we leave consideration of the plane and turn to the corresponding calculations for the sphere.

IV. Scattering from a Sphere

The analysis for spherical geometry closely parallels that for the plane except that certain details become more complicated. We assume as in the case of the plane that the incident electric field is directed along the z-axis (see Fig. 2 for the conventional definition of spherical coordinates). Since the field is assumed to have no spatial dependence, this axis must be an axis of symmetry. From these it is assumed that $\vec{H}$ has only one component, the magnitude of which is independent of the angle ϕ. Thus we may write immediately

$$\vec{H} = \hat{\phi}\, H_{\phi}\,(r,\theta,t) \tag{IV-1}$$

where $\hat{\phi}$ is a unit vector in the ϕ direction

$$\hat{\phi} = \hat{y}\sin\phi - \hat{x}\cos\phi . \tag{IV-2}$$

For the low conductivity region the field is expanded according to Eq. (II-6), the Toulios eigenfunction expansion. Direct substitution of each mode of this expansion into the Helmholtz Eq. (II-10) can readily be shown to give

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial H_n^{e,o}}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial H_n^{e,o}}{\partial\theta}\right) - \frac{H_n^{e,o}}{r^2\sin^2\theta} + \frac{\lambda_n^{e,o}}{c^2}H_n^{e,o} = 0. \tag{IV-3}$$

The superscripts "e" and "o" refer respectively to the even and odd Toulios eigenfunctions as discussed in Section II. This equation can be separated further and the details can be examined in many textbooks, i.e., Panofsky and Phillips.[1] We shall temporarily suppress the n, e and o designations on $H_n^{e,o}$

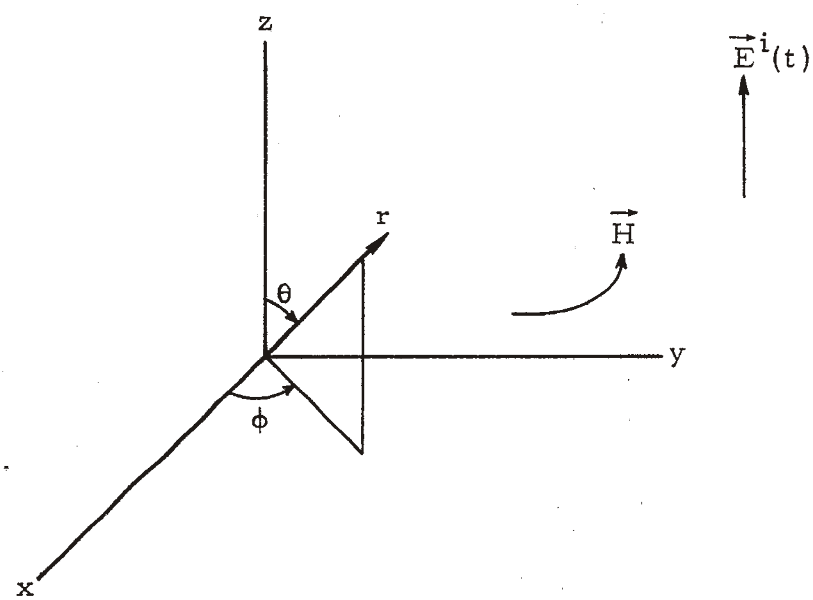

Fig. 2 Spherical Co-ordinate System

in Eq. (IV-3) to avoid unnecessary complication. The separated form of H can be shown to be

$$H = \sum_{\ell} P_{\ell}^{1} (\cos\theta)\, R_{\ell}(r) \qquad \text{(IV-4)}$$

where $P_{\ell}^{1}(\cos\theta)$ is the associated Legendre function[1,5] solving the separated differential equation

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{dP_{\ell}^{1}}{d\theta}\right) + \left[\ell(\ell+1) - \frac{1}{\sin^2\theta}\right] P_{\ell}^{1} = 0 \qquad \text{(IV-5)}$$

The separation constant is $\ell(\ell+1)$ where ℓ is a positive integer. The function R_{ℓ} is the solution of the differential equation

$$\frac{d^2R_{\ell}}{dr^2} + \frac{2}{r}\frac{dR_{\ell}}{dr} - \frac{\ell(\ell+1)R_{\ell}}{r^2} + \frac{\lambda R_{\ell}}{c^2} = 0. \qquad \text{(IV-6)}$$

We now distinguish two cases: $\lambda = 0$ and $\lambda \neq 0$. For $\lambda = 0$, the linearly independent solutions of Eq. (IV-6) are r^{ℓ} and $r^{-(\ell+1)}$. For $\lambda \neq 0$ the solutions are[1]

$$R_{\ell} = a_{\ell}\, j_{\ell}\left(\frac{\sqrt{\lambda}}{c} r\right) + b_{\ell}\, y_{\ell}\left(\frac{\sqrt{\lambda}}{c} r\right) \qquad \text{(IV-7)}$$

where j_{ℓ} and y_{ℓ} are spherical Bessel functions[4], and a_{ℓ} and b_{ℓ} are constants.

We now show how the expansion for each mode of the H-field given by Eq. (IV-4) reduces to a single term. The incident electric field is

$$\vec{E}^{i}(t) = \hat{z}\, E^{i}(t) \qquad \text{(III-1)}$$

or

$$\vec{E}^{i} = (\hat{r}\cos\theta - \hat{\theta}\sin\theta)\, E^{i}(t). \qquad \text{(IV-8)}$$

Maxwell's Eq. (II-2) has a θ-component

$$-\frac{1}{\varepsilon_o}\frac{1}{r}\frac{\partial}{\partial r}(r\, H_{\phi}) = \frac{\sigma}{\varepsilon_o} E_{\theta} + \dot{E}_{\theta}. \qquad \text{(IV-9)}$$

The boundary condition that the total tangential (θ-component) electric field vanish on the sphere $r = a$ is written

$$E_\theta - \sin\theta \, E^i(t) = 0. \qquad \text{(IV-10)}$$

Thus

$$-\frac{1}{\varepsilon_o a}\frac{\partial}{\partial r}(rH_\phi)_{r=a} = \sin\theta\left(\frac{\sigma}{\varepsilon_o}E^i + \dot{E}^i\right). \qquad \text{(IV-11)}$$

Using the fact that

$$P_1^1(\cos\theta) = \sin\theta \qquad \text{(IV-12)}$$

and the orthogonality[1] of the associated Legendre functions, we can conclude that the only non-zero term of Eq. (IV-4) is for $\ell = 1$.

With this simplification the scattered $\vec{H}$ field can be written as a sum only over those indices (n, e and o) which distinguish the Toulios eigenfunctions:

$$\vec{H} = \hat{\phi}\sin\theta\left[\frac{a_o}{r^2} + b_o r + \sum y_n^e(t)\left\{a_n^e j_1(k_n^e r) + b_n^e y_1(k_n^e r)\right\} + \sum y_n^o(t)\left\{-b_n^o j_1(k_n^o r) + a_n^o y_1(k_n^o r)\right\}\right] \equiv \hat{\phi}\sin\theta \, H(r,t). \qquad \text{(IV-13)}$$

The definition

$$k_n^{e,o} \equiv \frac{\sqrt{\lambda_n^{e,o}}}{c} \qquad \text{(IV-14)}$$

has been introduced and the expansion has been given to display explicitly the division according to the eigenvalues $\lambda_o^e = 0$, λ_o^e and λ_n^o. It is emphasized that the relative simplicity of Eq. (IV-13) is due directly to the assumed nature of the incident electric field. The advantage of avoiding multiple expansions is obvious for numerical computations.

The derivation of Eq. (V-13) has been somewhat abbreviated but the important details have been pointed out, and one can always verify that indeed the solution given by Eq. (IV-13) satisfies the general wave Eq. (II-5). Uniqueness of the solution can be inferred from the imposition of the proper boundary conditions and it is to this which we now turn.

It is convenient to define

$$F(x) = \frac{d}{dx}(xj_1), \qquad \text{(IV-15)}$$

$$G(x) = \frac{d}{dx}(xy_1), \qquad \text{(IV-16)}$$

and

$$F_n^{e,o} = F(k_n^{e,o}\, a) \qquad \text{(IV-17)}$$

$$G_n^{e,o} = G(k_n^{e,o}\, a). \qquad \text{(IV-18)}$$

Then substituting the expression for the field Eq. (IV-13) into Eq. (IV-11), we find

$$-\frac{1}{\varepsilon_o}\left[-\frac{a_o}{a^2} + 2a\, b_o + \sum y_n^e(t)\,(a_n^e F_n^e + b_n^e G_n^e) + \sum y_n^o(t)\,(-b_n^o F_n^o + a_n^o G_n^o)\right] = \frac{a}{g}\frac{d}{dt}(g\, E^i). \qquad \text{(IV-19)}$$

From the orthonormality of the Toulios eigenfunctions, Eq. (II-15), the following set of equations can be derived

$$\frac{a_o}{a^2} - 2a\, b_o = \frac{a\, \varepsilon_o\, g(t_o)\, E^i(t_o)}{2\int_o^{t_o} g(t)\, dt}, \qquad \text{(IV-20)}$$

$$a_n^o G_n^o - b_n^o F_n^o = a\varepsilon_o\, E_n^o, \qquad \text{(IV-21)}$$

and

$$a_n^e F_n^e + b_n^e G_n^e = a\varepsilon_o\, E_n^e \qquad \text{(IV-22)}$$

where the definitions

$$E_n^{e,o} = \int_o^{t_o} y_n^{e,o} \frac{d}{dt} (g\ E^i)\ dt \qquad \text{(IV-23)}$$

have been introduced. That is to say, Eqs. (IV-20), (IV-21) and (IV-22) incorporate the boundary information at the surface of the sphere as discussed in Section II.

The other condition imposed on the solution (IV-13) is that the scattered wave front travel outward from the sphere no faster than the speed of light,

$$H\ (a + ct,\ t) = 0. \qquad \text{(IV-24)}$$

However, as discussed in Section II, from a calculational standpoint it has been found more convenient to minimize

$$I = \int_o^{t_o} |\ H\ (a + ct,\ t)\ |^2\ dt \qquad \text{(IV-25)}$$

with respect to the undetermined coefficients in the expansion (IV-13) (see comment under Eq. (II-24)). The sum (IV-13) is, of course, truncated after a certain number of terms. In the actual calculations the set (b_o, b_n^e, b_n^o) is chosen as the undetermined set of coefficients. The minimization of the integral (IV-25) together with Eqs. (IV-20), (IV-21), and (IV-22) provide just enough equations so that the a's and b's can be found from a simple matrix inversion. As mentioned in Section II, experience with this technique has given satisfactory results and the examples presented below were all calculated in this manner.

As in the case of the plane, we have retained both the terms proportional to r^{-2} and r corresponding to the zero eigenvalue. The reason for retaining these may be reviewed under the discussion of scattering from the plane, Section

III. In the special situation for zero conductivity and $t_o \to \infty$, the solution of the wave Eq. (II-5) must have an outgoing character proportional to r^{-1} asymptotically. In this case g = 1 and Eq. (IV-20) becomes

$$\frac{a_o}{a^2} - 2a\, b_o = \frac{a\, \varepsilon_o\, E^i(t_o)}{2t_o} \qquad \text{(IV-26)}$$

and in the limit $t_o \to \infty$,

$$a_o = 2a^3\, b_o. \qquad \text{(IV-27)}$$

Thus the part of the H-field solution (IV-13) corresponding to the zero eigenvalue is

$$H_o = b_o \left(r + \frac{2a^3}{r^2} \right) \qquad \text{(IV-28)}$$

Now asymptotic arguments can be invoked to require $b_o = 0$. For the remainder of the solution, note the asymptotic form of the spherical Bessel functions[4] are

$$j_1(x) \xrightarrow[x \to \infty]{} -\frac{\cos x}{x} \qquad \text{(IV-29)}$$

and

$$y_1(x) \xrightarrow[x \to \infty]{} -\frac{\sin x}{x}. \qquad \text{(IV-30)}$$

These with the Toulios eigenfunctions for $\sigma = 0$, Eq. (II-17), indicate that the proper outgoing solution at asymptotic distances is for the choices

$$a_n^e = a_n^o$$
$$b_n^e = b_n^o. \qquad \text{(IV-31)}$$

Thus the solution for $\sigma = 0$ and $t_o \to \infty$ is easy to specify directly and the involved process for $\sigma \neq 0$ can be avoided.

