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Application of the Singularity Expansion Method to Scattering From Imperfectly Conducting Bodies and Perfectly Conducting Bodies Within a Parallel Plate Region

by

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Abstract

This note is a continuation of previous work on the singularity expansion method. Different integral-equation formulations describing electromagnetic scattering from imperfectly conducting bodies are considered. A set of volume-surface integral equations is used to determine the analytical properties in the complex frequency plane of the field scattered from imperfectly conducting, finite bodies. Conditions are determined for the constitutive parameters of the scattering body so that the scattered field can be described only by damped sinusoidal oscillations when the incident field is a delta function plane wave.

Scattering from a perfectly conducting, finite body within a parallel plate region is also considered. It is shown that the singularities in the complex frequency plane of the scattered field are poles and branch cuts. The locations of the branch cuts depend only on the separation between the parallel plates.

I. Introduction

The singularity expansion method for solving electromagnetic interaction problems is first introduced in Interaction Note 88[1]. Scattering from a perfectly conducting body of finite extent is considered in Interaction Note 92[2]. It is shown in [2] that the operator inverse to the integral operator of the magnetic field formulation is an analytic operator-valued function in the complex frequency (s) plane except at certain points in the left half plane where it has poles. A representation of the inverse operator in terms of the natural modes and frequencies is also given in reference [2]. A scheme for the numerical evaluation of the natural modes and frequencies for perfectly conducting bodies of revolution has been developed in reference [27]. In [27] the magnetic field integral equation is simplified to account for rotational symmetry. A computer code for the numerical evaluation of the natural frequencies and modes is also presented in [27] together with detailed numerical calculations of some of the natural frequencies and modes for a perfectly conducting prolate spheroid.

The natural frequencies and modes of a thin cylinder has been calculated numerically in reference [25] by using the electric field integral equation. Some approximate analytical results of the locations of some of the natural frequencies of a thin cylinder have been derived in reference [26]. The integral equation is solved approximately by using Fourier transform methods combined with the Wiener-Hopf technique. The solution thus obtained gives an accurate prediction of the locations of the natural frequencies of perfectly conducting cylinders with diameter-to-length ratio less than 1/100[25]. However, due to the approximations introduced, the method fails to give the correct analytical properties in the complex frequency plane of the field scattered from a thin wire. For example, the approximations introduce additional branch points which should not be present.

In this note we will consider the general problems of scattering from imperfectly conducting, inhomogeneous bodies in free space and of scattering from perfectly conducting bodies within a parallel plate region. The mathematical methods that we use resemble in a way those used in reference [2].

In section III, electromagnetic scattering from an imperfectly conducting body is first formulated in terms of a volume integral equation for the fields inside the body. The mathematical formalism used in section III is similar to the one used in deriving the so-called Oseen extinction theorem[9-12]. The integral equation is of the second kind. However, due to the singularity in the kernel of the integral equation we are unable to draw any conclusions about the analytical properties of the solution of the integral equation. Therefore, in section IV, we reformulate the electromagnetic scattering problem in terms of a set of volume-surface integral equations. When the scattering body is a pure dielectric body this set of integral equations reduces to an integral equation previously used in connection with considerations of uniqueness and existence of the solution of certain electromagnetic scattering problems[13].

This set of volume-surface integral equations is solved by using the Fredholm determinant theory. From this solution it follows that the scattered field has two types of singularities in the complex frequency plane. The first type is due to the singularities of the incident field (waveform singularities). The second type is due to the scattering body itself. Sufficient conditions for this latter type of singularities to consist of only poles are (1) the scattering body is of finite extent and (2) the constitutive parameters σ, ε, μ together with $\nabla\varepsilon'/\varepsilon'$ and $\nabla\mu/\mu$ are analytic functions in the entire complex frequency plane. Here, $\varepsilon' = \varepsilon + i\sigma/\omega$, and σ, ε, μ are the conductivity, permittivity and permeability, respectively, of the scattering body.

A knowledge of the order of each pole is necessary when solving transient electromagnetic interaction problems by using the singularity expansion method. As of now, no general analytical method of determining the order of each pole has been found. Therefore, the order of each pole has to be determined numerically when solving a specific problem. However, a general method for the numerical evaluation of the order of each pole has been discussed in reference [27].

Scattering from a perfectly conducting cylinder within a parallel plate region has been treated in references [20] through [24] by using the conventional method of first solving the integral equation numerically in the frequency domain and then applying a numerical inverse Fourier transform. In section V of this note we will investigate the analytical properties of the field scattered from a perfectly conducting, finite body, located within a parallel plate region. The effect of the parallel plates can be taken into account by using the method of images. Due to the interaction between the body and the two parallel plates we will get an infinite set of image bodies. Therefore, the theory developed in reference [2] cannot be applied directly.

The series representation of the Green's function, derived from the method of images, converges only in the right half of the complex frequency plane. In order to make use of the singularity expansion method we first have to analytically continue the Green's function into the entire complex frequency plane. This is done by finding an integral representation of the series defining the Green's function. From this integral representation it follows that the kernel of the integral equation has branch cuts in the left half plane. The locations of these branch cuts are uniquely determined by the distance separating the parallel plates.

Once a representation of the Green's function, valid in the entire complex frequency plane, is available, it is easy to show that the operator inverse to the integral operator of the magnetic field formulation has two types of singularities. One type is poles at those frequencies where there exist nontrivial solutions of the homogeneous integral equation. The other type is branch cuts coinciding with the branch cuts of the Green's function.

Before starting with the analysis outlined here we will in section II briefly consider a surface integral equation describing scattering from a homogeneous, imperfectly conducting body.

II. Scattering From a Finite, Imperfectly Conducting, Homogeneous Body

Before we consider scattering from an inhomogeneous, imperfectly conducting body we will discuss, in this section, briefly the special case of scattering from a homogeneous, imperfectly conducting body. Mathematically this scattering problem can be formulated in terms of a set of two coupled surface integral equations.

Let the scattering body occupy the region V of finite extent. The boundary surface of V is denoted by S and the outward unit normal of S is denoted by $\underline{n}$. We also assume that the electromagnetic properties of the scattering body can be described by the constitutive parameters σ, ε, μ. These parameters are allowed to vary with frequency, but they do not vary with position within V. (See figure 1). Throughout this section we use harmonic time dependence $\exp(-i\omega t)$.

Introduce the effective electric surface current $\underline{j}$ and the effective magnetic surface current $\underline{k}$, defined by

$$\underline{j} = \underline{n}\times\underline{H} \tag{2.1}$$

$$\underline{k} = \underline{n}\times\underline{E}. \tag{2.2}$$

Assuming that all sources of the electromagnetic field are outside V we can derive the following set of two coupled integral equations for $\underline{j}$ and $\underline{k}$[3]

$$\begin{aligned} &\frac{1}{2}\underline{j} - \underline{\underline{L}}\cdot\underline{j} - Z_o^{-1}\,\underline{\underline{M}}\cdot\underline{k} = \underline{j}^{inc} \\ &\frac{1}{2}\underline{k} + \underline{\underline{L}}_c\cdot\underline{k} - Z_c\underline{\underline{M}}_c\cdot\underline{j} = 0 \end{aligned} \tag{2.3}$$

where

$$(\underline{\underline{L}}\cdot\underline{g})(\underline{r}) = \int_S \underline{n}(\underline{r})\times[\nabla G(\underline{r},\underline{r}';ik)\times\underline{g}(\underline{r}')]dS', \tag{2.4}$$

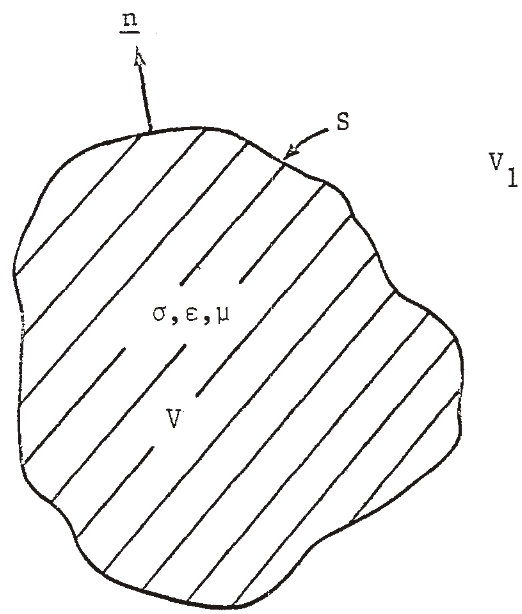

Figure 1. A homogeneous, imperfectly conducting, finite body.

$$(\underline{\underline{M}}\cdot\underline{g})(\underline{r}) = (ik)^{-1} \fint_S \underline{n}(\underline{r})\times\Big\{k^2\, G(\underline{r},\underline{r}';ik)\underline{g}(\underline{r}')$$

$$- [\kappa_u(\underline{r}') + \kappa_v(\underline{r}')][\nabla G(\underline{r},\underline{r}';ik)][\underline{n}(\underline{r}') - \underline{n}(\underline{r})]\cdot\underline{g}(\underline{r})$$

$$- [\underline{g}(\underline{r}') - \underline{g}(\underline{r})]\cdot\nabla_s'\nabla G(\underline{r},\underline{r}';ik)\Big\}\, dS', \tag{2.5}$$

where

$$G(\underline{r},\underline{r}';ik) = (4\pi|\underline{r} - \underline{r}'|)^{-1}\exp(ik|\underline{r} - \underline{r}'|)$$

the free-space Green's function, ∇_s' is the tangential gradient with respect to $\underline{r}'$, κ_u and κ_v are the principal curvatures of S[4], and $\fint_S$ indicates the principal value of the integral, i.e.,

$$\fint_S \cdots = \lim_{\delta\to 0}\int_{S-S_\delta}\cdots$$

Here $S-S_\delta$ is the part of the surface S outside a sphere with radius δ and center at $\underline{r}$. The subscript c in (2.3) on the operators $\underline{\underline{L}}_c$ and $\underline{\underline{M}}_c$ indicates that the wave number k_c of the medium should be used in the Green's function instead of the free-space wave number k. Also,

$$k_c = \omega[\mu(\varepsilon + i\sigma/\omega)]^{\frac{1}{2}}, \tag{2.6}$$

$$Z_c = [\mu/(\varepsilon + i\sigma/\omega)]^{\frac{1}{2}}, \tag{2.7}$$

Z_c being the wave impedance of the medium, $Z_o \approx 377\Omega$, and c is the vacuum speed of light.

We will now investigate some of the properties of the integral operators $\underline{\underline{L}}$ and $\underline{\underline{M}}$ defined by equations (2.4) and (2.5), respectively. In reference [2] it has been shown that the kernel of the integral expression (2.4) behaves like $|\underline{r} - \underline{r}'|^{-1}$ as $\underline{r}' \to \underline{r}$. From this fact it follows that the operator $\underline{\underline{L}}^2$, defined by $\underline{\underline{L}}^2\cdot\underline{g} = \underline{\underline{L}}\cdot(\underline{\underline{L}}\cdot\underline{g})$, is of Hilbert-Schmidt type[2].

In order to investigate the properties of the operator $\underline{\underline{M}}$ we assume that κ_u and κ_v exist everywhere on S. Let P be a point on S with coordinates

$\underline{r} = (x,y,z)$. (See figure 2). We choose the coordinate system in such a way that, in the vicinity of P, the surface S can be described in the following way:

$$S = \Big\{ \underline{r}' = (x',y',z') : x' - x = \rho \cos \psi,$$

$$y' - y = \rho \sin \psi,\ z' - z = \rho^2 f(\psi) + O(\rho^3) \Big\}. \tag{2.8}$$

Here $0 \leq \psi \leq 2\pi$ and $0 < \rho \leq \varepsilon$, where $\varepsilon\kappa_u \ll 1$ and $\varepsilon\kappa_v \ll 1$. The representation (2.8) is valid everywhere on S since S is a smooth surface. Assuming that the function $\underline{g}(\underline{r})$ is differentiable on S we then have

$$\underline{g}(\underline{r}') - \underline{g}(\underline{r}) = \rho \underline{g}^{(1)}(\underline{r})\cos \psi + \rho \underline{g}^{(2)}(\underline{r})\sin \psi + O(\rho^2) \tag{2.9}$$

where

$$\underline{g}^{(1)}(\underline{r}) = (\partial/\partial x)\underline{g}(\underline{r}),$$

$$\underline{g}^{(2)}(\underline{r}) = (\partial/\partial y)\underline{g}(\underline{r}).$$

Moreover, for an arbitrary vector $\underline{a}$ we have

$$(\underline{a}\cdot\nabla)\nabla G = \underline{a} f(R) + \underline{R}(\underline{a}\cdot\underline{R})[R^{-1}\ f'(R)] \tag{2.10}$$

where

$$f(R) = (4\pi R)^{-3}(ikR - 1)\exp(ikR), \qquad \underline{R} = \underline{r} - \underline{r}'$$

and the prime denotes differentiation with respect to R. After some algebraic manipulations we get

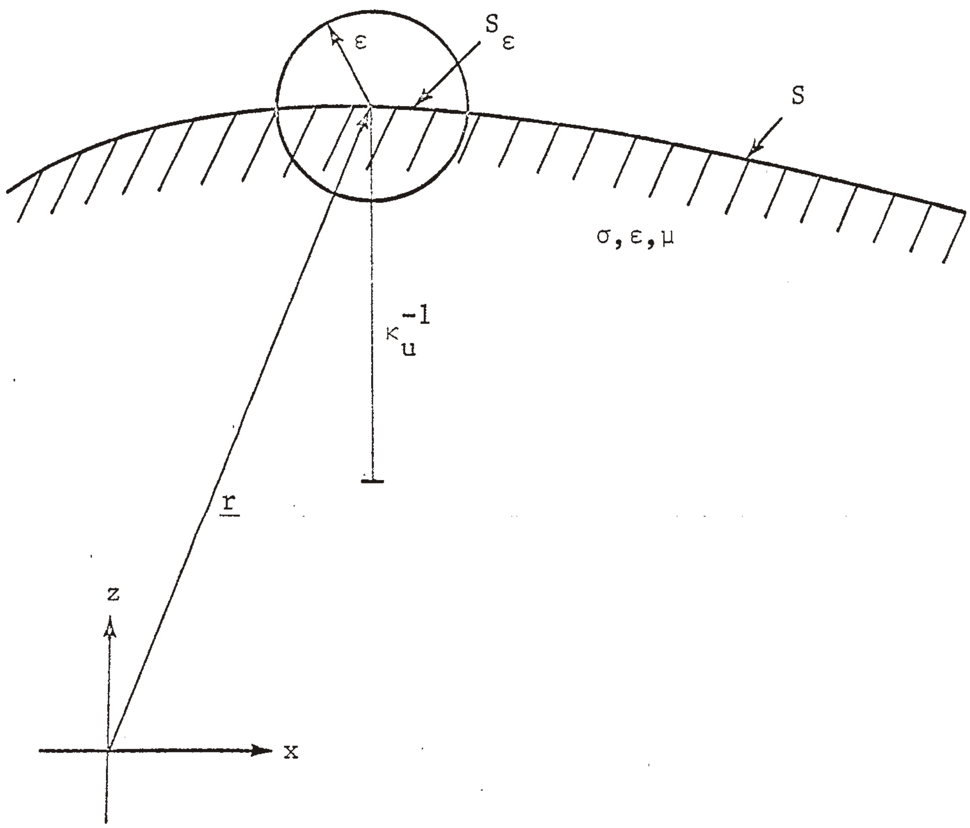

Figure 2. The local geometry of the surface of the scattering body.

$$[\underline{g}(\underline{r}') - \underline{g}(\underline{r})]\cdot\nabla_s'\nabla G(\underline{r},\underline{r}';ik) = (4\pi\rho^2)^{-1}\Big\{\underline{g}^{(1)}(\underline{r})\cos\psi + \underline{g}^{(2)}(\underline{r})\sin\psi$$

$$- 3\Big[g_x^{(1)}(\underline{r})\cos^2\psi + g_y^{(2)}(\underline{r})\sin^2\psi$$

$$+ \cos\psi\sin\psi[g_y^{(1)}(\underline{r}) + g_x^{(2)}(\underline{r})]\Big]$$

$$[\hat{x}\cos\psi + \hat{y}\sin\psi]\Big\} + O(\rho^{-1}). \qquad (2.11)$$

In order to evaluate the integral (2.5) we proceed as follows. Let S_ε be the part of S inside a sphere with radius ε and center at $\underline{r}$. Moreover, let us denote the integrand of the integral expression (2.5) by $\underline{H}(\underline{r},\underline{r}')$ so that

$$(\underline{\underline{M}}\cdot\underline{g})(\underline{r}) = \int_S \underline{H}(\underline{r},\underline{r}')dS'. \qquad (2.12)$$

We then have

$$(\underline{\underline{M}}\cdot\underline{g})(\underline{r}) = \int_{S_\varepsilon} \underline{H}(\underline{r},\underline{r}')dS' + \int_{S-S_\varepsilon} \underline{H}(\underline{r},\underline{r}')dS'. \qquad (2.13)$$

Since ε is finite it is easy to see from equation (2.5) that $\underline{H}(\underline{r},\underline{r}')$ is finite when $\underline{r}'$ belongs to $S-S_\varepsilon$. Thus, the last integral of equation (2.13) is a well-defined function of $\underline{r}$. From equation (2.11) it follows that

$$\int_{S_\varepsilon} \underline{H}(\underline{r},\underline{r}')dS' = \lim_{\delta\to 0}\int_{S_\varepsilon - S_\delta} (4\pi)^{-1}[\underline{\underline{N}}(\psi)\cdot\underline{g}^{(1)}(\underline{r}) + \underline{\underline{Q}}(\psi)\cdot\underline{g}^{(2)}(\underline{r})]\rho^{-2}\, dS'$$

$$+ \lim_{\delta\to 0}\int_{S_\varepsilon - S_\delta} \underline{H}_1(\underline{r},\underline{r}')dS'. \qquad (2.14)$$

Here $\underline{H}_1(\underline{r},\underline{r}') \sim |\underline{r} - \underline{r}'|^{-1}$ as $\underline{r}' \to \underline{r}$. Therefore, the principal value integral,

$$\lim_{\delta\to 0}\int_{S_\varepsilon - S_\delta} \underline{H}_1(\underline{r},\underline{r}')dS',$$

can be replaced by an ordinary integral over S_ε. Furthermore, $\underline{\underline{N}}(\psi)$ and $\underline{\underline{Q}}(\psi)$ are two matrices with elements

$$n_{xx}(\psi) = \cos\psi - 3\cos^3\psi$$

$$n_{xy}(\psi) = -3\cos^2\psi\sin\psi$$

$$n_{yx}(\psi) = n_{xy}(\psi) \tag{2.15}$$

$$n_{yy}(\psi) = \cos\psi - 3\cos\psi\sin^2\psi$$

$$q_{xx}(\psi) = \sin\psi - 3\cos^2\psi\sin\psi$$

$$q_{xy}(\psi) = -3\cos\psi\sin^2\psi$$

$$q_{yx}(\psi) = q_{xy}(\psi) \tag{2.16}$$

$$q_{yy}(\psi) = \sin\psi - 3\sin^3\psi.$$

From the expressions (2.15) and (2.16) it follows that

$$\int_0^{2\pi} \underline{\underline{N}}(\psi)d\psi = \int_0^{2\pi} \underline{\underline{Q}}(\psi)d\psi = 0. \tag{2.17}$$

Thus, we have

$$\lim_{\delta\to 0} \int_{S_\varepsilon - S_\delta} (4\pi)^{-1}[\underline{\underline{N}}(\psi)\cdot\underline{g}^{(1)}(\underline{r}) + \underline{\underline{Q}}(\psi)\cdot\underline{g}^{(2)}(\underline{r})]\rho^{-2}\, dS' = 0. \tag{2.18}$$

Equation (2.18) implies that the operator $\underline{\underline{M}}$ is a bounded operator when operating on differentiable functions and the surface of integration is a smooth surface.

From equations (2.14) through (2.16) it follows that

$$|\underline{\underline{H}}(\underline{r},\underline{r}')| \sim |\underline{r} - \underline{r}'|^{-2} \qquad \text{as} \quad \underline{r}' \to \underline{r} \tag{2.19}$$

and that the integral

$$\int_{S_\varepsilon} |\underline{\underline{H}}(\underline{r},\underline{r}')|dS' \tag{2.20}$$

does not exist. From equation (2.19) it follows that the kernel of any iterated operator $\underline{\underline{M}}^n$, defined by

$$\underline{\underline{M}}^n \cdot \underline{g} = \underline{\underline{M}} \cdot (\underline{\underline{M}}^{n-1} \cdot \underline{g}), \qquad n > 1$$

also behaves like $|\underline{r} - \underline{r}'|^{-2}$ as $\underline{r}' \to \underline{r}$. (c.f. reference [19]). Thus, there exists no n such that the operator $\underline{\underline{M}}^n$ is of Hilbert-Schmidt type. Therefore, the Fredholm determinant theory cannot be used when solving the set of integral equations (2.3).

Next, we will study the analytical behavior, in the complex frequency plane, of the operators $\underline{\underline{L}}$, $\underline{\underline{L}}_c$, $\underline{\underline{M}}$, $\underline{\underline{M}}_c$. By making the substitution $\gamma = -ik$, $k = i\gamma$ where γ is any complex number we can extend the validity of the set of integral equations (2.3) to all complex frequencies $s = c\gamma$. From equations (2.4) and (2.5) it easily follows that $\underline{\underline{L}}$ and $\underline{\underline{M}}$ are analytic operator-valued functions of γ in the entire complex frequency plane. It also follows from equations (2.3) through (2.5) that the integral operator defined by the left hand side of the set of integral equations (2.3) is an analytic operator-valued function of γ provided that γ_c and Z_c are analytic functions of γ. Here

$$\gamma_c = c\{\mu\varepsilon\gamma[\gamma + \sigma/(c\varepsilon)]\}^{\frac{1}{2}} \tag{2.21}$$

$$Z_c = \{\mu\varepsilon^{-1}\ \gamma/[\gamma + \sigma/(c\varepsilon)]\}^{\frac{1}{2}}. \tag{2.22}$$

Unfortunately, the fact that σ, ε and μ are all analytic functions does not necessarily ensure that γ_c and Z_c are analytic functions of γ. For example, when σ, ε and μ are constants we notice from equations (2.21) and (2.22) that γ_c and Z_c have branch points at $\gamma_1 = 0$ and $\gamma_2 = -\sigma/(c\varepsilon)$. Thus, the operators $\underline{\underline{L}}_c$ and $\underline{\underline{M}}_c$ are analytic operators in the entire γ-plane except along the branch line C, defined by

$$C = \{\gamma : -\sigma/(c\varepsilon) \le \mathrm{Re}\{\gamma\} \le 0,\ \mathrm{Im}\{\gamma\} = 0\}.$$

In sections III and IV we will study electromagnetic scattering from an inhomogeneous, imperfectly conducting body. When σ, ε, μ and the incident field are analytic functions of γ we will show that the only singularities of the induced volume current density are poles. This fact implies that the solution of the set of integral equations (2.3) is an analytic function of γ except for poles when σ, ε, μ and the incident field are analytic functions of γ. Unfortunately we can not draw this conclusion from the set of integral equations (2.3) for two reasons. The first reason stems from the fact that Z_c and γ_c are not necessarily analytic functions of γ even if σ, ε and μ are analytic functions of γ. The second reason is due to the fact that the Fredholm determinant theory can not be used when solving the set of integral equations (2.3). However, in view of the results in section IV the set of integral equations (2.3) can be used in finding the natural frequencies and modes of a homogeneous, imperfectly conducting, finite body. It is a well-known fact that a surface integral equation is better suited for numerical treatment than a volume integral equation.

In the next section we will derive a volume integral equation for calculating the scattering from an inhomogeneous imperfectly conducting body of finite size.

III. Volume Integral Equation Describing Scattering From an Inhomogeneous, Imperfectly Conducting Body

In this section we will derive a volume integral equation that describes electromagnetic scattering from a body whose conductivity (σ), permittivity (ε), and permeability (μ) are functions of space and frequency. Throughout this section we will suppress a harmonic time dependence factor $\exp(-i\omega t)$.