This completes the discussion for the calculation of the H-field in the low conductivity region. Once this field is found, the surface current flowing on the sphere is, from Eq. (I-1),

$$I\ (\theta, t) = 2\pi a \sin\theta\ H \tag{IV-32}$$

Note that from this and Eq. (IV-13) that the current has a simple $\sin^2\theta$-dependence.

At higher conductivities where the condition (II-25) can be employed, it is necessary to solve the diffusion equation

$$\nabla^2\ \vec{H} = \frac{\partial\vec{H}}{\partial\xi} \tag{II-27}$$

where the variable ξ has been defined by Eq. (II-28). From the arguments given at the beginning of the discussion for the low conductivity region, the $\vec{H}$-field can immediately be written

$$\vec{H}(\vec{r}, \xi) = \hat{\phi} \sin\theta\ H(r, \xi) \tag{IV-33}$$

It is straightforward to show that when this is substituted in Eq. (II-27) (using Eq. (IV-2) as before) that the result is

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial H}{\partial r}\right) - \frac{2H}{r^2} = \frac{\partial H}{\partial\xi}. \tag{IV-34}$$

It is convenient to define

$$u = r\ H. \tag{IV-35}$$

Then

$$\frac{\partial^2 u}{\partial r^2} - \frac{2u}{r} = \frac{\partial u}{\partial\xi}\ . \tag{IV-36}$$

To obtain a unique solution[3] to this last equation it is necessary to have proper boundary and initial conditions, and they are the next order of business. At time $t = t_o$

$$u = r\ H_o(r) \tag{IV-37}$$

where H_o is the field given by the Toulios expansion at this time. The boundary condition is obtained from Eq. (II-26), the relevant component of which is

$$(\nabla \times \vec{H})_\theta = \sigma E_\theta \tag{IV-38}$$

or

$$\left.\frac{1}{r}\frac{\partial u}{\partial r}\right|_{r=a} = -\sigma(t)\, E^i(t) \tag{IV-39}$$

where the requirement that the tangential electric field vanish on the sphere (Eq. (IV-10)) has been incorporated. The boundary condition in terms of ξ is written (really a definition of ϕ (ξ))

$$\frac{\partial u}{\partial r} = -\phi(\xi). \tag{IV-40}$$

Now as in the case of scattering from a plane, the Laplace transform method is used to solve Eq. (IV-36). The notation used has already been indicated in Eq. (III-24). Application of the transform gives the equation

$$\frac{d^2\bar{u}}{dr^2} - \frac{2\bar{u}}{r^2} - s\bar{u} = -u(o) = -r\, H_o(r). \tag{IV-41}$$

The homogeneous equation is

$$\frac{d^2\bar{u}_h}{dr^2} - \frac{2\,\bar{u}_h}{r^2} - s\,\bar{u}_h = 0. \tag{IV-42}$$

The solution of this which is bounded as $r \to \infty$ can be seen to be

$$\bar{u}_h = e^{-r\sqrt{s}}\left(\frac{1}{\sqrt{s}} + \frac{1}{sr}\right) . \tag{IV-43}$$

To obtain a particular solution we try

$$\bar{u}_p = B(r)\ \bar{u}_h(r) \tag{IV-44}$$

and proceed as we did for the plane. The total solution is now

$$\bar{u} = \bar{u}_h \; (A + B(r)) \tag{IV-45}$$

where A is a constant. The transform of the boundary condition Eq. (IV-40) is

$$\left.\frac{d\bar{u}}{dr}\right|_{r=a} = -\bar{\phi}(s). \tag{IV-46}$$

Then from this

$$-\bar{\phi}(s) = \left[A + B(a)\right] \bar{u}_h'(a) + \bar{u}_h(a) \; B'(a). \tag{IV-47}$$

The quantities A and B(r) are as yet unspecified. To obtain a simple expression for A we impose on B(r).

$$\left.\frac{d}{dr} (B \; \bar{u}_h)\right|_a = 0. \tag{IV-48}$$

Then this requires

$$A = \frac{\bar{\phi}(s) \; sa^2 \; e^{a\sqrt{s}}}{1 + a\sqrt{s} + a^2 s}. \tag{IV-49}$$

To find B(r) substitution of Eq. (IV-45) into Eq. (IV-41) gives

$$\frac{d}{dr} \; (B' \; \bar{u}_h^2) = -\bar{r} \; \bar{u}_h \; H_o. \tag{IV-50}$$

Since $\bar{u}_h$ is a decreasing exponential of r, the solutions for B′(r) and B(r) are

$$B'(r) = \frac{1}{\bar{u}_h^2} \int_r^\infty \bar{u}_h(r') \; r' \; H_o(r') \; dr', \tag{IV-51}$$

and

$$B(r) = \int_a^r \frac{dr''}{\bar{u}_h^2(r'')} \int_{r''}^\infty \bar{u}_h(r') \; r' \; H_o(r') \; dr' - \frac{1}{\bar{u}_h'(a) \; \bar{u}_h(a)} \int_a^\infty \bar{u}_h(r') \; r' \; H_o(r') \; dr'. \tag{IV-52}$$

Any doubt about the boundedness of the particular solution $\bar{u}_h B$ as $r \to \infty$ can be removed by an examination of its asymptotic behavior. After some manipulation (omitted here) this can be shown to be

$$\bar{u}_p \xrightarrow[r \to \infty]{} \frac{r\, H_o(r)}{s} . \qquad \text{(IV-53)}$$

Also note that Eqs. (IV-51) and (IV-52) satisfy Eq. (IV-48).

The expression for B(r) is somewhat formidable. However, the concern of this note is the current induced on the sphere r = a so that the first term of Eq. (IV-52) vanishes. Then, after a little algebra, the Laplace ransform of the solution can be written as

$$\bar{u}(a,s) = \frac{a\,\bar{\phi}(s)\,(a\sqrt{s} + 1)}{1 + a\sqrt{s} + a^2 s} + a^2 \int_a^\infty dr'\, H_o(r')\, \frac{e^{-\sqrt{s}(r' - a)}\,(r'\sqrt{s} + 1)}{1 + a\sqrt{s} + a^2 s} . \qquad \text{(IV-54)}$$

To perform the inverse transform, Eq. (IV-54) is put into simpler form by means of the partial fraction reduction

$$\frac{\sqrt{s}\, r' + 1}{1 + a\sqrt{s} + a^2 s} = \frac{D^1(r')}{a\sqrt{s} + h} + \text{complex conjugate} \qquad \text{(IV-55)}$$

where

$$h = \frac{1 - i\sqrt{3}}{2} \qquad \text{(IV-56)}$$

and

$$D_1(r') = -\frac{i\sqrt{3}}{3}\left(1 - \frac{hr'}{a}\right). \qquad \text{(IV-57)}$$

By employing the convolution theorem, Eq. (III-40) and a table[5] of inverse transforms, we can write the final solution as

$$u(a,\xi) = 2\,R_e \left[E_1 \int_o^{\xi} \phi(\xi')\, d\xi' \left\{ \frac{1}{[\pi(\xi - \xi')]^{1/2}} - \frac{h}{a}\, e^{\frac{h^2}{a^2}(\xi - \xi')} \operatorname{erfc}\left(\frac{h}{a} \sqrt{\xi - \xi'} \right) \right\} \right.$$

$$\left. + a \int_o^{\infty} dr'\, H_o(r')\, D_1(r') \left\{ \frac{1}{(\pi\,\xi)^{1/2}}\, e^{-\frac{(r' - a)^2}{4\xi}} - \frac{h}{a}\, e^{\frac{h}{a}(r' - a) + \frac{\xi h^2}{a^2}} \operatorname{erfc}\left(\frac{r' - a}{2\sqrt{\xi}} + \frac{h\sqrt{\xi}}{a} \right) \right\} \right], \qquad \text{(IV-58)}$$

where

$$E_1 = \frac{3 - i\sqrt{3}}{6} \qquad \text{(IV-59)}$$

and

$$\operatorname{erfc}(Z) = \frac{2}{\sqrt{\pi}} \int_Z^{\infty} e^{-t^2}\, dt \qquad \text{(IV-60)}$$

is the complementary error function.[5]

The current as before is

$$I(\theta,t) = 2\pi \sin^2\theta\; u(a,\xi). \qquad \text{(IV-61)}$$

Note there is the explicit need for the transformation from t to ξ as given by Eq. (II-25).

As a check, the solution (IV-58) should reduce to

$$u(a,\xi) = a\, H_o(a) \qquad \text{(IV-62)}$$

in the limit $\xi \to 0$. To see this it is sufficient to consider the third integral (all the others are zero in this limit). Let

$$x^2 = \frac{(r' - a)^2}{4\xi} . \tag{IV-63}$$

Then the third integral becomes

$$I_3 = \frac{2a}{\sqrt{\pi}} \int_0^\infty e^{-x^2} H_o(a + 2\, x \sqrt{\xi}\,)\, dx \tag{IV-64}$$

which obviously has the proper limit.

V. Discussion and Examples

As pointed out in the Introduction, the prime concern of this note is to describe an analysis of currents induced on metallic bodies in the presence of a nearby nuclear surface burst. These were estimated from surface currents induced on perfect conductors according to

$$\vec{K} = \hat{n} \times \vec{H} \tag{I-1}$$

which means it is necessary to compute $\vec{H}$, the magnetic field intensity. Accurate calculations of this field come, of course, from solutions to Maxwell's Eqs. (II-1), (II-2) and (II-3) (or equivalently, the wave Eq. (II-5)) with the proper boundary conditions and source terms. There are two common ways to approach the problem. One is to attack the equations directly by means of numerical finite difference techniques. The second approach, adopted here, is to analyze Eq. (II-5) with the classical techniques of separation of variables (at least when the conductivity is low enough). The discussion in Section II centered around the separation of the equation into temporal and spatial factors, and Sections III and IV were concerned with further separation of the equations resulting from the spatial part. At later times when the conductivity is expected to grow in magnitude, a very good approximation is to ignore the displacement current altogether as discussed in Section II. The resulting equations were then solved directly as shown in the latter parts of Sections III and IV.

At earlier times, the separated temporal differential equation was chosen to be a Sturm-Liouville Equation (with proper boundary conditions) on a finite time interval. Because of the presence of the time-varying conductivity and its functional dependence on the time, analytic solutions of this system are out of the question and numerical techniques have to be employed to generate the necessary eigenvalues and eigenfunctions. This approach was first described by P. Toulios.[2] We might remark that in general the separated differential equations result in Sturm-Liouville systems and that the time part of the problem is no exception.