From Maxwell's equations one can derive the following set of partial differential equations for the electromagnetic potentials ($\Phi,\underline{A}$) in an arbitrary medium

$$\begin{aligned} \nabla^2\Phi + k^2\Phi &= - \varepsilon_o^{-1}(\rho - \nabla\cdot\underline{P}) \\ \nabla^2\underline{A} + k^2\underline{A} &= - \mu_o(\underline{i} + \nabla\times\underline{M} - i\omega\underline{P}). \end{aligned} \tag{3.1}$$

Here $k = \omega/c$, $\underline{P}$ is the polarization of the medium, $\underline{M}$ is the magnetization of the medium, and ρ and $\underline{i}$ represent the charge and current density, respectively, of exterior sources. The electromagnetic fields $\underline{E}$, $\underline{B}$ are obtained from

$$\begin{aligned} \underline{E} &= i\omega\underline{A} - \nabla\Phi \\ \underline{B} &= \nabla\times\underline{A}. \end{aligned} \tag{3.2}$$

We also have the relationships

$$\begin{aligned} \underline{E} &= \varepsilon_o^{-1}(\underline{D} - \underline{P}) \\ \underline{B} &= \mu_o(\underline{H} + \underline{M}) \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} \nabla\cdot\underline{D} &= \rho \\ \nabla\times\underline{H} &= \underline{i} - i\omega\underline{D}. \end{aligned} \tag{3.4}$$

From equations (3.1) and (3.4) it follows that

$$\nabla \cdot \underline{A} - ikc^{-1}\Phi = 0. \tag{3.5}$$

Next, we consider a region V of finite extent with some polarization $\underline{P}$ and magnetization $\underline{M}$. Futhermore, we assume that there are no exterior sources of the electromagnetic field in V implying that $\rho = 0$ and $\underline{i} = 0$ in V. Let S be the boundary surface of V and let $\underline{n}$ be the outward unit normal of S. The complement of V with respect to the three dimensional Euclidean space is denoted by V_1. We also assume that the region V_1 is vacuum as far as the electromagnetic properties are concerned. Assuming that all the exterior sources of the electromagnetic field are at infinity we have the following differential equations for the electromagnetic potentials

$$\begin{aligned} &\nabla^2\Phi + k^2\Phi = \varepsilon_o^{-1}\nabla\cdot\underline{P} \\ &\nabla^2\underline{A} + k^2\underline{A} = \mu_o(i\omega\underline{P} - \nabla\times\underline{M}) \qquad \text{in} \quad V \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} &\nabla^2\Phi + k^2\Phi = 0 \\ &\nabla^2\underline{A} + k^2\underline{A} = 0 \qquad \text{in} \quad V_1. \end{aligned} \tag{3.7}$$

Let G be the free-space Green's function, and let $\underline{r}$ be a point inside V. We then have

$$\begin{aligned} \int_{V-V_\delta} (G\nabla^2\Phi - \Phi\nabla'^2G)dV' &= \int_{V-V_\delta} \nabla'\cdot(G\nabla\Phi - \Phi\nabla'G)dV' \\ &= \int_S \underline{n}'\cdot(G\nabla\Phi - \Phi\nabla'G)dS' - \int_{S_\delta} \hat{R}\cdot(G\nabla\Phi - \Phi\nabla'G)dS' \end{aligned} \tag{3.8}$$

where ∇' operates on the second argument of $G(\underline{r},\underline{r}';ik)$, $\underline{n}' = \underline{n}(\underline{r}')$, and V_δ is a sphere with surface S_δ, radius δ, and center at $\underline{r}$ (see figure 3). It is easy to show that

$$\lim_{\delta\to 0} \int_{S_\delta} \hat{R}\cdot(G\nabla\Phi - \Phi\nabla'G)dS' = \Phi. \tag{3.9}$$

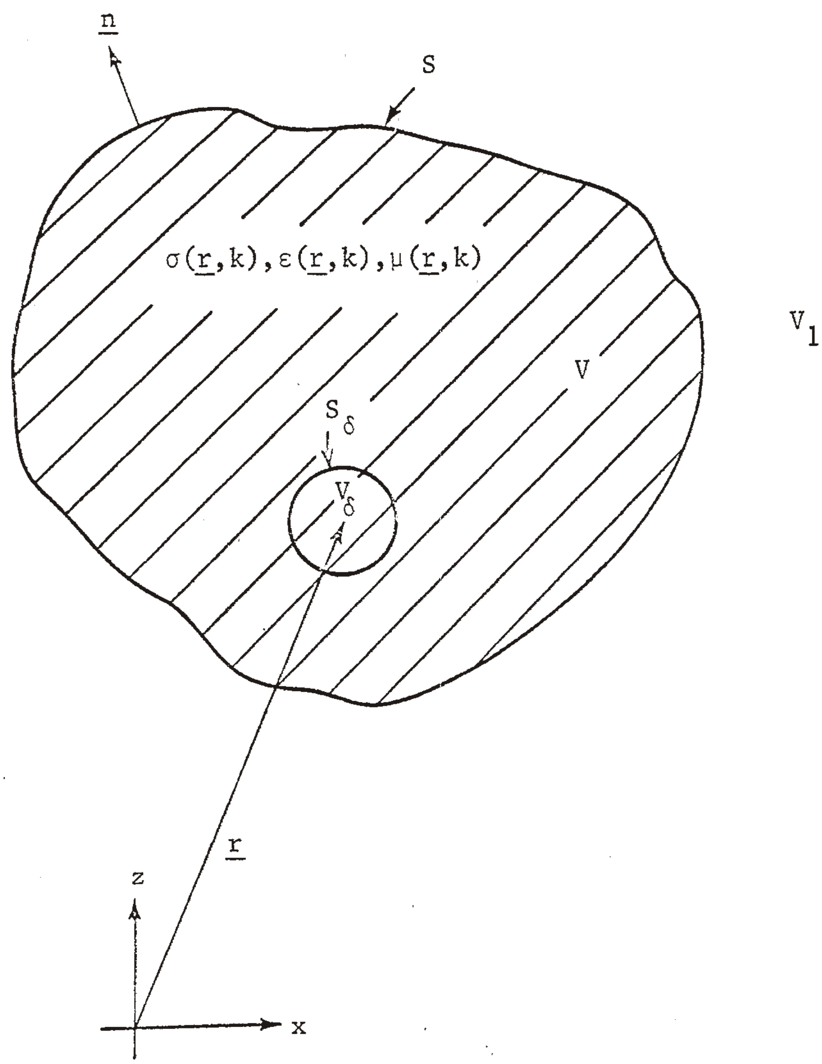

Figure 3. An inhomogeneous, imperfectly conducting, finite body.

Equations (3.6), (3.8) and (3.9) then give

$$\Phi + \varepsilon_o^{-1} \int_V G\nabla\cdot\underline{P}dV' + \int_S (\Phi\partial G/\partial n' - G\partial\Phi/\partial n')dS' = 0. \tag{3.10}$$

Similarly, it can be shown that

$$\underline{A} + \mu_o \int_V G(i\omega\underline{P} - \nabla\times\underline{M})dV' + \int_S [\underline{A}\partial G/\partial n' - G(\underline{n}'\cdot\nabla')\underline{A}]dV' = 0. \tag{3.11}$$

Next, we consider an actual scattering situation. We split the electromagnetic field into two parts:

$$\begin{aligned} \Phi &= \Phi^{inc} + \Phi^{sc} \\ \underline{A} &= \underline{A}^{inc} + \underline{A}^{sc} \end{aligned} \tag{3.12}$$

where Φ^{sc} and $\underline{A}^{sc}$ satisfy the radiation condition at infinity and the differential equations

$$\begin{aligned} &\nabla^2\Phi^{sc} + k^2\Phi^{sc} = 0 \\ &\nabla^2\underline{A}^{sc} + k^2\underline{A}^{sc} = 0 \qquad \text{in} \quad V_1, \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} &\nabla^2\Phi^{sc} + k^2\Phi^{sc} = \varepsilon_o^{-1}\nabla\cdot\underline{P} \\ &\nabla^2\underline{A}^{sc} + k^2\underline{A}^{sc} = \mu_o(i\omega\underline{P} - \nabla\times\underline{M}) \qquad \text{in} \quad V. \end{aligned} \tag{3.14}$$

The incident field satisfies the differential equations

$$\begin{aligned} &\nabla^2\Phi^{inc} + k^2\Phi^{inc} = 0 \\ &\nabla^2\underline{A}^{inc} + k^2\underline{A}^{inc} = 0 \qquad \text{in} \quad V \cup V_1. \end{aligned} \tag{3.15}$$

We now go on to apply the integral formulas (3.10) and (3.11) to Φ^{sc}

and $\underline{A}^{sc}$. Equation (3.10) gives

$$\Phi^{sc} + \varepsilon_o^{-1} \int_V G\nabla\cdot\underline{P}dV' + \int_S [(\Phi_+^{sc} - \Phi_-^{sc})\partial G/\partial n'$$

$$- G(\partial\Phi_+^{sc}/\partial n' - \partial\Phi_-^{sc}/\partial n')]dS' = 0, \qquad \underline{r} \in V \qquad (3.16)$$

where

$$\Phi_+(\underline{r}) = \lim_{\substack{\underline{r}\to S \\ \underline{r}\in V}} \Phi(\underline{r})$$

and

$$\Phi_-(\underline{r}) = \lim_{\substack{\underline{r}\to S \\ \underline{r}\in V_1}} \Phi(\underline{r}).$$

On the surface S we impose the boundary conditions that Φ, $\underline{A}$ and $\underline{D}\cdot\underline{n}$ are continuous. Since Φ^{inc} is continuous everywhere these boundary conditions imply that

$$\Phi_+^{sc} = \Phi_-^{sc}$$

and

$$\partial\Phi_+^{sc}/\partial n - \partial\Phi_-^{sc}/\partial n = \varepsilon_o^{-1}\underline{n}\cdot(\underline{P}_+ - \underline{P}_-) = \varepsilon_o^{-1}\underline{n}\cdot\underline{P}_+$$

where we have made use of the fact that $\underline{P} \equiv 0$ in V_1. Suppressing the index + we arrive at the following expression

$$\Phi = \Phi^{inc} - \varepsilon_o^{-1} \int_V G\nabla\cdot\underline{P}dV' + \varepsilon_o^{-1} \int_S G\underline{n}'\cdot\underline{P}dS' \qquad (3.17)$$

and equation (3.17) is valid at points inside V.

In the same way, equation (3.11) gives

$$\underline{A}^{sc} + \mu_o^{-1} \int_V (i\omega\underline{P} - \nabla\times\underline{M})dV' + \int_S [(\underline{A}_+^{sc} - \underline{A}_-^{sc})\partial G/\partial n'$$

$$- G(\underline{n}'\cdot\nabla)(\underline{A}_+^{sc} - \underline{A}_-^{sc})]dS' = 0, \qquad \underline{r} \in V. \qquad (3.18)$$

We also impose the boundary conditions that $\underline{A}$, $\underline{n}\cdot\underline{B}$ and $\underline{n}\times\underline{H}$ are continuous on S. Making use of these boundary conditons, equation (3.5), and the vector formula

$$\underline{n}\times(\nabla\times\underline{A}) = \nabla A_n - (\underline{n}\cdot\nabla)\underline{A}$$

we get

$$\underline{A} = \underline{A}^{inc} + \mu_o \int_V G(\nabla\times\underline{M} - i\omega\underline{P})dV' - \mu_o \int_S G(\underline{n}'\times\underline{M})dS'. \tag{3.19}$$

Equation (3.19) is valid at points inside V. In the surface integral of equation (3.19) we have used the definition

$$\underline{M}(\underline{r}) = \lim_{\substack{\underline{r}\to S\\ \underline{r}\in V}} \underline{M}(\underline{r}).$$

To sum up, we have the following set of equations for the electromagnetic potentials, valid in the region V,

$$\begin{aligned} \underline{A} &= \underline{A}^{inc} + \mu_o \int_V G(\nabla\times\underline{M} - i\omega\underline{P})dV' - \mu_o \int_S G(\underline{n}'\times\underline{M})dS' \\ \Phi &= \Phi^{inc} - \varepsilon_o^{-1} \int_V G\nabla\cdot\underline{P}dV' + \varepsilon_o^{-1} \int_S G\underline{n}'\cdot\underline{P}dS'. \end{aligned} \tag{3.20}$$

Combining equations (3.2) and (3.20) we get the following set of equations for the electromagnetic field quantities $\underline{E}$, $\underline{B}$, also valid in V,

$$\begin{aligned} \underline{E} &= \underline{E}^{inc} + \varepsilon_o^{-1}\nabla \int_V G\nabla\cdot\underline{P}dV' - \varepsilon_o^{-1}\nabla \int_S G\underline{n}'\cdot\underline{P}dS' \\ &\quad + i\omega\mu_o \int_V G(\nabla\times\underline{M} - i\omega\underline{P})dV' - i\omega\mu_o \int_S G(\underline{n}'\times\underline{M})dS' \\ \underline{B} &= \underline{B}^{inc} + \mu_o\nabla\times\int_V G(\nabla\times\underline{M} - i\omega\underline{P})dV' - \mu_o\nabla\times\int_S G(\underline{n}'\times\underline{M})dS' \end{aligned} \tag{3.21}$$

where

$$\underline{E}^{inc} = -\nabla\Phi^{inc} + i\omega\underline{A}^{inc}$$

and

$$\underline{B}^{inc} = \nabla\times\underline{A}^{inc}.$$

Employing the integral formulas

$$\int_V G\nabla\cdot\underline{P}dV' = \int_S G\underline{n}'\cdot\underline{P}dS' + \nabla\cdot\int_V G\underline{P}dV' \tag{3.22}$$

and

$$\int_V G\nabla\times\underline{M}dV' = \int_S G\underline{n}'\times\underline{M}dS' + \nabla\times\int_V G\underline{M}dV' \tag{3.23}$$

to equation (3.21) we get

$$\begin{aligned}\underline{E} &= \underline{E}^{inc} + \varepsilon_o^{-1}\nabla\nabla\cdot\int_V G\underline{P}dV' + \omega^2\mu_o\int_V G\underline{P}dV' + i\omega\mu_o\nabla\times\int_V G\underline{M}dV' \\ \underline{B} &= \underline{B}^{inc} + \mu_o\nabla\times\nabla\times\int_V G\underline{M}dV' - i\omega\mu_o\nabla\times\int_V G\underline{P}dV'.\end{aligned} \tag{3.24}$$

Making use of equation (3.3) and the facts that

$$\underline{D}^{inc} = \varepsilon_o\underline{E}^{inc} \tag{3.25}$$

and

$$(\nabla^2 + k^2)\int_V G\underline{P}dV' = -\underline{P} \tag{3.26}$$

we can replace the set of equations (3.24) by the following set of equations

$$\begin{aligned}\underline{D} &= \underline{D}^{inc} + \nabla\times\nabla\times\int_V G\underline{P}dV' + i\omega\mu_o\varepsilon_o\nabla\times\int_V G\underline{M}dV' \\ \underline{B} &= \underline{B}^{inc} + \mu_o\nabla\times\nabla\times\int_V G\underline{M}dV' - i\omega\mu_o\nabla\times\int_V G\underline{P}dV'.\end{aligned} \tag{3.27}$$