Scattering from a plane and sphere were considered in Sections III and IV respectively. As remarked in the Introduction these are complementary in the sense that the plane approximates the scattering from a body of which the dimensions are large in comparison to the relevant wavelengths of the fields while the sphere approximates the opposite situation. At higher conductivities, however, the geometry of the scatterer is not too important. Physically, what may occur is a skin-depth phenomenon. That is, for high conductivities the important dimensions of the conducting medium may become much smaller than the diameter of the sphere for the range of relevant frequencies and the locally induced currents do not depend on the features of the sphere far removed from any small segment under consideration. Another way of saying this is that for higher conductivities, the sphere acts more and more like a plane. Therefore it can also be anticipated that the current densities induced on a sphere may approach those induced on a plane for larger and larger radii (the examples below illustrate this). These considerations can be made quantitative. Inspection of Eqs. (III-44) and (IV-58) reveals the common expression (aside from a trivial sign change and the multiplying factor a, the radius of sphere, in the definition Eq. (IV-40)),

$$\frac{1}{\sqrt{\pi}} \int_0^{\xi} \frac{\phi(\xi')\, d\xi'}{(\xi - \xi')^{\frac{1}{2}}} \,. \qquad \text{(V-1)}$$

Trial calculations with the computer code have confirmed the dominance of this term in the region of higher conductivity (see the examples which follow). Note that Eq. (V-1) does not contain any reference to a particular geometry, i.e., Eq. (V-1) could be used to estimate induced currents on any body provided the skin depths are smaller than the relevant dimensions. Since the integral involved is a contribution to the H-field, the total current could be found by including an appropriate perimeter-like factor. The definition of the variable Eq. (II-28), requires the choice of t_o, the point at which the conduction current can be considered to be much greater than the displacement current. This can usually be estimated from the input data $E^i(t)$ and $\sigma(t)$. Note also that using the integral Eq. (V-1) is equivalent to ignoring the contributions from $H_o(r)$. Thus at later times at least (higher conductivities), the entire Toulios eigenfunctions expansion can be ignored and the induced current can be estimated from Eqs. (V-1) and(II-28) alone.

However, at early times when the conductivity is low the eigenfunction expansion is necessary. In addition to the two geometries above, similar but unpublished calculations have been performed for the infinite cylinder and the prolate spheroid when the incident electric field is parallel to the axis of rotational symmetry. The need for application along with simplicity dictated the concentrated effort spent on programming the sphere solutions for a digital computer.

In general, the importance of the conductivity cannot be emphasized too highly. It is evident from the fundamental Eqs. (II-1) and (II-2) that the solutions scale linearly with the magnitude of the incident electric field. The conductivity, on the other hand, enters non-linearly and can have an enormous effect in which currents and current rates can increase by an order of magnitude or so. A metallic object always has some reactive (capacitive, inductive) relationship to its environment. When the conductivity of surrounding medium rises, this relationship can change to a more resistive one with consequently large changes in the induced currents. It is important to realize that not only is it inconsistent with Maxwell's equations to try to estimate induced currents from fields alone but that these estimates will be grossly underestimated if the conductivity is omitted. These points are illustrated in the examples below.

Before turning to the examples, however, we would like to make a few comments on the analysis and the computer code in general. An important question is how many terms are to be included in the Toulios eigenfunction expansion. Usually one can estimate what high frequencies are likely to result from an analysis of the incident electric field by considering those regions in which the field is changing most rapidly. It is well known[3,4] that the higher order functions in a Sturm-Liouville system become sinusoidal. Thus a direct comparison can be made and the number of even and odd eigenfunctions to be included can be determined. On this basis, for a few test cases, the eigenfunction expansions were examined term-by-term and it was found that indeed the low order terms were orders of magnitude greater than the latter ones. This, of course, is a somewhat case-dependent conclusion and one could always make provision for this examination in every case.

Comparisons were made between the known eigenvalues and eigenfunctions for $\sigma = 0$ and σ = constant (see Eqs. (II-17) thru (II-22)) and those generated numerically. The low order eigenvalues agreed to about 5 significant figures while the highest order agreed to about 3 (agreement here was always much better than 1%. See, for example, Table II). The values of the eigenfunctions agreed about as well. The reason the higher order comparisons are poorer than

n	λ_n^o	λ_n^e	$(\pi n/t_o)^2$
1	.034052	.021644	.028528
2	.13064	.10077	.11411
3	.27922	.24375	.25675
4	.48111	.44291	.45645
5	.74294	.69657	.71320
6	1.0597	1.0109	1.0270
7	1.4292	1.3841	1.3978
8	1.8585	1.8103	1.8258
9	2.3467	2.2936	2.3108
10	2.8876	2.8381	2.8528
11	3.4855	3.4381	3.4519
12	4.1447	4.0919	4.1081
13	4.8591	4.8059	4.8213
14	5.6275	5.5794	5.5915
15	6.4566	6.4063	6.4188
16	7.3449	7.2900	7.3032
17	8.2867	8.2355	8.2446
18	9.2861	9.2378	9.2431
19	10.348	10.294	10.298
20	11.465	11.411	11.411

TABLE I COMPARISON OF EIGENVALUES FOR t_o = 18.6 ns AND σ (FIG. 3)

n	$\lambda_n^{e,o}$	$(n\pi/t_o)^2$
1	.0061685	.0061685
2	.024674	.024674
3	.055517	.055517
4	.098696	.098696
5	.15421	.15421
6	.22207	.22207
7	.30226	.30226
8	.39480	.39478
9	.49968	.49965
10	.61692	.61685
11	.74651	.74639
12	.88846	.88826
13	1.0428	1.0425
14	1.2095	1.2090
15	1.3887	1.3879
16	1.5802	1.5791
17	1.7842	1.7827
18	2.0008	1.9986
19	2.2298	2.2268
20	2.4714	2.4674

TABLE II COMPARISON OF EIGENVALUES FOR t_o = 40 ns AND σ = 0 the low order ones can be traced to the fineness with which the fundamental time interval $(0, t_o)$ is divided. This division is made so that the highest eigenfunctions are represented by about five points per half-cycle. Substantiation of this comes from choosing different numbers of points in $(0, t_o)$ and then comparing eigenvalues and eigenfunctions of the same order for the two different cases. Since the eigenfunction expansions are performed over a time interval for which the conductivity is not too large, the calculations for the case of a variable can be assumed to be satisfactory.

To illustrate the results of the computer code, consider the input data in Fig. 3. It should be emphasized that this data is pure invention. It is not even consistent with Maxwell's equations and does not reflect any known real situation resulting from a nuclear burst. It was invented, however, with these four features in mind. First, there is the rapid rise time of both the incident electric field and the air conductivity. Second, the electric field rises first and is then followed later by the conductivity. Third, the electric field peaks before the conductivity, and the decay of the field is then more rapid than the conductivity. Fourth, the data ranges span orders of magnitude. While this is not'real' data, it does serve to illustrate the workings of the computer program and the kinds of results obtained from the foregoing analysis. The electric field is plotted with an arbitrary scale because the solutions scale linearly as mentioned above. The conductivity given in Fig. 3 is referred to as $\sigma(t)$ in the discussion which follows. For comparison, examples are also given for $.25\sigma$, $.5\sigma$ and 2σ. The switchover points from the eigenfunction expansion to the diffusion approximation are given in Table III along with the corresponding ratios of the conduction current to the displacement current.

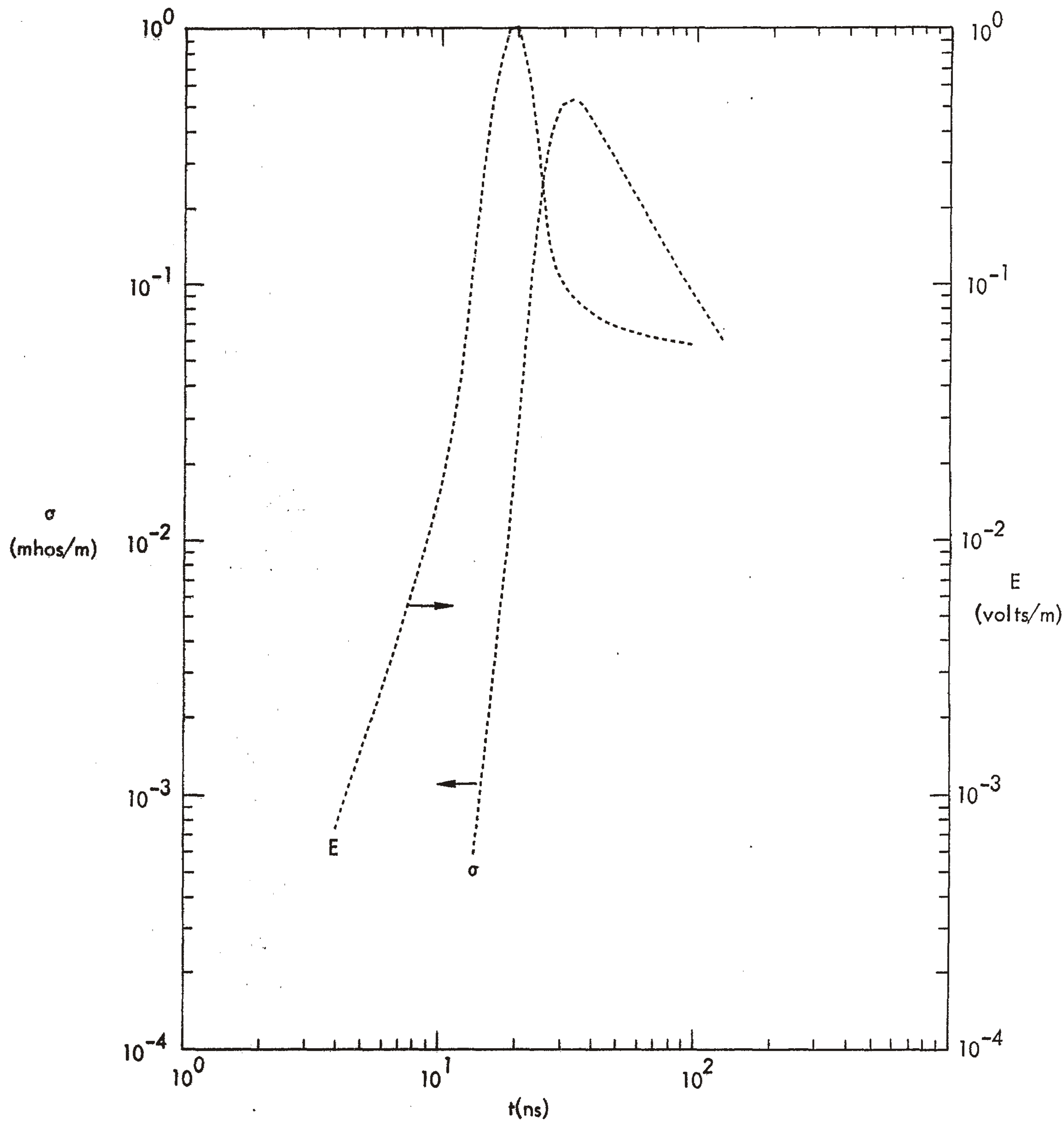

Fig. 3 Time Varying Conductivity and Incident Electric Field

CONDUCTIVITY	t_o (ns)	I_{cond}/I_{disp}
.25 σ	19.2	11.0
.5 σ	19.0	9.68
σ	18.6	7.49
2 σ	18.2	7.82

TABLE III SWITCHOVER TIMES FOR THE NON-ZERO CONDUCTIVITIES ALONG WITH THE RATIOS OF CONDUCTION TO DISPLACEMENT CURRENTS AT THESE TIMES

Figures 4 through 7 show typical computer-generated eigenfunctions for the conductivity of Figure 3 ($\sigma(t)$ and t_o = 18.6 ns). Note they display damped behavior toward the end of the interval as expected (see Eqs. (II-21) and (II.22) in this regard). Figure 8 illustrates the variation of the variable ξ, Eq. (II-28), with t from 18.6 ns to 100.5 ns.

Figure 9 shows the equatorial currents induced on spheres of radii .3M, .5M and 1M out to 40 ns. These were computed with 20 even and 20 odd eigenfunctions. Of course, the diffusion approximation is not appropriate here and the entire interval was used for the eigenfunction expansion. Note the oscillations of the current which can be interpreted as a natural ringing of the sphere when subjected to the incident field. The damping of the oscillations toward the end of the interval **indicate** at later times the spheres re-radiate as expected.

Figures 10, 12, 14 and 16 show equatorial currents for .25σ, .5σ, and 2σ respectively for the three perfectly conducting spheres. The currents in the first three cases were calculated with 20 even and 20 odd eigenfunctions while the last was calculated with 30 even and odd. These values were estimated to be sufficient to give high enough frequencies to reproduce the response of the spheres. Note that the curves have some small oscillations superimposed on the general trend of the current. These have a period about that corresponding to the first ignored eigenfunctions. Since these oscillations are down by least one order of magnitude or two in comparison to the currents at later times (the region of interest), no attempt was made to suppress them.

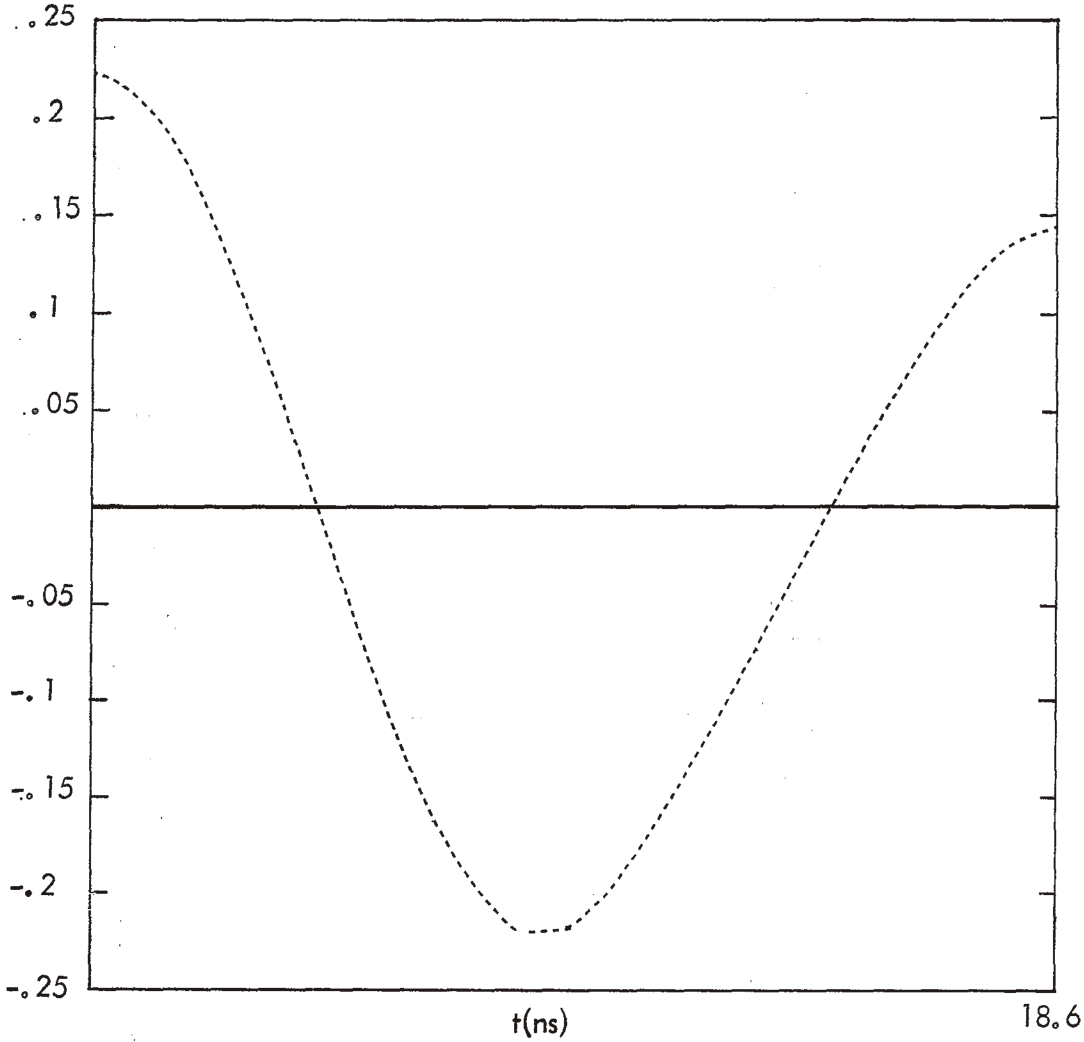

Fig. 4 Second Even Normalized Eigenfunction

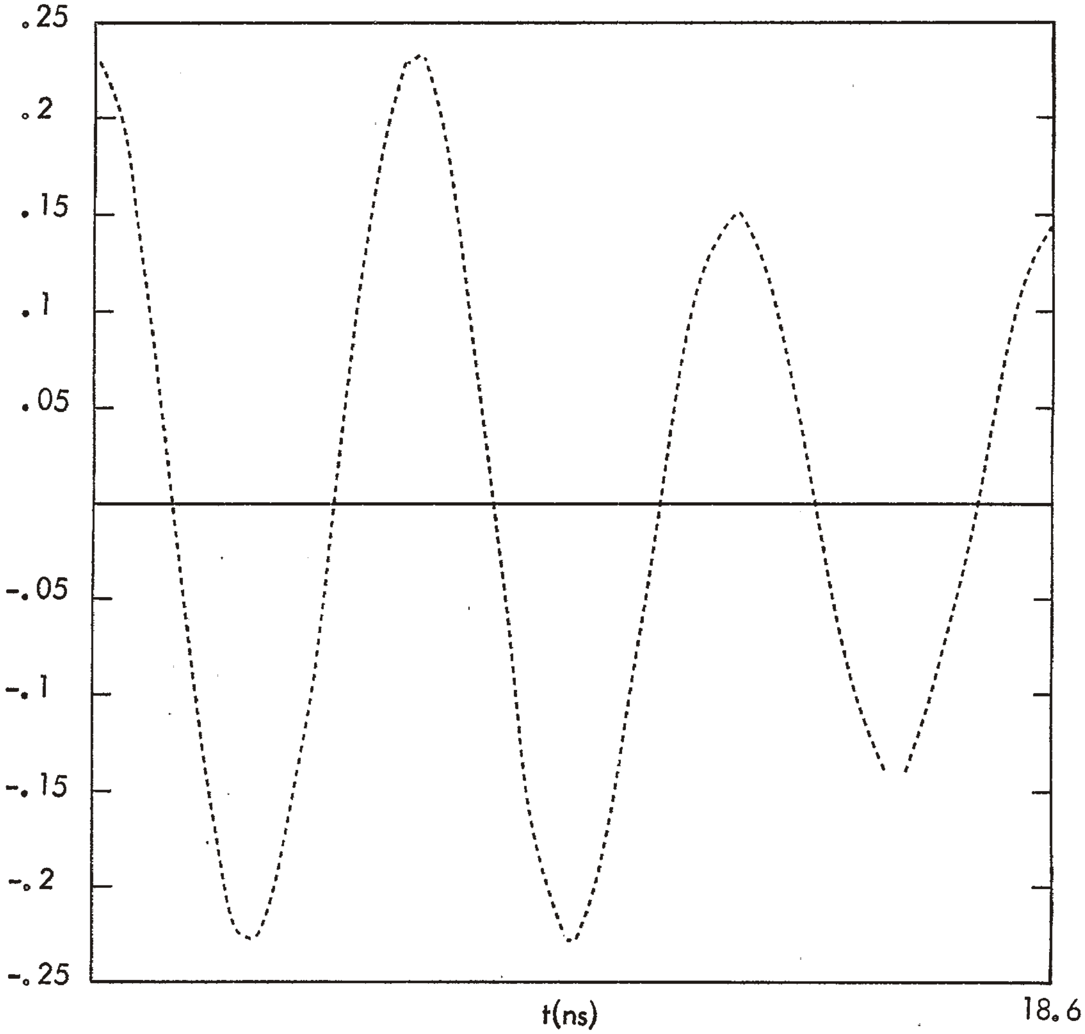

Fig. 5 Sixth Even Normalized Eigenfunction

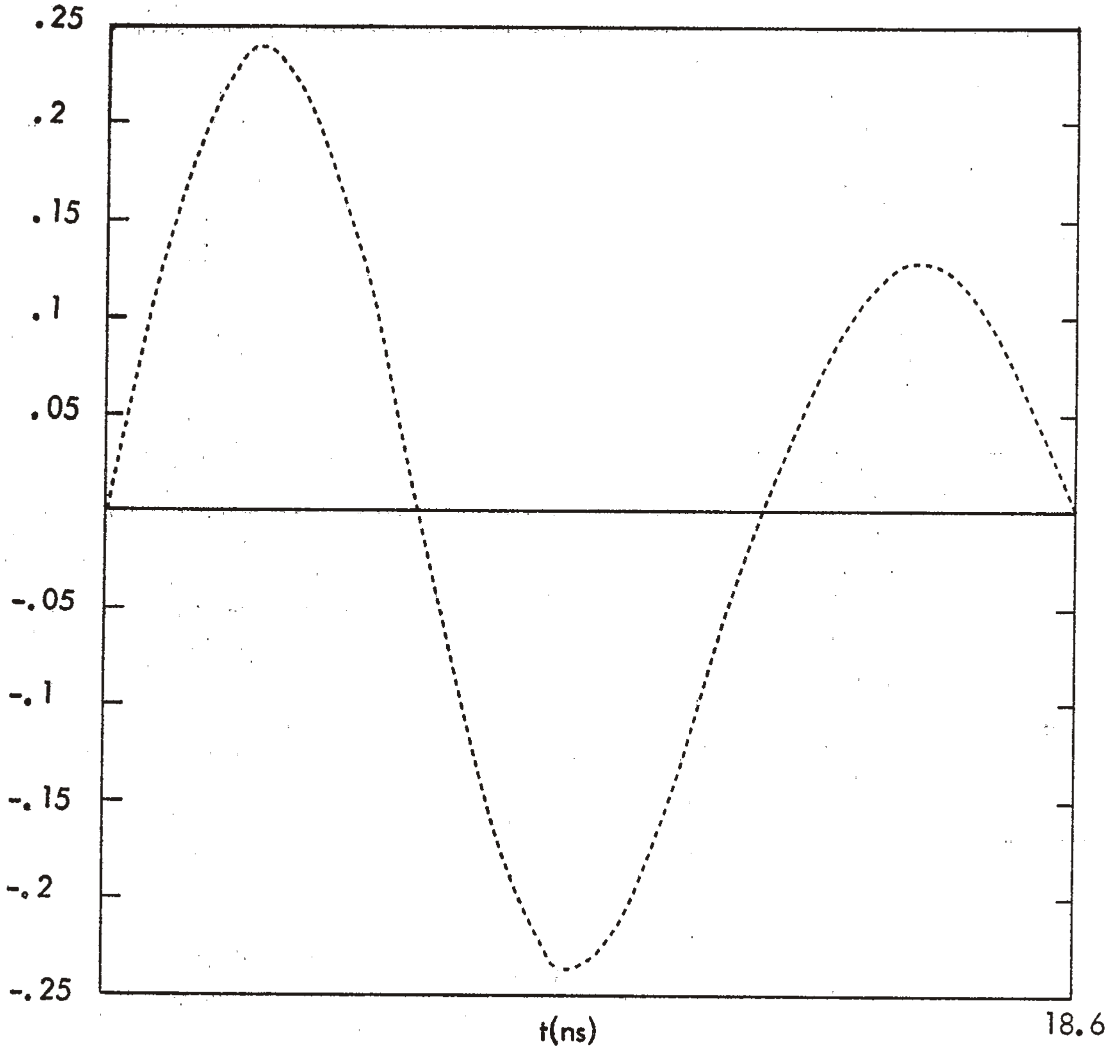

Fig. 6 Third Odd Normalized Eigenfunction

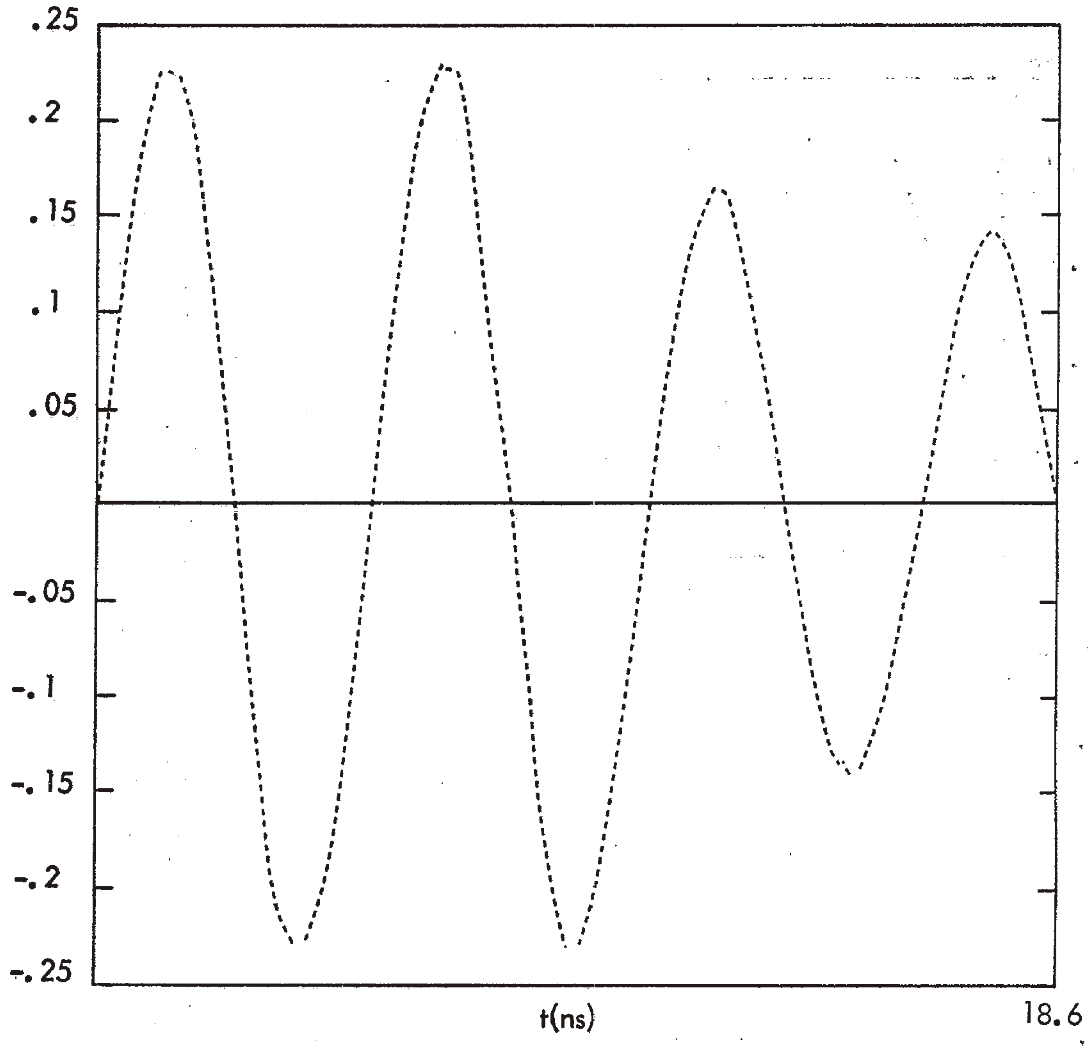