From now on we restrict our calculations to linear media, i.e., there exist $\tilde{\varepsilon}(\underline{r},k)$ and $\tilde{\mu}(\underline{r},k)$ such that

$$\underline{D}(\underline{r},k) = \varepsilon_o \tilde{\varepsilon}(\underline{r},k)\underline{E}(\underline{r},k)$$

and (3.28)

$$\underline{B}(\underline{r},k) = \mu_o \tilde{\mu}(\underline{r},k)\underline{H}(\underline{r},k).$$

Here $\tilde{\varepsilon}(\underline{r},k)$ accounts for both the permittivity, $\varepsilon(\underline{r},k)$, and the conductivity, $\sigma(\underline{r},k)$, of the medium,

$$\tilde{\varepsilon}(\underline{r},k) = [\varepsilon(\underline{r},k) + i\sigma(\underline{r},k)/(ck)]/\varepsilon_o$$

With

$$\chi(\underline{r},k) = \tilde{\varepsilon}(\underline{r},k)/[\tilde{\varepsilon}(\underline{r},k) - 1]$$

and (3.29)

$$\kappa(\underline{r},k) = \tilde{\mu}(\underline{r},k)/[\tilde{\mu}(\underline{r},k) - 1]$$

we get the following differential volume integral equation for $\underline{P}$ and $\underline{M}$

$$\chi\underline{P} = \underline{D}^{inc} + \nabla\times\nabla\times\int_V G\underline{P}dV' + i\omega c^{-2}\nabla\times\int_V G\underline{M}dV'$$

(3.30)

$$\kappa\underline{M} = \underline{H}^{inc} + \nabla\times\nabla\times\int_V G\underline{M}dV' - i\omega\nabla\times\int_V G\underline{P}dV'.$$

To derive a set of integral equations from the set of equations (3.29) we notice that[9]

$$\nabla\times\nabla\times\int_V G\underline{Q}dV' = 2\underline{Q}/3 + ⨍_V \nabla\times\nabla\times(G\underline{Q})dV' \qquad (3.31)$$

and

$$\nabla\times\int_V G\underline{Q}dV' = \int_V \nabla\times(G\underline{Q})dV'. \qquad (3.32)$$

Here $⨍$ denotes the principal value integral, i.e.,

$$⨍_V g dV' = \lim_{\delta\to 0} \int_{V-V_\delta} g dV'$$

where V_δ is a sphere with radius δ and center at $\underline{r}$. Applying equations (3.31) and (3.32) to the set of equations (3.30) we arrive at the following set of coupled integral equations for $\underline{P}$ and $\underline{M}$

$$(\tilde{\varepsilon} + 2)/[3(\tilde{\varepsilon} - 1)]\underline{P} = \underline{D}^{inc} + \int_V \nabla\times\nabla\times(G\underline{P})dV' + ikc^{-1} \int_V \nabla\times(G\underline{M})dV'$$

$$(\tilde{\mu} + 2)/[3(\tilde{\mu} - 1)]\underline{M} = \underline{H}^{inc} - ikc^{-1} \int_V \nabla\times(G\underline{P})dV' + \int_V \nabla\times\nabla\times(G\underline{M})dV' \qquad (3.33)$$

The integral

$$\int_V \nabla\times\nabla\times(G\underline{Q})dV' \qquad (3.34)$$

is a well-defined quantity for any point inside V provided that $\underline{Q}$ is a Hölder continuous function in V. This can be seen from the fact that

$$\nabla\times\nabla\times(G\underline{Q}) = [R^{-3}\underline{\underline{F}}(\theta',\phi') + \underline{\underline{S}}(R,\theta,\phi')]\cdot\underline{Q}(\underline{r}') \qquad (3.35)$$

as $R = |\underline{r} - \underline{r}'| \to 0$. Here θ' and ϕ' are the polar angles of the vector $\underline{R} = \underline{r} - \underline{r}'$. Moreover, $\underline{\underline{F}}$ and $\underline{\underline{S}}$ are 3×3 matrices such that

$$|\underline{\underline{S}}(R,\theta',\phi')| = O(R^{-2}) \qquad \text{as} \quad R \to 0 \qquad (3.36)$$

and

$$\int_S \underline{\underline{F}}(\theta',\phi')\sin\theta' d\theta' d\phi' = 0 \qquad (3.37)$$

where S is the surface of the unit sphere. The principal value integral (3.34) has been discussed previously in the literature and we refer the interested reader to references [5] through [8].

The singularity inherent in the set of integral equations (3.33) makes it difficult to determine the analytical properties in the complex frequency plane of $\underline{P}$ and $\underline{M}$ from the set of integral equations (3.33). In the next section we will start with the set of differential integral equations (3.27) and derive a set of volume-surface integral equations for $\underline{E}$ and $\underline{H}$. This set of integral equations can be treated by means of the Fredholm determinant theory.

IV. Analytical Properties of the Field Scattered From an Imperfectly Conducting, Finite Body

Based on the differential integral equation (3.27) we will in this section first derive a volume-surface integral equation describing scattering from an inhomogeneous, imperfectly conducting, finite body. We will then discuss the analytical properties in the complex frequency plane of the solution of this integral equation.

Employing the integral formulas (3.22) and (3.26) to the set of equations (3.24) we arrive at the following set of volume-surface integral equations

$$\underline{E} = \underline{E}^{inc} + \int_V [k^2(\tilde{\varepsilon} - 1)G\underline{E} + \nabla G\tilde{\varepsilon}^{-1}\nabla\tilde{\varepsilon}\cdot\underline{E}]dV' - \int_S (\tilde{\varepsilon} - 1)\nabla G(\underline{n}'\cdot\underline{E})dS' + ikZ_o \int_V (\tilde{\mu} - 1)\nabla G\times\underline{H}dV'$$

$$\underline{H} = \underline{H}^{inc} + \int_V [k^2(\tilde{\mu} - 1)G\underline{H} + \nabla G\tilde{\mu}^{-1}\nabla\tilde{\mu}\cdot\underline{H}]dV' - \int_S (\tilde{\mu} - 1)\nabla G(\underline{n}'\cdot\underline{H})dS' - ikZ_o^{-1} \int_V (\tilde{\varepsilon} - 1)\nabla G\times\underline{E}dV'. \qquad (4.1)$$

In deriving the set of equations (4.1) we have made use of the fact that

$$\underline{P} = \varepsilon_o(\tilde{\varepsilon} - 1)\underline{E} = \tilde{\varepsilon}^{-1}(\tilde{\varepsilon} - 1)\underline{D},$$

$$\underline{M} = (\tilde{\mu} - 1)\underline{H} = \mu_o^{-1}\tilde{\mu}^{-1}(\tilde{\mu} - 1)\underline{B}.$$

For the special case where $\tilde{\mu} = 1$ and $\sigma = 0$ the set of equations (4.1) reduces to the integral equations

$$\underline{E} = \underline{E}^{inc} + \int_V [k^2(\tilde{\varepsilon} - 1)G\underline{E} + \nabla G\tilde{\varepsilon}^{-1}\nabla\tilde{\varepsilon}\cdot\underline{E}]dV' - \int_S (\tilde{\varepsilon} - 1)\nabla G(\underline{n}'\cdot\underline{E})dS'$$

$$\underline{H} = \underline{H}^{inc} - ikZ_o^{-1} \int_V (\tilde{\varepsilon} - 1)\nabla G\times\underline{E}dV' \qquad (4.2)$$

which have been discussed previously in the literature[13].

Instead of considering the set of integral equations (4.1) we will consider the following more general set of integral equations

$$\begin{aligned}
&\underline{E} - \int_V \underline{\underline{L}}_1 \cdot \underline{E} dV' - \int_S \underline{L}_1 F dS' - \int_V \underline{\underline{M}}_1 \cdot \underline{H} dV' = \underline{E}^{inc}, && \underline{r} \in V \\
&F - \int_S L_1 F dS' - \int_V \underline{L}_2 \cdot \underline{E} dV' - \int_V \underline{M}_1 \cdot \underline{H} dV' = F^{inc}, && \underline{r} \in S \\
&\underline{H} - \int_V \underline{\underline{L}}_2 \cdot \underline{H} dV' - \int_S \underline{L}_3 J dS' - \int_V \underline{\underline{M}}_2 \cdot \underline{E} dV' = \underline{H}^{inc}, && \underline{r} \in V \\
&J - \int_S L_2 J dS' - \int_V \underline{L}_4 \cdot \underline{H} dV' - \int_V \underline{M}_2 \cdot \underline{E} dV' = J^{inc}, && \underline{r} \in S
\end{aligned} \tag{4.3}$$

where

$$\underline{\underline{L}}_1 = k^2(\tilde{\varepsilon} - 1)G\underline{\underline{I}} + \tilde{\varepsilon}^{-1}\nabla G \nabla \tilde{\varepsilon}$$

$$\underline{\underline{L}}_2 = k^2(\tilde{\mu} - 1)G\underline{\underline{I}} + \tilde{\mu}^{-1}\nabla G \nabla \tilde{\mu}$$

$$\underline{L}_1 = (1 - \tilde{\varepsilon})\nabla G$$

$$\underline{L}_2 = k^2(\tilde{\varepsilon} - 1)G\underline{n} + \tilde{\varepsilon}^{-1}(\underline{n}\cdot\nabla G)\nabla\tilde{\varepsilon}$$

$$\underline{L}_3 = (1 - \tilde{\mu})\nabla G$$

$$\underline{L}_4 = k^2(\tilde{\mu} - 1)G\underline{n} + \tilde{\mu}^{-1}(\underline{n}\cdot\nabla G)\nabla\tilde{\mu}$$

$$L_1 = (1 - \tilde{\varepsilon})(\underline{n}\cdot\nabla G)$$

$$L_2 = (1 - \tilde{\mu})(\underline{n}\cdot\nabla G)$$

$$\underline{\underline{M}}_1 \cdot = ikZ_o(\tilde{\mu} - 1)\nabla G \times$$

$$\underline{\underline{M}}_2 \cdot = - ikZ_o^{-1}(\tilde{\varepsilon} - 1)\nabla G \times$$

$$\underline{M}_1 = ikZ_o(\tilde{\mu} - 1)(\underline{n}\times\nabla G)$$

$$\underline{M}_2 = - ikZ_o^{-1}(\tilde{\varepsilon} - 1)(\underline{n}\times\nabla G).$$

Substituting

$$F^{inc} = \underline{n}\cdot\underline{E}^{inc},$$
$$J^{inc} = \underline{n}\cdot\underline{H}^{inc} \tag{4.4}$$

into the set of equations (4.3) one can verify that the solution of the set of integral equations (4.1) also satisfies (4.3). In fact, having the solution of (4.1) we only have to substitute

$$F = \underline{n}\cdot\underline{E},$$
$$J = \underline{n}\cdot\underline{H} \tag{4.5}$$

into (4.3) in order to show that the solution of (4.1) satisfies (4.3).

Next, we will show that the solution of the set of equations (4.3) satisfies the set of equations (4.1). The uniqueness of the solution of the integral equation (4.2) for real values of k has been shown in reference [13]. Without going into the details of the proof we claim here that the uniqueness of the solutions of the set of equations (4.1) and (4.3) can be shown by using the technique employed in reference [13]. Let $\underline{E}_1$, $\underline{H}_1$ be a solution of (4.1) with the right hand sides equal to $\underline{E}_o$ and $\underline{H}_o$. Moreover, let $\underline{E}_2$, F_2, $\underline{H}_2$, J_2 be a solution of (4.3) with the right hand sides equal to $\underline{E}_o$, $F_o = \underline{n}\cdot\underline{E}_o$, $\underline{H}_o$, $J_o = \underline{n}\cdot\underline{H}_o$. Introduce the quantities

$$\begin{aligned} \underline{E}' &= \underline{E}_1 - \underline{E}_2 \\ F' &= \underline{n}\cdot\underline{E}_1 - F_2 \\ \underline{H}' &= \underline{H}_1 - \underline{H}_2 \\ J' &= \underline{n}\cdot\underline{H}_1 - J_2 \end{aligned} \tag{4.6}$$

Then one can easily see that $\underline{E}'$, F', $\underline{H}'$, J' satisfy the set of homogeneous integral equations

$$\begin{aligned}
&\underline{E}' - \int_V \underline{\underline{L}}_1 \cdot \underline{E}' dV' - \int_S \underline{L}_1 F' dS' - \int_V \underline{\underline{M}}_1 \cdot \underline{H}' dV' = 0 \\
&F' - \int_S L_1 F' dS' - \int_V \underline{L}_2 \cdot \underline{E}' dV' - \int_V \underline{M}_1 \cdot \underline{H}' dV' = 0 \\
&\underline{H}' - \int_V \underline{\underline{L}}_2 \cdot \underline{H}' dV' - \int_S \underline{L}_3 J' dS' - \int_V \underline{\underline{M}}_2 \cdot \underline{E}' dV' = 0 \\
&J' - \int_S L_2 J' dS' - \int_V \underline{L}_4 \cdot \underline{H}' dV' - \int_V \underline{M}_2 \cdot \underline{E}' dV' = 0.
\end{aligned} \tag{4.7}$$

From the uniqueness theorem it follows, however, that the only solution of the set of integral equations (4.7) is the trivial solution $\underline{E}' = F' = \underline{H}' = J' = 0$. Thus, by substituting the set of equations (4.4) into the set of integral equations (4.3) it follows that the solution of (4.3) satisfies (4.1) for real values of k. From the principle of analytic continuation it then follows that the solution of equation (4.3) satisfies equation (4.1) for all complex values of k.

We will now continue with the solution of the set of integral equations (4.3). First we will transform the set of vector integral equations (4.2) into a set of two coupled scalar integral equations. For that reason we introduce a set of volumes, V_ℓ, and a surface, S_1 defined by

$$V_\ell = \{\underline{r} : \underline{r} = \ell \underline{r}_o + r', \; r' \in V\}$$

$$S_1 = \{\underline{r} : \underline{r} = \underline{r}_o + \underline{r}', \; \underline{r}' \in S\}.$$

Here $0 \le \ell \le 5$ and

$$\underline{r}_o = \hat{t} \max_{\underline{r}'' \in S} [\hat{t} \cdot \underline{r}'' + d]$$

where $\hat{t}$ is an arbitrary unit vector and $d > 0$. Notice that $V_o = V$ and that the regions V are nonintersecting regions and that $S_o = S$ and S_1 are two nonintersecting surfaces. Moreover, define

$$\eta(\underline{r}) = \begin{cases} E_x(\underline{r}), & \underline{r} \in V_o \\ E_y(\underline{r} - \underline{r}_o), & \underline{r} \in V_1 \\ E_z(\underline{r} - 2\underline{r}_o), & \underline{r} \in V_2 \\ H_x(\underline{r} - 3\underline{r}_o), & \underline{r} \in V_3 \\ H_y(\underline{r} - 4\underline{r}_o), & \underline{r} \in V_4 \\ H_z(\underline{r} - 5\underline{r}_o), & \underline{r} \in V_5 \end{cases}$$

and

$$\sigma(\underline{r}) = \begin{cases} F(\underline{r}), & \underline{r} \in S_o \\ J(\underline{r} - \underline{r}_o), & \underline{r} \in S_1. \end{cases}$$

The set of integral equations (4.3) can then be transformed into the following set of two coupled integral equations

$$\begin{aligned} \eta(\underline{r}) - \int_\Omega \Gamma_{11}(\underline{r},\underline{r}';\gamma)\eta(\underline{r}')dV' - \int_\Sigma \Gamma_{12}(\underline{r},\underline{r}';\gamma)\sigma(\underline{r}')dS' &= \eta^{inc}(\underline{r}); \quad \underline{r} \in \Omega \\ \sigma(\underline{r}) - \int_\Omega \Gamma_{21}(\underline{r},\underline{r}';\gamma)\eta(\underline{r}')dV' - \int_\Sigma \Gamma_{22}(\underline{r},\underline{r}';\gamma)\sigma(\underline{r}')dS' &= \sigma^{inc}(\underline{r}); \quad \underline{r} \in \Sigma \end{aligned} \tag{4.8}$$

Here

$$\Omega = \bigcup_{i=0}^{5} V_i,$$

$$\Sigma = S_o \cup S_1,$$

and Γ_{11}, Γ_{12}, Γ_{21}, Γ_{22} are constructed from the kernels of the set of integral equations (4.3) in the same way as the quantity Γ used in section IV of reference [2]. Since we are not interested in the actual analytical expressions of Γ_{11}, Γ_{12}, Γ_{21} and Γ_{22} we leave it to the interested reader to find these expressions. From the definitions of the kernels of the set of integral equations (4.3) we can derive the following asymptotic expressions

$$|\Gamma_{11}(\underline{r} - \underline{r}')| \sim |\underline{r} - \underline{r}' - (i - j)\underline{r}_o|^{-2}$$

when $\underline{r} \in V_i$ and $\underline{r}' \in V_j$ and $\underline{r}' \to \underline{r} - (i - j)\underline{r}_o$;

$$|\Gamma_{12}(\underline{r} - \underline{r}')| \sim |\underline{r} - \underline{r}' - (i - j)\underline{r}_o|^{-2}$$

when $\underline{r} \in V_i$ and $\underline{r}' \in S_j$ and $\underline{r}' \to \underline{r} - (i - j)\underline{r}_o$;

$$|\Gamma_{21}(\underline{r} - \underline{r}')| \sim |\underline{r} - \underline{r}' - (i - j)\underline{r}_o|^{-2}$$

when $\underline{r} \in S_i$ and $\underline{r}' \in V_j$ and $\underline{r}' \to \underline{r} - (i - j)\underline{r}_o$; and

$$|\Gamma_{22}(\underline{r} - \underline{r}')| \sim |\underline{r} - \underline{r}' - (i - j)\underline{r}_o|^{-1}$$

when $\underline{r} \in S_i$ and $\underline{r}' \in S_j$ and $\underline{r}' \to \underline{r} - (i - j)\underline{r}_o$.

Introducing $\underline{\phi} = (\eta,\sigma)$ the set of integral equations (4.8) can be cast into the following operator form

$$(\underline{\underline{I}} - \underline{\underline{\Gamma}})\cdot\underline{\phi} = \underline{\phi}^{inc}. \tag{4.9}$$

The operator $\underline{\underline{\Gamma}}$ is determined by the kernels Γ_{11}, Γ_{12}, Γ_{21}, Γ_{22} and the domains of integration Ω,Σ. It is easy to show that the kernels defining the iterated operator $\underline{\underline{\Gamma}}^3$ have logarithmic singularities at those values, of $(\underline{r},\underline{r}')$ where Γ_{11}, Γ_{12}, Γ_{21} and Γ_{22} are singular. Since any solution of the integral equation (4.9) also satisfies the integral equation

$$(\underline{\underline{I}} - \underline{\underline{\Gamma}}^3)\cdot\underline{\phi} = \underline{\phi}_1^{inc} \tag{4.10}$$

where

$$\underline{\phi}_1^{inc} = (\underline{\underline{I}} + \underline{\underline{\Gamma}} + \underline{\underline{\Gamma}}^2)\cdot\underline{\phi}^{inc}$$

we will consider the integral equation

$$(\underline{\underline{I}} - \underline{\underline{\Gamma}}^3)\cdot\underline{\phi} = \underline{\psi}. \tag{4.11}$$

A solution of the integral equation (4.11) is derived in Appendix C. Provided that $\tilde{\varepsilon}$, $\nabla\tilde{\varepsilon}/\tilde{\varepsilon}$, $\tilde{\mu}$ and $\nabla\tilde{\mu}/\tilde{\mu}$ are analytic functions in the entire complex frequency plane it follows from the analysis presented in Appendix C that poles are the only singularities of the operator inverse to the integral operator defining the left hand side of (4.11). The location of these poles are given by those values of γ, γ_n, where there exists a nontrivial solution of the homogeneous equation[2]

$$(\underline{\underline{I}} - \underline{\underline{\Gamma}}^3)\cdot\underline{\phi} = 0. \tag{4.12}$$

To conclude this section we have shown that there are two types of singularities in the complex frequency plane of the field scattered from an imperfectly conducting, inhomogeneous body. One type is due to the singularities of the incident field. The other type is due to the scattering body. Sufficient conditions for the latter type of singularities to consist of only poles are that the scattering body is of finite extent and that $\tilde{\varepsilon}$, $\nabla\tilde{\varepsilon}/\tilde{\varepsilon}$, $\tilde{\mu}$, and $\nabla\tilde{\mu}/\tilde{\mu}$ are analytic functions throughout the entire complex frequency plane.

The approach we have used is based on a formulation of the electromagnetic scattering problem in terms of a set of coupled Fredholm integral equations of the second kind. This approach resembles the approach used in deriving the so-called Oseen extinction theorem[9-12]. When solving for the natural frequencies and natural modes of a particular scattering body it might be advantageous from the numerical point of view to use other formulations. One method that can be used is an integral equation of the first kind for the induced polarization current and magnetization current used in reference [17]. In comparison to different formulations used when solving electromagnetic scattering problems involving perfectly conducting bodies, the formulation used in this note resembles the magnetic field formulation whereas the formulation used in reference [17] resembles the electric field formulation. Finially, as we have seen in section II, scattering from a homogeneous, imperfectly conducting body can be formulated in terms of a surface integral equation.

V. Scattering From a Perfectly Conducting, Finite Body Within a Parallel Plate Region

In this section we will consider electromagnetic scattering from a perfectly conducting, finite body located between two parallel plates. With the use of image theory we will first formulate an integral equation for the scattering problem valid for real frequencies. Next, we will find the analytic continuation of this integral equation to the entire complex frequency plane. Finally, we will discuss some of the analytic properties of the solution of the integral equation.

The geometry of the scattering problem considered is depicted in figure 4. From the theory of images it follows that the integral equation derived from the magnetic field formulation can be cast into the following form:

$$(\frac{1}{2}\underline{\underline{I}} - \underline{\underline{L}})\cdot\underline{j} = \underline{j}^{inc}. \qquad (5.1)$$

Here $\underline{j}$ is the induced surface current density on the scattering object, $\underline{\underline{I}}$ is the identity operator, $\underline{\underline{L}}$ is an integral operator defined by

$$\underline{\underline{L}}\cdot\underline{j} = \int_S \underline{n}\times[\nabla\times(\underline{\underline{G}}\cdot\underline{j})]dS', \qquad (5.2)$$

and $\underline{n}$ is the outward unit normal of the surface S of the scattering object. Moreover,

$$\underline{\underline{G}} = \underline{\underline{G}}^- + \underline{\underline{G}}^+, \qquad (5.3)$$

$$\underline{\underline{G}}^- = \underline{\underline{I}}G^-, \qquad (5.4)$$

$$G^-(\underline{r},\underline{r}';ik) = \sum_{-\infty}^{\infty} (4\pi R_n^-)^{-1} \exp(ikR_n^-), \qquad (5.5)$$

$$R_n^- = [(x - x' - 2nd)^2 + (y - y')^2 + (z - z')^2]^{\frac{1}{2}}, \qquad (5.6)$$

$$\underline{\underline{G}}^+ = \underline{\underline{I}}^+G^+, \qquad (5.7)$$

$$\underline{\underline{I}}^+ = - \hat{x}\hat{x} + \hat{y}\hat{y} + \hat{z}\hat{z}, \qquad (5.8)$$

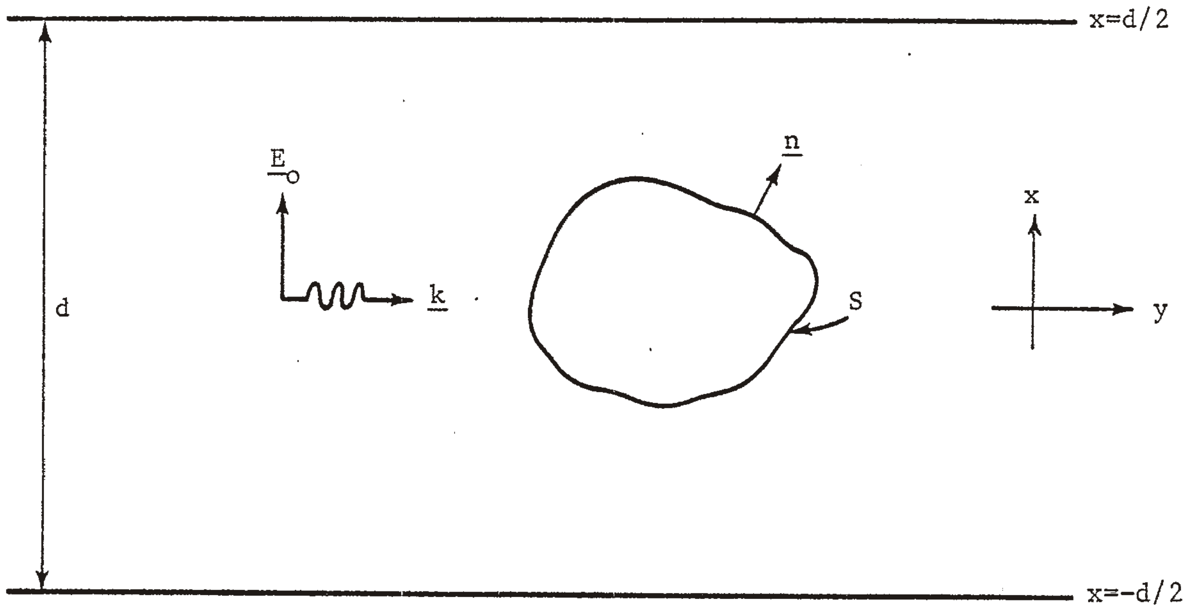

Figure 4. A perfectly conducting body within a parallel plate region.

$$G^{+}(\underline{r},\underline{r}';ik) = \sum_{-\infty}^{\infty} (4\pi R_n^{+})^{-1} \exp(ikR_n^{+}), \tag{5.9}$$

$$R_n^{+} = [(x + x' + 2nd)^2 + (y - y')^2 + (z - z')^2]^{\frac{1}{2}}, \tag{5.10}$$

and d is the spacing between the perfectly conducting, parallel plates.