Fig. 7 Seventh Odd Normalized Eigenfunction

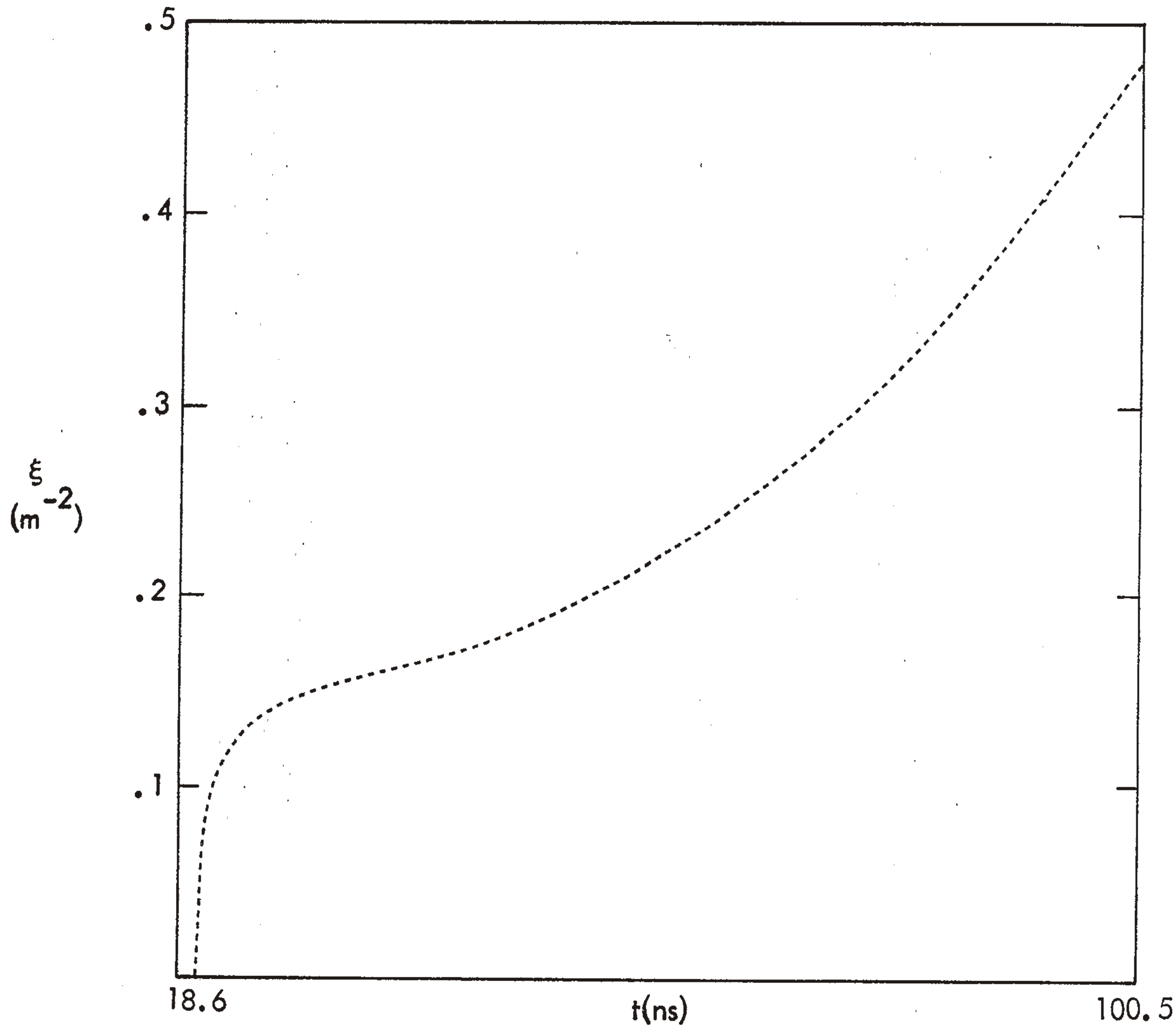

Fig. 8 Variation of ξ (Eq. (II-28)) with Time

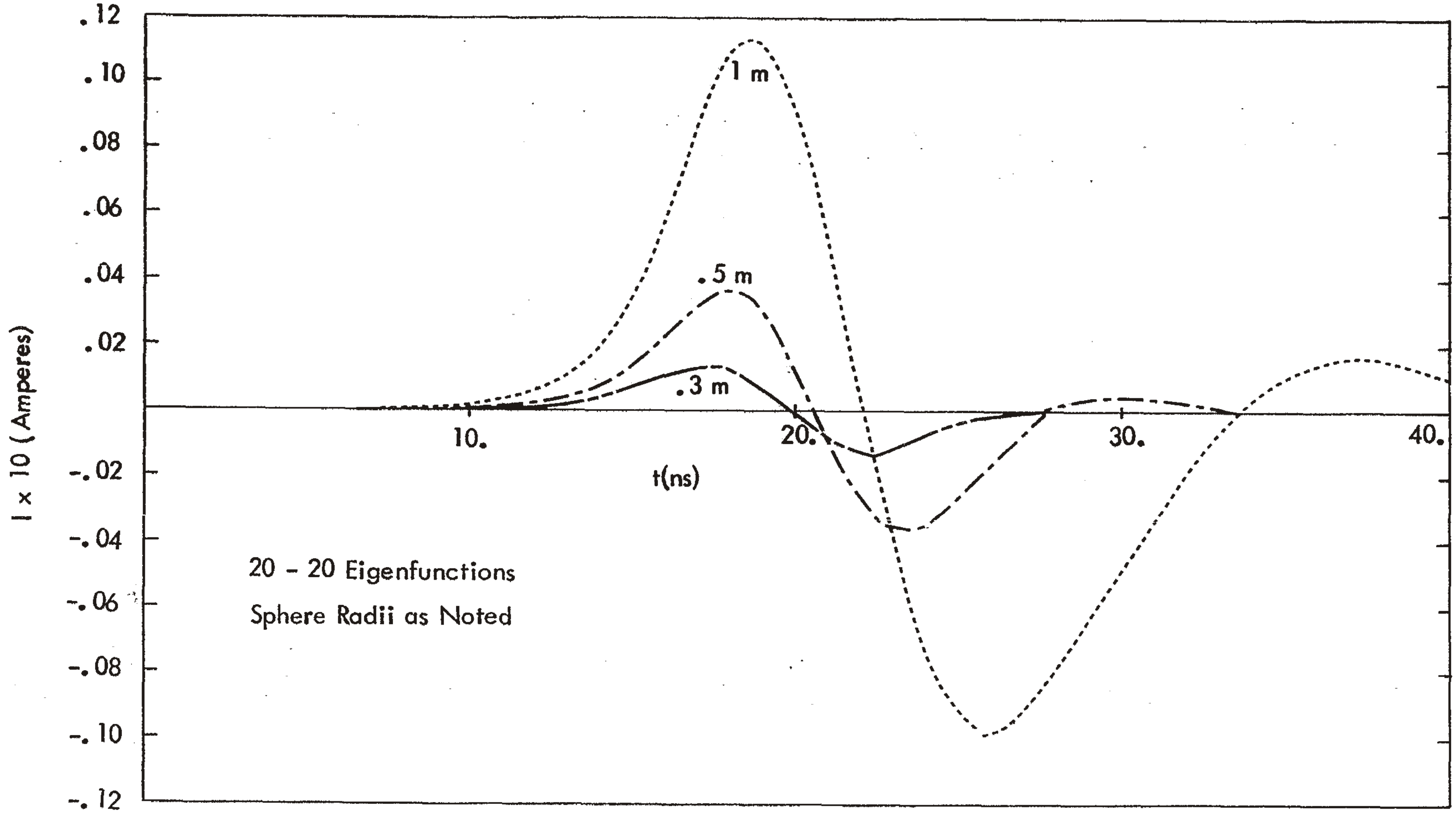

Fig. 9 Equatorial Currents Induced on Spheres for Zero Conductivity

Figures 11, 13, 15 and 17 show the equatorial currents calculated with the diffusion approximation from 19 ns to 28 ns (where the currents peak). These currents are the continuations of the current plots discussed in the previous paragraph. A comparison of these figures with Figure 9 serves to illustrate the decided importance with which the presence of conductivity can influence results. The enhancement factors given in Table IV and plotted in Figure 18 are defined as the ratio of the peak values of the currents and current-rates for $\sigma \neq 0$ to those for $\sigma = 0$ for each of the three spheres. Finally, Figures 11, 13, 15 and 17 also illustrate the approximation Eq. (V-1) to the current in the high conductivity region. As mentioned above, this approximation is better for higher conductivities and larger spheres.

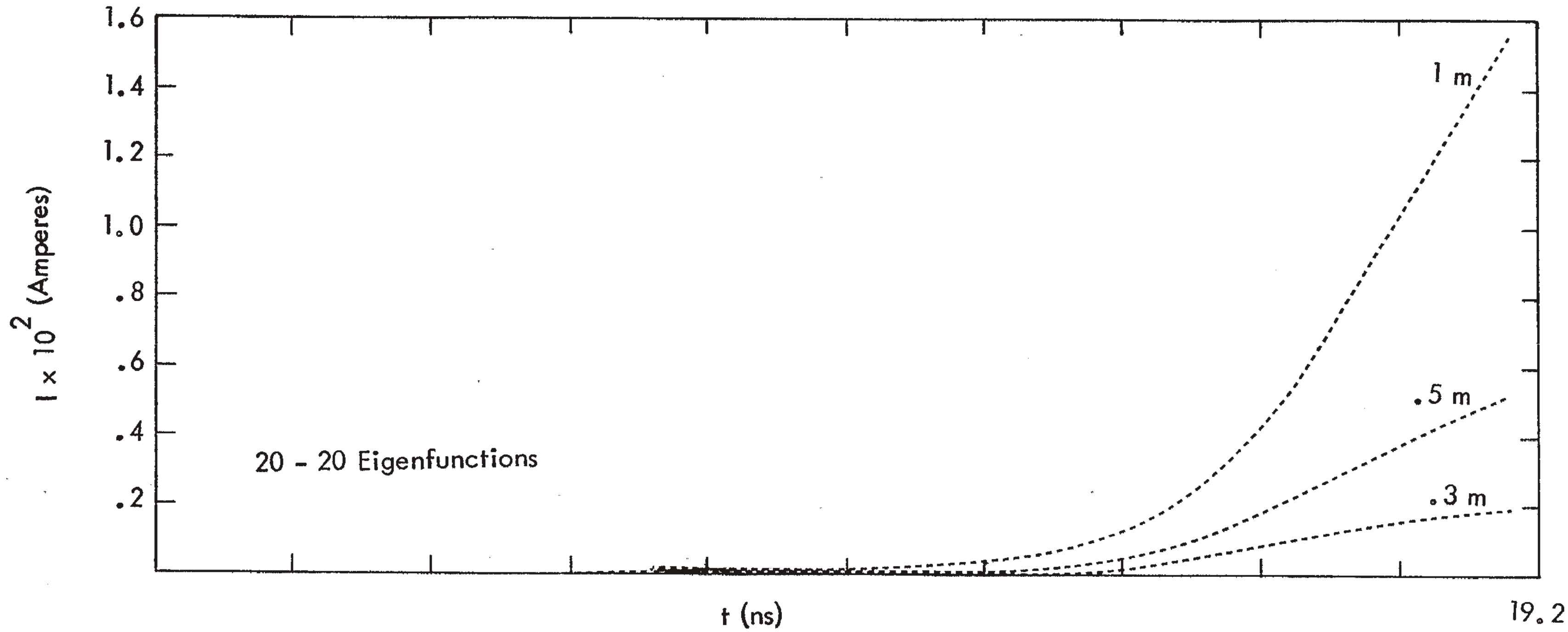

Fig. 10 Equatorial Currents Induced on Spheres at Early Times for .25 σ

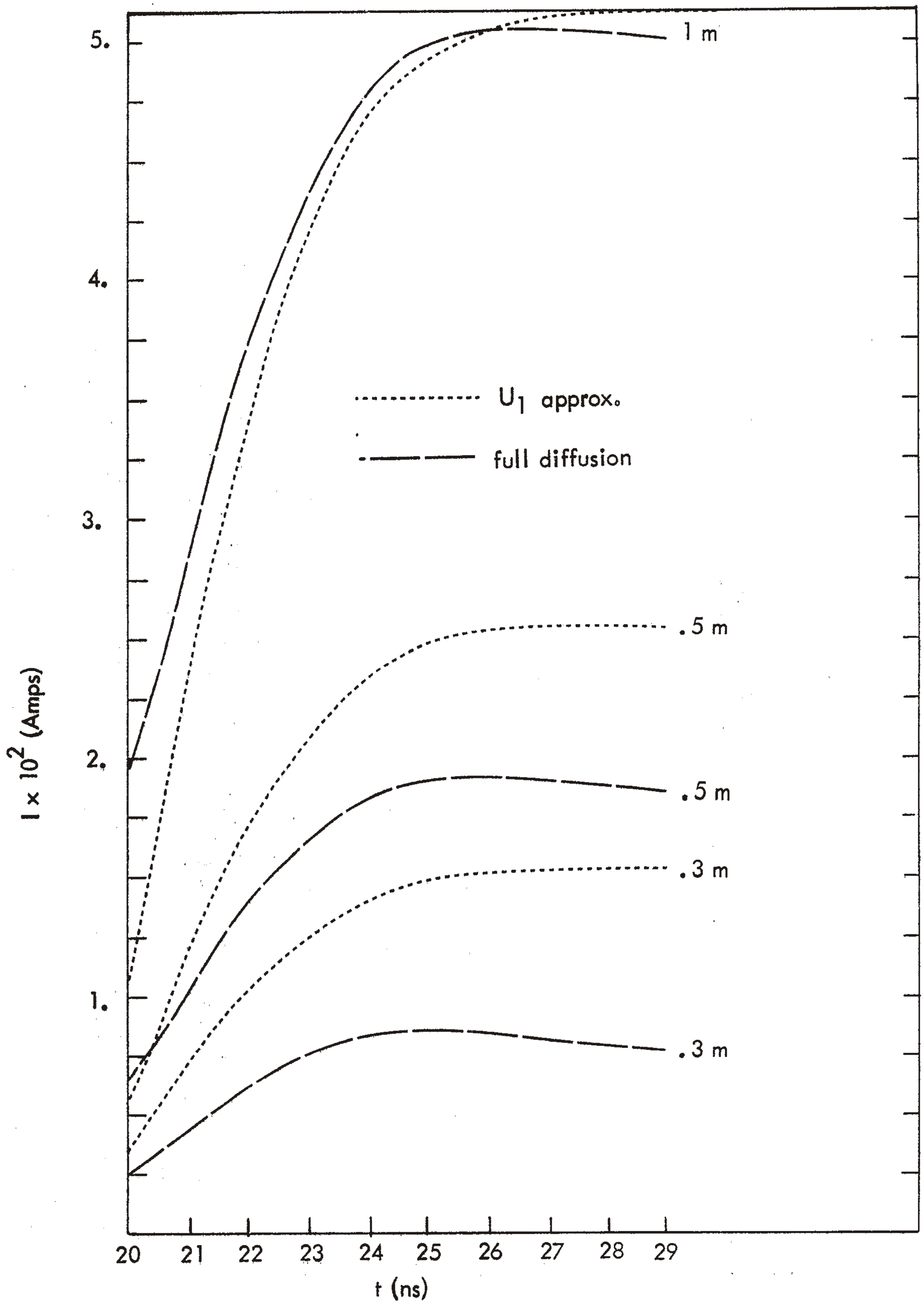

Fig. 11 Equatorial Currents Induced at Later Times for .25 σ

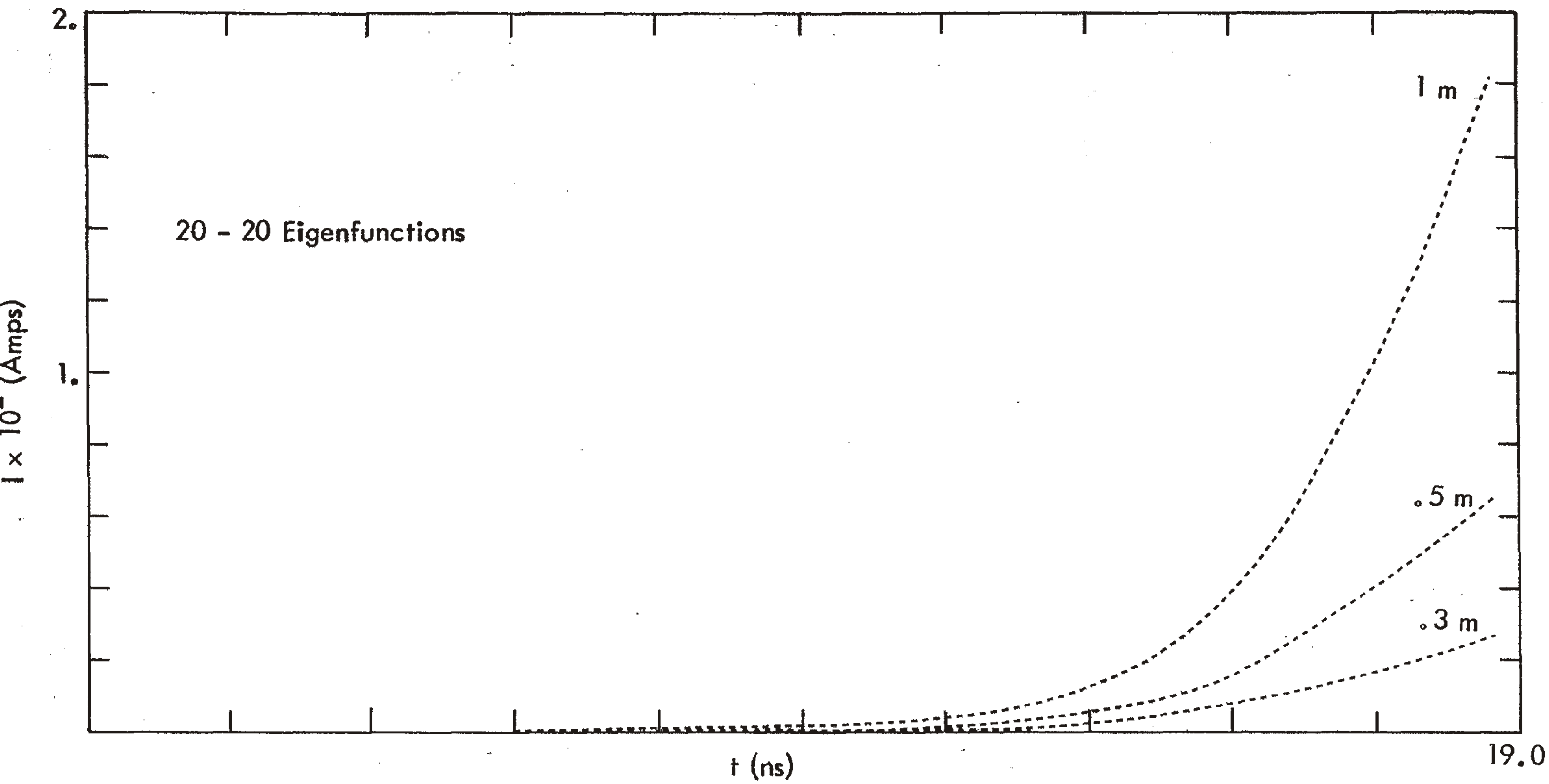

Fig. 12 Equatorial Currents Induced at Early Times for .5 σ

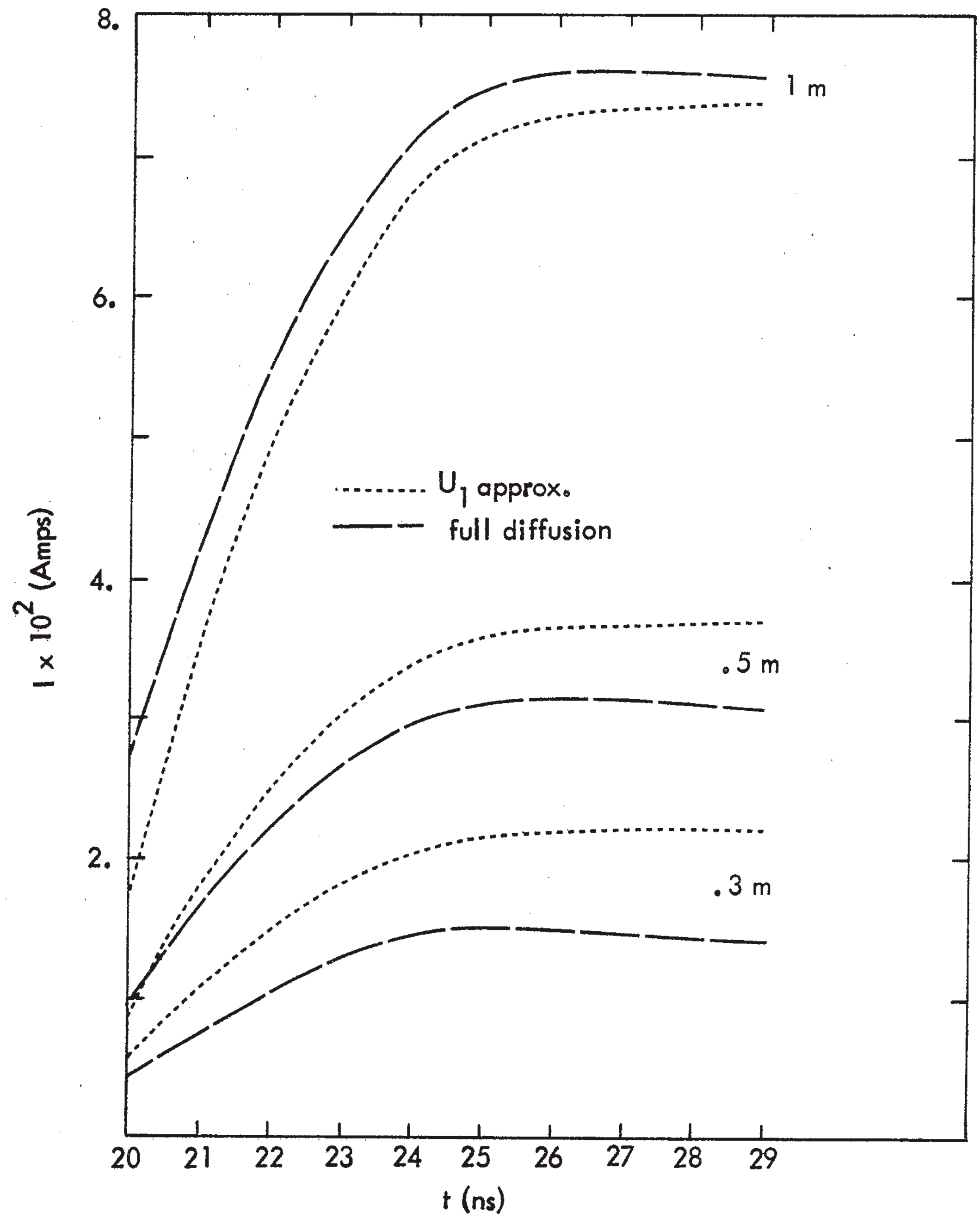

Fig. 13 Equatorial Currents Induced at Later Times for .5 σ

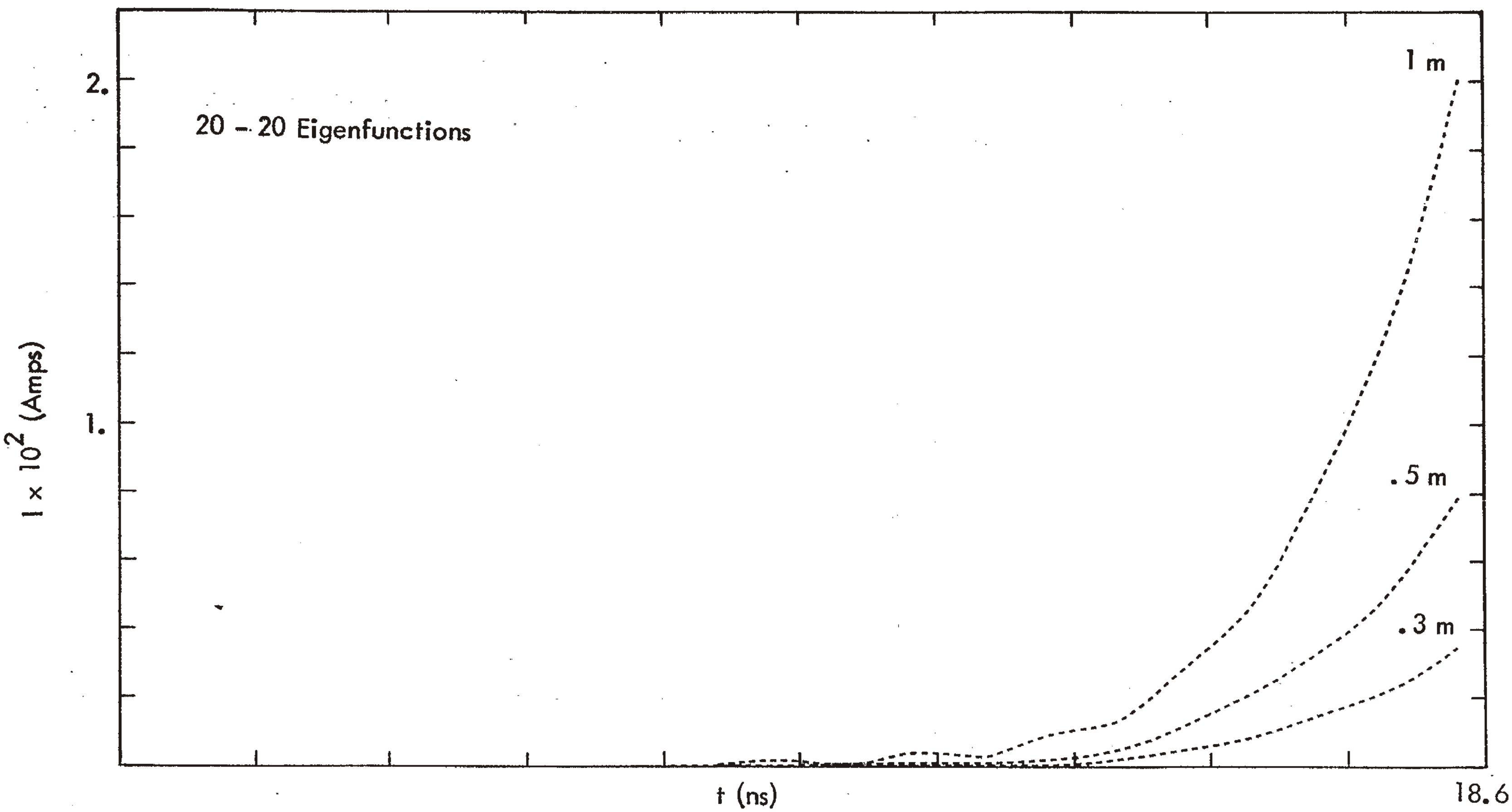

Fig. 14 Equatorial Currents at Early Times for σ

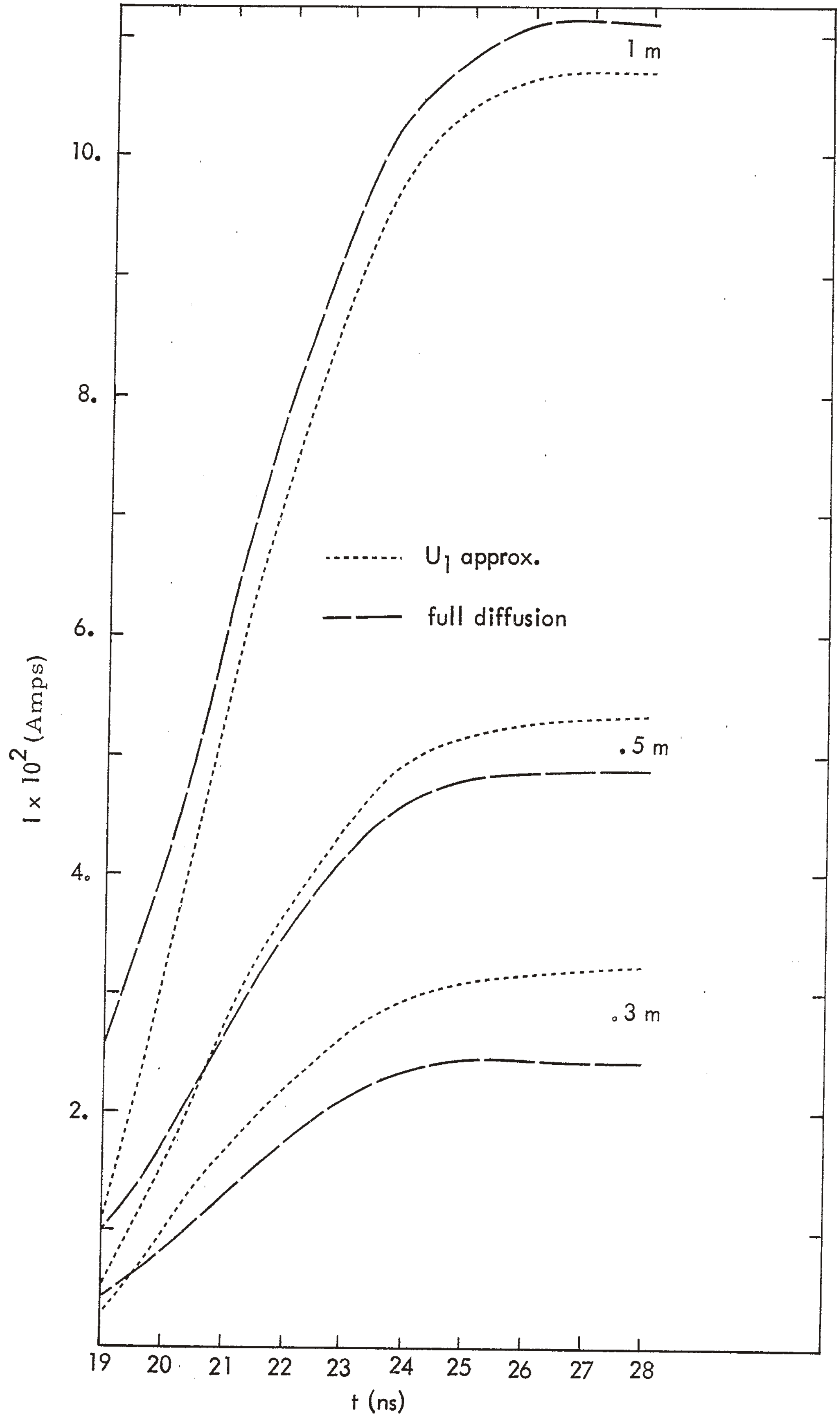

Fig. 15 Equatorial Currents at Later Times for σ

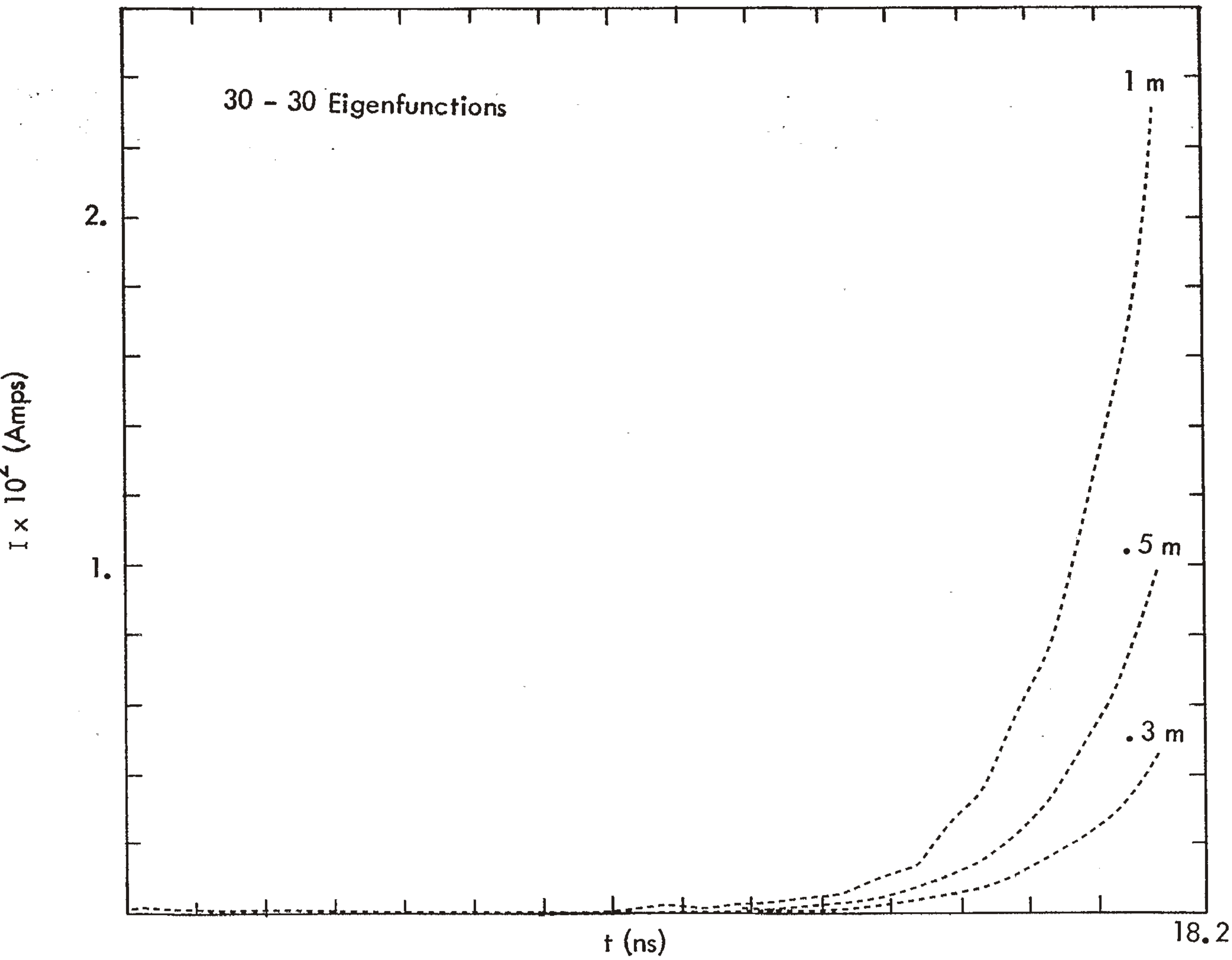

Fig. 16 Equatorial Currents at Early Times for 2 σ