We observe that the series representations (5.5) and (5.9) of $G^{-}(\underline{r},\underline{r}';ik)$ and $G^{+}(\underline{r},\underline{r}';ik)$, respectively, converge for real values of k except for $k = n\pi/d$ (n is an integer)[24]. We will later return to the question of convergence of the series representation of the Green's function in connection with the problem of finding representations of $G^{\pm}(\underline{r},\underline{r}';\gamma)$ which are valid in the entire γ-plane.

We will now investigate the kernel of the integral equation (5.1),

$$\nabla\times(\underline{\underline{G}}\cdot\underline{j}) = \nabla\times(\underline{\underline{G}}^{-}\cdot\underline{j}) + \nabla\times(\underline{\underline{G}}^{+}\cdot\underline{j}). \tag{5.11}$$

Here ∇ operates on the first argument of $G^{\pm}(\underline{r},\underline{r}';ik)$. Making use of Cartesian coordinates (x,y,z) one has

$$\nabla\times(\underline{\underline{G}}^{\pm}\cdot\underline{j}) = \sum_{-\infty}^{\infty} f(R_n^{\pm})\underline{\underline{D}}_n^{\pm}\cdot\underline{j} \tag{5.12}$$

where

$$f(R) = (4\pi R^3)^{-1}(ikR - 1)\exp(ikR)$$

and $\underline{\underline{D}}_n^{\pm}$ have the dyadic representations:

$$\underline{\underline{D}}_n^{-}(\underline{r},\underline{r}') = -(z - z')\hat{x}\hat{y} + (y - y')\hat{x}\hat{z} + (z - z')\hat{y}\hat{x} - (x - x' - 2nd)\hat{y}\hat{z}$$
$$- (y - y')\hat{z}\hat{x} + (x - x' - 2nd)\hat{z}\hat{y},$$

$$\underline{\underline{D}}_n^{+}(\underline{r},\underline{r}') = -(z - z')\hat{x}\hat{y} + (y - y')\hat{x}\hat{z} - (z - z')\hat{y}\hat{x} - (x + x' + 2nd)\hat{y}\hat{z}$$
$$+ (y - y')\hat{z}\hat{x} + (x + x' + 2nd)\hat{z}\hat{y}.$$

Thus, $\underline{\underline{L}}$ can be split into two parts (c.f. (5.2), (5.3) and (5.12)),

$$\underline{\underline{L}} = \underline{\underline{L}}^{-} + \underline{\underline{L}}^{+} \tag{5.13}$$

where $\underline{\underline{L}}^{\pm}$ are two integral operators with kernels $\underline{\underline{K}}^{\pm}$, having the representations,

$$\underline{\underline{K}}^{\pm}(\underline{r},\underline{r}';ik) = \sum_{-\infty}^{\infty} \underline{\underline{K}}_n^{\pm}(\underline{r},\underline{r}';ik). \tag{5.14}$$

Here

$$\underline{\underline{K}}_n^{\pm}(\underline{r},\underline{r}';ik) = f(R_n^{\pm})\underline{\underline{F}}^{\pm}(\underline{r},\underline{r}'),$$

and $\underline{\underline{F}}_n^{\pm}$ are two dyadics,

$$\begin{aligned}
\underline{\underline{F}}_n^{-}(\underline{r},\underline{r}') = &- [(y - y')n_y + (z - z')n_z]\hat{x}\hat{x} + (x - x' - 2nd)n_y\hat{x}\hat{y} \\
&+ (x - x' - 2nd)n_z\hat{x}\hat{z} + (y - y')n_x\hat{y}\hat{x} \\
&- [(z - z')n_z + (x - x' - 2nd)n_x]\hat{y}\hat{y} + (y - y')n_z\hat{y}\hat{z} \\
&+ (z - z')n_x\hat{z}\hat{x} + (z - z')n_y\hat{z}\hat{y} \\
&- [(x - x' - 2nd)n_x + (y - y')n_y]\hat{z}\hat{z},
\end{aligned}$$

$$\begin{aligned}
\underline{\underline{F}}_n^{+}(\underline{r},\underline{r}') = &[(y - y')n_y + (z - z')n_z]\hat{x}\hat{x} + (x + x' + 2nd)n_y\hat{x}\hat{y} \\
&+ (x + x' + 2nd)n_z\hat{x}\hat{z} - (y - y')n_x\hat{y}\hat{x} \\
&- [(z - z')n_z + (x + x' + 2nd)n_x]\hat{y}\hat{y} + (y - y')n_z\hat{x}\hat{z} \\
&- (z - z')n_x\hat{z}\hat{x} + (z - z')n_y\hat{z}\hat{y} \\
&- [(x + x' + 2nd)n_x + (y - y')n_y]\hat{z}\hat{z}.
\end{aligned}$$

From (5.14) one can derive the following asymptotic expression for $\underline{\underline{K}}_n^{\pm}(\underline{r},\underline{r}';ik)$,

$$\underline{\underline{K}}_n^{\pm}(\underline{r},\underline{r}';ik) = \pm \underline{\underline{A}}n^{-1} \exp(2ikd|n|) + \underline{\underline{B}}^{\pm}(\underline{r},\underline{r}')b_n \tag{5.15}$$

where $b_n = O(n^{-2})$ as $n \to \infty$ and $\underline{\underline{A}}$ is a constant dyadic. Thus, we have

$$\underline{\underline{K}}^{\pm}(\underline{r},\underline{r}';ik) = \pm \underline{\underline{A}} \sum_{n \neq 0} n^{-1} \exp(2ikd|n|) + \underline{\underline{R}}^{\pm}(\underline{r},\underline{r}';ik) \qquad (5.16)$$

where $\underline{\underline{R}}^{\pm}(\underline{r},\underline{r}';ik)$ is well defined for all real values of k and $\underline{r} \neq \underline{r}'$. Moreover, since

$$\sum_{n=1}^{\infty} n^{-1} \exp(2ikdn) = - \ln[1 - \exp(2ikd)] \qquad (5.17)$$

the series representation (5.14) of $\underline{\underline{K}}^{\pm}(\underline{r},\underline{r}';ik)$ converges for all real values of k such that $k \neq n\pi/d$. Also from equation (5.15) the series (5.14) converges conditionally for $k = n\pi/d$. In passing we want to point out that the series

$$\underline{\underline{K}}(\underline{r},\underline{r}';ik) = \sum_{-\infty}^{\infty} [\underline{\underline{K}}_n^{-}(\underline{r},\underline{r}';ik) + \underline{\underline{K}}_n^{+}(\underline{r},\underline{r}';ik)] \qquad (5.18)$$

converges absolutely for all real values of k. This can be seen from the fact that for large n we have asymptotically:

$$\underline{\underline{K}}_n^{-}(\underline{r},\underline{r}';ik) + \underline{\underline{K}}_n^{+}(\underline{r},\underline{r}';ik) = \underline{\underline{C}}(\underline{r},\underline{r}')c_n \qquad (5.19)$$

and $c_n = O(n^{-2})$ as $n \to \infty$.

To conclude our considerations for real values of k we will briefly investigate the convergence of the series representation (5.4) through (5.10) of the dyadics $\underline{\underline{G}}^{\pm}(\underline{r},\underline{r}';ik)$. Employing the methods used when deriving (5.16) it can be shown that one representation of $\underline{\underline{G}}^{\pm}(\underline{r},\underline{r}';ik)$ is given by

$$\underline{\underline{G}}^{\pm}(\underline{r},\underline{r}';ik) = \underline{\underline{G}}_1^{\pm}(\underline{r},\underline{r}') \sum_{n=1}^{\infty} n^{-1} \exp(2iknd) + \underline{\underline{G}}_2^{\pm}(\underline{r},\underline{r}';ik) \qquad (5.20)$$

where $\underline{\underline{G}}_2^{\pm}(\underline{r},\underline{r}';ik)$ is well defined for all values of k and $\underline{r} \neq \underline{r}'$, and $\underline{\underline{G}}_1^{\pm}(\underline{r},\underline{r}')$ is well defined for $\underline{r} \neq \underline{r}'$. Thus, from equations (5.17) and (5.20) it can be seen that the series representations (5.3) through (5.10) of $\underline{\underline{G}}^{\pm}(\underline{r},\underline{r}';ik)$ diverge for $k = n\pi/d$ (in contrast to the series representation of $\underline{\underline{K}}^{\pm}(\underline{r},\underline{r}';ik)$).

Until now we have only considered the operator $\underline{\underline{L}} = \underline{\underline{L}}(\gamma)$ when γ is a purely imaginary number. In order to find an analytic continuation of $\underline{\underline{L}}$ from the imaginary axis into the complex γ-plane we first make the following observation. The series

$$G^{\pm}(\underline{r},\underline{r}';\gamma) = \sum_{-\infty}^{\infty} (4\pi R_n^{\pm})^{-1} \exp(-\gamma R_n^{\pm}) \tag{5.21}$$

converges for $Re\{\gamma\} > 0$. Therefore, the definition of $\underline{\underline{L}}(\gamma)$ from $G^{\pm}(\underline{r},\underline{r}';\gamma)$ by equations (5.2) through (5.10) can be extended to all complex values of γ such that $Re\{\gamma\} \geq 0$. Next, we will find an operator $\underline{\underline{\Lambda}}(\gamma)$, defined almost everywhere in the entire complex γ-plane, and such that $\underline{\underline{\Lambda}}(\gamma) = \underline{\underline{L}}(\gamma)$ for $Re\{\gamma\} \geq 0$. In order to find $\underline{\underline{\Lambda}}$ we start with the following representation of $\underline{\underline{L}}$,

$$\underline{\underline{L}} = \underline{\underline{L}}_o + \sum_{\substack{m=2,3 \\ \ell=0,1}} (\underline{\underline{L}}_{m\ell}^{-} + \underline{\underline{L}}_{m\ell}^{+}) \tag{5.22}$$

where

$$(\underline{\underline{L}}_o \cdot \underline{f})(\underline{r}) = \int_S [\underline{\underline{K}}_o^{-}(\underline{r},\underline{r}';\gamma) + \underline{\underline{K}}_o^{+}(\underline{r},\underline{r}';\gamma)] \cdot f(\underline{r}')dS',$$

$$(\underline{\underline{L}}_{m\ell}^{\pm} \cdot \underline{f})(\underline{r}) = \int_S S^{\pm}(\gamma;\underline{r},\underline{r}';m,\ell)\underline{\underline{B}}_{m\ell}(\underline{r},\underline{r}') \cdot \underline{f}(\underline{r}')dS',$$

and

$$S^{\pm}(\gamma;\underline{r},\underline{r}';m,\ell) = \sum_{n=1}^{\infty} n^{\ell}(R_n^{\pm})^{-m} \exp(-\gamma R_n^{\pm}).$$

The analytic properties in the complex γ-plane of the functions $S^{\pm}(\gamma;\underline{r},\underline{r}';m,\ell)$ are investigated in Appendix A. In this appendix we have shown the existence and given a method of constructing two functions, $\Sigma^{\pm}(\gamma;\underline{r},\underline{r}';m,\ell)$, defined almost everywhere in the entire γ-plane, and such that

$$\Sigma^{\pm}(\gamma;\underline{r},\underline{r}';m,\ell) = S^{\pm}(\gamma;\underline{r},\underline{r}';m,\ell) \tag{5.23}$$

for $Re\{\gamma\} \geq 0$. Moreover $\Sigma^{\pm}(\gamma;\underline{r},\underline{r}';m,\ell)$ are analytic functions in the entire γ-plane except along the lines C_n, defined by

$$C_n = \{\gamma : Re\{\gamma\} \leq 0,\ Im\{\gamma\} = n\pi/d,\ n \text{ integer}\},$$

where $\Sigma^{\pm}(\gamma;\underline{r},\underline{r}';m,\ell)$ are discontinuous (see figure 5). Thus, we can define an operator $\underline{\underline{\Lambda}} = \underline{\underline{\Lambda}}(\gamma)$,

$$\underline{\underline{\Lambda}} = \underline{\underline{L}}_o + \sum_{\substack{m=2,3\\ \ell=0,1}} (\underline{\underline{\Lambda}}^-_{m\ell} + \underline{\underline{\Lambda}}^+_{m\ell}) \tag{5.24}$$

where

$$(\underline{\underline{\Lambda}}^{\pm}_{m\ell}\cdot\underline{f})(\underline{r}) = \int_S \Sigma^{\pm}(\gamma;\underline{r},\underline{r}';m,\ell)\underline{\underline{B}}_{m\ell}(\underline{r},\underline{r}')\cdot\underline{f}(\underline{r}')dS'$$

and $\underline{\underline{L}}_o$ is defined by equation (5.22). From equations (5.22) through (5.24) it follows that

$$\underline{\underline{\Lambda}}(\gamma) = \underline{\underline{L}}(\gamma)$$

for $\mathrm{Re}\{\gamma\} \geq 0$. Moreover, $\underline{\underline{\Lambda}}(\gamma)$ is an analytic operator-valued function of γ except on the lines C_n. Of course, the lines C_n can be considered as branch cuts in the complex γ-plane of the operator $\underline{\underline{\Lambda}}(\gamma)$. The operator $\underline{\underline{\Lambda}}$ is the analytic continuation of $\underline{\underline{L}}$ into the complex γ-plane.