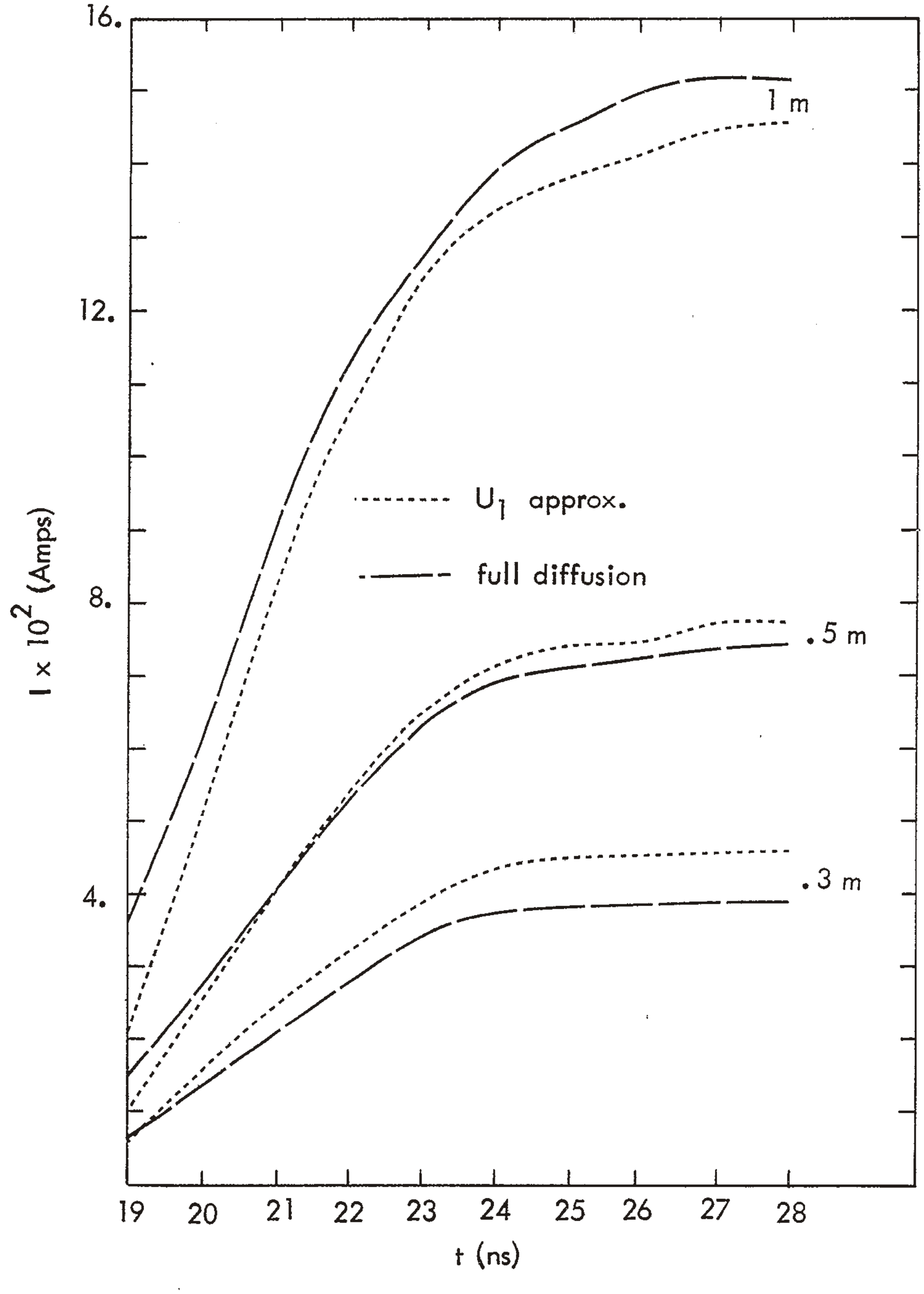

Fig. 17 Equatorial Currents at Later Times for 2 σ

ā (Meters)		.25σ	.5σ	σ	2σ
.3	Current	6.39	11.0	18.2	28.8
	Rate	2.15	3.53	4.98	7.49
.5	Current	5.18	8.44	13.1	20.0
	Rate	1.79	2.82	3.89	5.99
.1	Current	4.34	6.52	9.55	13.8
	Rate	1.88	2.85	3.85	5.70

TABLE IV ENHANCEMENT FACTORS (SEE TEXT) FOR CURRENTS AND CURRENT-RATES

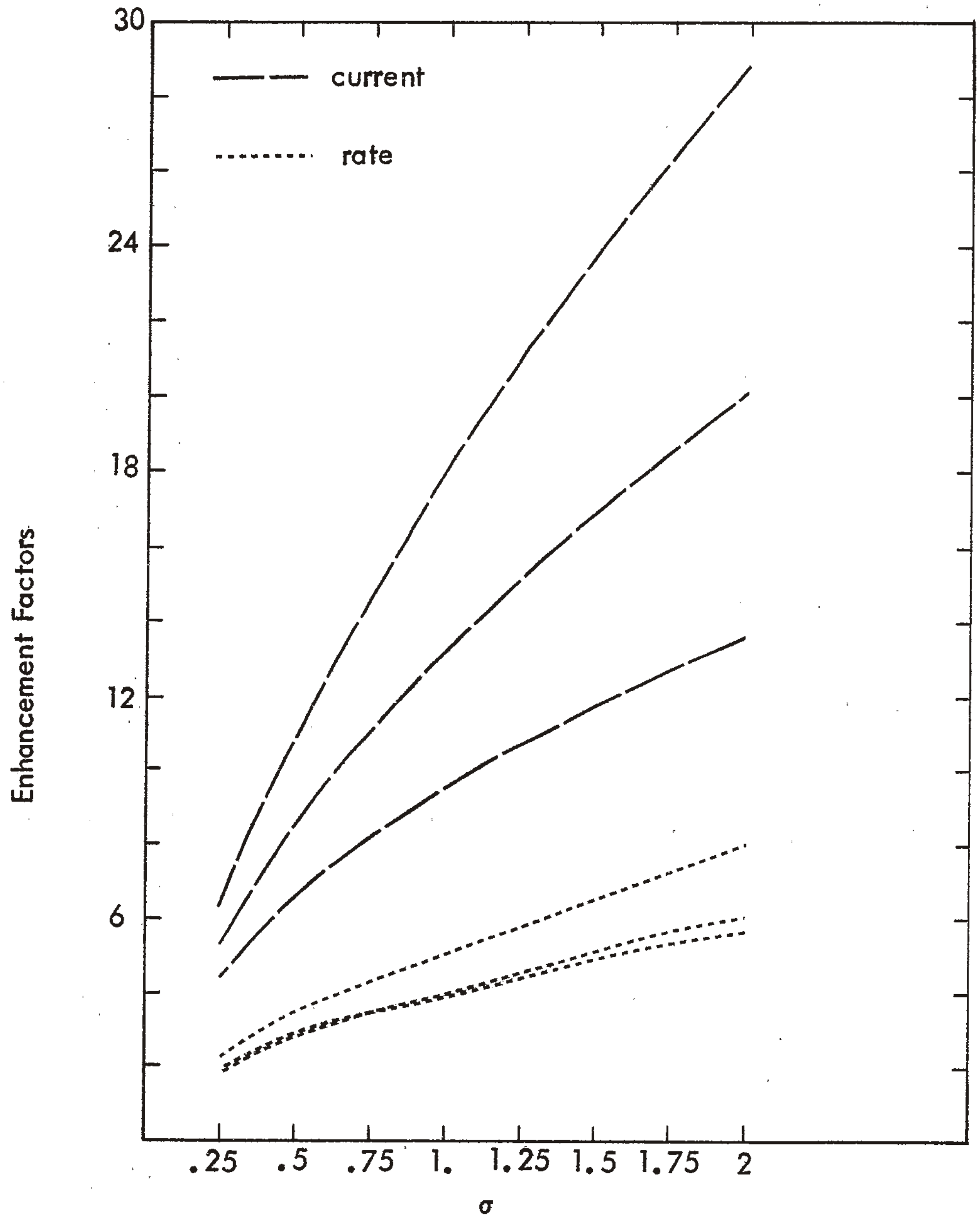

Fig. 18 Enhancement Factors (see text) as a Function of Conductivity

REFERENCES

1. Panofsky, W. K. H. and M. Phillips, Classical Electricity and Magnetism, Addison-Wesley, 1955

2. Toulios, P., Missile Flying Through a High Altitude Source Region, Interaction Note 58, February 1970

3. Courant, R. and D. Hilbert, Methods of Mathematical Physics, Interscience, 1953

4. Morse, P. M., and H. Feshback, Methods of Theoretical Physics, Mc-Graw-Hill, 1953

5. Abramowitz, M. and I. Stegun, ed., Handbook of Mathematical Functions, National Bureau of Standards, Applied Mathematics Series No. 55, U. S. Government Printing Office, 1964

6. Baum, Lt. C. E., Sensor and Simulation Note 15, Radiation and Conductivity Constraints on the Design of a Dipole Electric Field Sensor, Feb. 1965

7. Baum, Capt. C. E., Interaction Note 76, Electromagnetic Pulse Interaction Close to Nuclear Bursts and Associated EMP Environment Specification, July 1971