We now go on to consider the solution of the integral equation

$$(\frac{1}{2}\,\underline{\underline{I}} - \underline{\underline{\Lambda}})\cdot\underline{j} = \underline{j}^{inc} \tag{5.25}$$

for any complex value of γ. Since $\underline{\underline{\Lambda}}(ik) = \underline{\underline{L}}(ik)$ it follows from the uniqueness theorem that the solution of the integral equations (5.1) and (5.25) are identical for $\gamma = ik$. Next, we will consider the analytic behavior in the complex γ-plane of the solution of the integral equation (5.25). It is easy to show that the operator $\underline{\underline{\Lambda}}^2$, defined by $\underline{\underline{\Lambda}}^2\cdot\underline{f} = \underline{\underline{\Lambda}}\cdot(\underline{\underline{\Lambda}}\cdot\underline{f})$, is of Hilbert Schmidt type[2]. The integral equation

$$(\frac{1}{4}\,\underline{\underline{I}} - \underline{\underline{\Lambda}}^2)\cdot\underline{j} = (\frac{1}{2}\,\underline{\underline{I}} + \underline{\underline{\Lambda}})\cdot\underline{j}^{inc} \equiv \underline{f}^{inc}$$

can then be solved by using the Fredholm determinant theory[2]. Following the procedure in reference [2] one can easily see that the inverse operator $(\frac{1}{2}\,\underline{\underline{I}} - \underline{\underline{\Lambda}})^{-1}$ has two types of singularities. One type is poles at those values

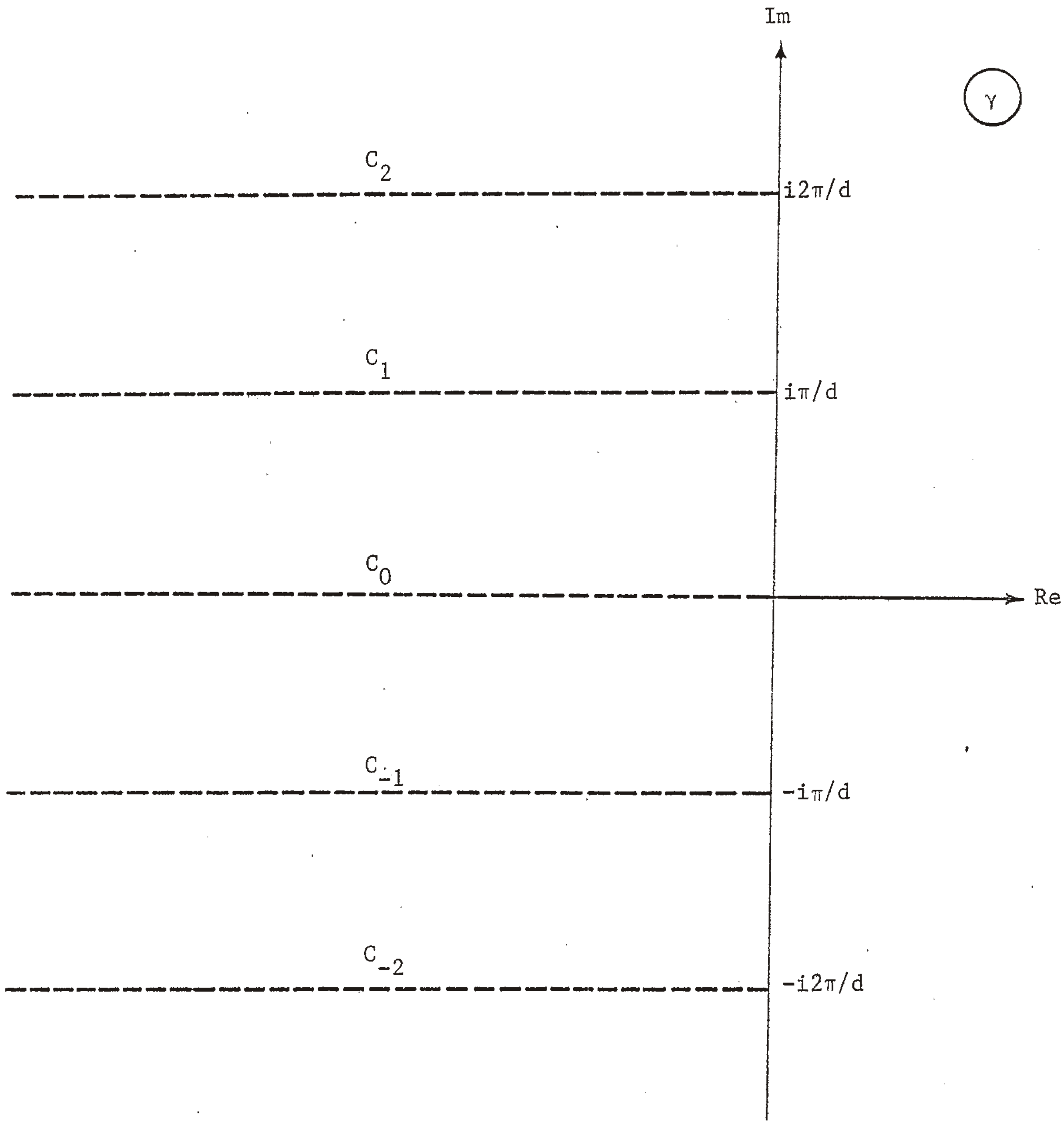

Figure 5. Branch cuts of the operator $\underline{\underline{\Lambda}}(\gamma)$.

of γ, γ_n, where there exists a nontrivial solution of the homogeneous integral equation

$$[\frac{1}{2}\underline{\underline{I}} - \underline{\underline{\Lambda}}(\gamma_n)]\cdot\underline{j}_n = 0, \qquad \underline{j}_n \neq 0.$$

The other type of singularities of the inverse operator $(\frac{1}{2}\underline{\underline{I}} - \underline{\underline{\Lambda}})^{-1}$ is discontinuities along the lines C_n, where the operator $\frac{1}{2}\underline{\underline{I}} - \underline{\underline{\Lambda}}$ is discontinuous.

For the case where all the poles of the inverse operator $(\frac{1}{2}\underline{\underline{I}} - \underline{\underline{\Lambda}})^{-1}$ are simple poles we get the following time-domain representation of the induced current, $J(\underline{r},t)$, due to a delta-function incident TEM wave,[2]

$$\underline{H}^{inc} = \underline{I}_o\delta(z - ct), \qquad \underline{I}_o = I_o\hat{y}$$

$$\underline{J}(\underline{r},t) = U(ct - z_o)\Bigg[\sum_{n,m}\langle\underline{j}^{inc}(\gamma_n)\exp(\gamma_n z_o),\underline{h}_{nm}\rangle[\langle\underline{\underline{B}}_n\cdot\underline{j}_{nm},\underline{h}_{nm}\rangle]^{-1}\underline{j}_{nm}\exp[\gamma_n(ct - z_o)]$$

$$+ (2\pi i)^{-1}\sum_{-\infty}^{\infty}\int_0^{\infty}[\underline{\underline{P}}_n(\xi)\cdot\underline{j}^{inc}(-\xi + i\pi n/d)\exp(-ct\xi + i\pi nct/d)]d\xi\Bigg]$$

Here

$$\underline{j}^{inc}(\gamma) = \underline{n}\times\hat{y}\,\exp(-\gamma z),$$

$$\underline{\underline{B}}_n = d\underline{\underline{\Lambda}}/d\gamma \quad \text{evaluated at} \quad \gamma = \gamma_n,$$

$\underline{j}_{nm}$ and $\underline{h}_{nm}$ are linearly independent nontrivial solutions of the homogeneous integral equations

$$[\frac{1}{2}\underline{\underline{I}} - \underline{\underline{\Lambda}}(\gamma_n)]\cdot\underline{j}_{nm} = 0,$$

$$[\frac{1}{2}\underline{\underline{I}} - \underline{\underline{\Lambda}}^{\dagger}(\gamma_n)]\cdot\underline{h}_{nm} = 0,$$

where $\underline{\underline{\Lambda}}^{\dagger}$ is the adjoint operator of $\underline{\underline{\Lambda}}$. Moreover,

$$\underline{\underline{P}}_n(\xi) = \underline{\underline{P}}_n^-(\xi) - \underline{\underline{P}}_n^+(\xi),$$

$$\underline{\underline{P}}_n^{\pm}(\xi) = \lim_{\eta \to 0\pm} [\tfrac{1}{2} \underline{\underline{I}} - \underline{\underline{\Lambda}}(-\xi + i\eta + i\pi n/d)]^{-1}$$

and we assume that the inverse operator $[\frac{1}{2} \underline{\underline{I}} - \underline{\underline{\Lambda}}(\gamma)]^{-1}$ exists as γ approaches C_n (from above or below).

To sum up, we have shown that the singularities in the complex frequency plane of the operator inverse to the integral operator of the magnetic field formulation are poles and branch cuts. The location of the branch cuts is uniquely determined by the distance between the parallel plates. Based on the theory developed here we might in the future undertake a numerical study of the location of the poles. As we have seen, the main problem is to find a representation of the integral operator that is valid in the entire complex frequency plane. There still remain some problems to be solved in the construction of this operator. Needless to say, the "ordinary" series representation, derived from the method of images, is not valid throughout the entire complex frequency plane.

Based on eigenfunction expansions different representation of the Green's function is given in Appendix B.

Appendix A

In this appendix we will consider the following sums

$$S(\zeta;m,\ell) = \sum_{n=1}^{\infty} n^{\ell} R_n^{-m} \exp(-\zeta R_n), \qquad m \geq \ell \tag{A.1}$$

where

$$R_n = (n^2 + \alpha n + \beta)^{\frac{1}{2}}.$$

The series (A.1) converges for $\text{Re}\{\zeta\} > 0$ when m and ℓ are any finite integers. It also converges for $\text{Re}\{\zeta\} \geq 0$ when $m \geq \ell + 2$. In this appendix we will find the analytical continuation of $S(\zeta;m,\ell)$ into the entire ζ-plane when $m \geq \ell$.

Let N be a finite integer such that

$$|\alpha| + |\beta| < N \leq |\alpha| + |\beta| + 1. \tag{A.2}$$

We then have

$$S(\zeta;m,\ell) = H(\zeta;m,\ell) + D(\zeta;m,\ell) \tag{A.3}$$

where

$$H(\zeta;m,\ell) = \sum_{n=1}^{N-1} n^{\ell} R_n^{-m} \exp(-\zeta R_n)$$

and

$$D(\zeta;m,\ell) = \sum_{n=N}^{\infty} n^{\ell} R_n^{-m} \exp(-\zeta R_n).$$

Since N is finite it easily follows that $H(\zeta;m,\ell)$ is an analytic function in the entire ζ-plane. Next we consider the expansion

$$R_n^{-m} \exp(-\zeta R_n) = \sum_{k=0}^{\infty} f_k(\zeta,m) n^{-m-k} \exp(-n\zeta). \tag{A.4}$$

The analytical expression for $f_k(\zeta,m)$ is rather complicated. We will here only consider the convergence of the series (A.4). We have

$$R_n^{-m} = n^{-m}\left[\sum_{k=0}^{\infty} \alpha_k n^{-k}\right]^{-m} \tag{A.5}$$

and the expansion (A.5) converges for $n \geq N$. Moreover, it is easy to show that there exists a finite constant A such that $R_n > nA$ for all n. We then have

$$\begin{aligned} |R_n^{-m} \exp(-\zeta R_n)| &< (nA)^{-m}|\exp(-\zeta n \sum_{k=0}^{\infty} \alpha_k n^{-k})| \\ &= (nA)^{-m}|\exp(-\zeta n - \zeta\alpha_1 - \zeta \sum_{k=2}^{\infty} \alpha_k n^{-k+1})| \\ &= (nA)^{-m}|[1 + C_n(\zeta)]\exp(-\zeta n - \zeta\alpha_1)| \end{aligned} \tag{A.6}$$

where

$$C_n(\zeta) = \sum_{\ell=1}^{\infty} \zeta^{\ell}(\ell!)^{-1}\left[\sum_{k=2}^{\infty} \alpha_k n^{-k+1}\right]^{\ell}$$

Furthermore, we have

$$|C_n(\zeta)| \leq \sum_{\ell=1}^{\infty} |\zeta|^{\ell}(\ell!)^{-1}\left[\sum_{k=2}^{\infty} |\alpha_k| N^{-k+1}(N/n)^{k-1}\right]^{\ell} \leq \sum_{\ell=1}^{\infty} C|N\zeta|^{\ell}(\ell!)^{-1}n^{-\ell} \tag{A.7}$$

where

$$C = \sum_{k=2}^{\infty} |\alpha_k| N^{-k+1} < \infty.$$

Comparing equations (A.4) and (A.7) we obtain

$$|f_k(\zeta,m)| < CA^{-m}|N\zeta|^{k}(k!)^{-1}|\exp(-\zeta n - \zeta\alpha_1|, \qquad k \geq 1. \tag{A.8}$$

From equation (A.8) it follows that the series (A.6) converges for all complex values of ζ. In passing we also want to point out that, from the construction of $f_k(\zeta,m)$, it follows immediately that $f_k(\zeta,m)$ is an analytic function in the entire ζ-plane.

From equation (A.8) it also follows that

$$D(\zeta;m,\ell) = \sum_{n=N}^{\infty}\sum_{k=0}^{\infty} f_k(\zeta,m)n^{\ell-m} \exp(-n\zeta) = \sum_{k=0}^{\infty}\sum_{n=N}^{\infty} f_k(\zeta,m)n^{\ell-m} \exp(-n\zeta). \tag{A.9}$$

From the definition of the Γ function it follows that (c.f. Appendix D of reference [28])

$$n^{-\ell} \exp(-n\zeta) = [\Gamma(\ell)]^{-1} \int_0^\infty x^{\ell-1} \exp[-(\zeta + x)n]dx \tag{A.10}$$

and

$$\sum_{n=N}^{\infty} n^{-\ell} \exp(-n\zeta) = [\Gamma(\ell)]^{-1} F_\ell(\zeta) \tag{A.11}$$

where

$$F_\ell(\zeta) = \int_0^\infty x^{\ell-1} \exp[-(x + \zeta)N]/[1 - \exp(-\zeta - x)]dx. \tag{A.11}$$

Equation (A.11) is valid for $\mathrm{Re}\{\zeta\} > 0$ when $\ell = 1$ and it is valid for $\mathrm{Re}\{\zeta\} \geq 0$ when $\ell \geq 2$. Thus, we have

$$D(\zeta;m,\ell) = \sum_{k=0}^{\infty} f_k(\zeta,m)F_{m-\ell}(\zeta)/\Gamma(m - \ell). \tag{A.12}$$

The convergence of the series (A.12) follows from the following consideration: for $\mathrm{Re}\{\zeta\} > 0$ there exists an $\varepsilon > 0$ such that

$$|1 - e^{-(\zeta+x)}| > \varepsilon$$

and from equation (A.11) it then follows that

$$|F_\ell(\zeta)| < \varepsilon^{-1}\Gamma(\ell)N^{-\ell}|\exp(-N\zeta)|. \tag{A.13}$$

The convergence of the series (A.12) is now clear in view of equations (A.8), (A.11) and (A.13).

Introduce the function

$$\Phi_\ell(\zeta) = \int_C z^{\ell-1} \exp[-(z + \zeta)N]/[1 - \exp(-z - \zeta)]dz \tag{A.14}$$

where the path of integration, C, is along the real axis in the complex z-plane. One easily sees that $\Phi_\ell(\zeta)$ is an analytic function in the entire ζ-plane except on the lines C_n,

$$C_n = \{\zeta : \mathrm{Re}\{\zeta\} < 0,\ \mathrm{Im}\{\zeta\} = 2\pi n,\ n \text{ integer}\}.$$

We also notice that

$$\Phi_\ell(\zeta + 2\pi i n) = \Phi_\ell(\zeta)$$

so that we can limit our investigation of $\Phi_\ell(\zeta)$ within the band $\{\zeta : -\infty < \mathrm{Re}\{\zeta\} < \infty, -\pi \le \mathrm{Im}\{\zeta\} \le \pi\}$. From equations (A.11) and (A.14) it follows that

$$\Phi_\ell(\zeta) = F_\ell(\zeta) \quad \text{for} \quad \mathrm{Re}\{\zeta\} > 0.$$

Thus, the function $\Phi_\ell(\zeta)$ is an analytic continuation of the function $F_\ell(\zeta)$. Making use of complex contour integration it is easy to show that the limit values

$$\lim_{\eta\to 0\pm} \Phi_\ell(\xi + i\eta), \qquad \xi < 0 \tag{A.15}$$

exist. Moreover, we have

$$\lim_{\eta\to 0-} \Phi_\ell(\xi + i\eta) - \lim_{\eta\to 0+} \Phi_\ell(\xi + i\eta) = 2\pi i(-\xi)^\ell \tag{A.16}$$

Thus, we can define a function $\Delta(\zeta;m,\ell)$,

$$\Delta(\zeta;m,\ell) = \sum_{k=0}^{\infty} f_k(\zeta,m)\Phi_{m-\ell}(\zeta)/\Gamma(m-\ell) \tag{A.17}$$

and for $\mathrm{Re}\{\zeta\} > 0$ we have

$$\Delta(\zeta;m,\ell) = D(\zeta;m,\ell). \tag{A.18}$$

It is easy to show that the series (A.17) converges for $\zeta \notin C_n$ (c.f. the proof above concerning the convergence of the series (A.12)). Thus, the function

$\Delta(\zeta;m,\ell)$ is an analytic function of ζ in the entire ζ-plane except on the lines C_n. We use $\Delta(\zeta;m,\ell)$ as the analytic continuation of $D(\zeta;m,\ell)$ into the entire ζ-plane. From equations (A.3) and (A.18) it then follows that the function $\Sigma(\zeta;m,\ell)$,

$$\Sigma(\zeta;m,\ell) = H(\zeta;m,\ell) + \Delta(\zeta;m,\ell), \qquad (A.19)$$

in an analytic continuation into the entire ζ-plane of the function $S(\zeta;m,\ell)$ defined by equation (A.1).

Appendix B

Using eigenfunction expansions we will in this Appendix find a representation of the scalar Green's function different from the one used in section V.

The Green's function $G_1(\underline{r},\underline{r}';\gamma)$ of a parallel plate region satisfies the differential equation

$$\left[\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2} - \gamma^2\right]G_1 = -\frac{\delta(\rho-\rho')}{\rho'}\delta(\phi - \phi')\delta(z - z') \qquad \text{(B.1)}$$

for $0 < z < d$ and the boundary conditions

$$\frac{\partial G_1}{\partial z} = 0 \quad \text{when} \quad z = 0,d.$$

To find G_1 that satisfies equation (B.1) we proceed in the usual way. Let

$$G_1(\underline{r},\underline{r}';\gamma) = \begin{cases} \sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} A_{nm}K_m(\lambda_n\rho)[K_m(\lambda_n\rho')]^{-1}\cos(n\pi z/d)\cos(n\pi z'/d)\exp[im(\phi-\phi')], & \rho > \rho' \\ \sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} A_{nm}I_m(\lambda_n\rho)[I_m(\lambda_n\rho')]^{-1}\cos(n\pi z/d)\cos(n\pi z'/d)\exp[im(\phi-\phi')], & \rho < \rho' \end{cases} \qquad \text{(B.2)}$$

where

$$\lambda_n^2 = \gamma^2 + (n\pi/d)^2$$

and $I_m(\zeta)$, $K_m(\zeta)$ are modified Bessel functions. Note that (B.2) satisfies the boundary conditions (B.1). To determine the constants A_{nm}, multiply the differential equation (B.1) with $\cos(n\pi z/d)$ and perform the integration

$$\lim_{\varepsilon\to 0}\int_o^d\int_o^{2\pi}\int_{\rho'-\varepsilon}^{\rho'+\varepsilon}(\cdots\cdot)\rho d\rho d\phi dz.$$

After some straightforward algebraic manipulations combined with the Wronskian for the modified Bessel functions we obtain

$$A_{nm} = (\varepsilon_n\pi d)^{-1}I_m(\lambda_n\rho')K_m(\lambda_n\rho') \qquad \text{(B.3)}$$

where $\varepsilon_n = 1$ if $n \geq 1$ and $\varepsilon_o = 2$. Substituting (B.3) into (B.2) we have

$$G_1(\underline{r};\underline{r}';\gamma) = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} (\varepsilon_n \pi d)^{-1} I_m(\lambda_n \rho_<) K_m(\lambda_n \rho_>) \cos(n\pi z/d) \cos(n\pi z'/d) \exp[im(\phi-\phi')] \tag{B.4}$$

where $\rho_<$ ($\rho_>$) denotes the smaller (larger) of ρ and ρ'.

Next, we consider the analytical properties of G_1 in the complex γ-plane. In doing so we first observe that the series representation (B.4) converges for all finite values of γ such that $\lambda_n(\gamma) \neq 0$ when $\underline{r} \neq \underline{r}'$. The convergence is easy to show from the asymptotic expansions of the Bessel functions. From other representations of $G_1(\underline{r},\underline{r}';\gamma)$ one has

$$G_1(\underline{r},\underline{r}';\gamma) \sim (4\pi|\underline{r} - \underline{r}'|)^{-1} \exp(-\gamma|\underline{r} - \underline{r}'|) \qquad \text{as} \quad \underline{r} \to \underline{r}'.$$

We also note that the function $\lambda_n(\gamma)$,

$$\lambda_n(\gamma) = \sqrt{\gamma^2 + (n\pi/d)^2}$$

is multiple-valued but can be made single-valued by introducing branch cuts parallel to the negative real axis and starting at $\pm i\pi n/d$. These branch cuts have been denoted by C_n in section V. (See figure 5). Since $\lambda_n = 0$ at $i\pi n/d$ it is clear that the branch cuts for λ_n also make $I_m(\lambda_n \rho)$ and $K_m(\lambda_n \rho)$ unique functions of γ. Thus, $G_1(\underline{r},\underline{r}';\gamma)$ is an analytic function in the entire γ-plane except on the branch cuts C_n.

Appendix C

In this appendix we will find a solution of the integral equation (4.11) of section IV,

$$(\underline{\underline{I}} - \underline{\underline{\Gamma}}^3)\cdot\underline{\phi} = \underline{\psi}. \tag{C.1}$$

The quantities $\underline{\underline{\Gamma}}^3$, $\underline{\phi}$ and $\underline{\psi}$ have been defined in section IV.

The operator equation (C.1) has the following explicit representation

$$\begin{aligned}
&\eta(\underline{r}) - \int_\Omega \Lambda_{11}(\underline{r},\underline{r}';\gamma)\eta(\underline{r}')dV' - \int_\Sigma \Lambda_{12}(\underline{r},\underline{r}';\gamma)\sigma(\underline{r}')dS' = \eta_1(\underline{r}), \quad \underline{r}\in\Omega \\
&\sigma(\underline{r}) - \int_\Omega \Lambda_{21}(\underline{r},\underline{r}';\gamma)\eta(\underline{r}')dV' - \int_\Sigma \Lambda_{22}(\underline{r},\underline{r}';\gamma)\sigma(\underline{r}')dS' = \sigma_1(\underline{r}), \quad \underline{r}\in\Sigma
\end{aligned} \tag{C.2}$$

Here Λ_{11}, Λ_{12}, Λ_{21}, Λ_{22} have a logarithmic singularity at those points where Γ_{11}, Γ_{12}, Γ_{21}, Γ_{22} are singular.

The method we will use in solving the set of integral equations (C.2) is based on the Fredholm determinant theory. Introducing the resolvent, $R_1(\underline{r},\underline{r}';\gamma)$, which satisfies the integral equation

$$R_1(\underline{r},\underline{r}';\gamma) - \int_\Omega \Lambda_{11}(\underline{r},\underline{r}'';\gamma)R_1(\underline{r}'',\underline{r}';\gamma)dV'' = \Lambda_{11}(\underline{r},\underline{r}';\gamma) \tag{C.3}$$

we can transform the first equation in (C.2) into the following equation[18]

$$\begin{aligned}
\eta(\underline{r}) = \eta_1(\underline{r}) &+ \int_\Sigma \Lambda_{12}(\underline{r},\underline{r}';\gamma)\sigma(\underline{r}')dS' + \int_\Omega R_1(\underline{r},\underline{r}';\gamma)\eta_1(\underline{r}')dV' \\
&+ \int_{\Omega\times\Sigma} R_1(\underline{r},\underline{r}';\gamma)\Lambda_{12}(\underline{r}',\underline{r}'';\gamma)\sigma(\underline{r}'')dV'dS''
\end{aligned} \tag{C.4}$$

Substituting the expression (C.4) into the second equation of (C.2) we arrive at the following integral equation for $\sigma(\underline{r})$,

$$\sigma(\underline{r}) - \int_{\Sigma} \Lambda'(\underline{r},\underline{r}';\gamma)\sigma(\underline{r}')dS' = \sigma'(\underline{r}), \qquad \underline{r} \in \Sigma. \tag{C.5}$$

Here,

$$\Lambda'(\underline{r},\underline{r}';\gamma) = \Lambda_{22}(\underline{r},\underline{r}';\gamma) + \Lambda_1(\underline{r},\underline{r}';\gamma) + \Lambda_2(\underline{r},\underline{r}';\gamma)$$

$$\Lambda_1(\underline{r},\underline{r}';\gamma) = \int_{\Omega} \Lambda_{21}(\underline{r},\underline{r}'';\gamma)\Lambda_{12}(\underline{r}'',\underline{r}';\gamma)dV''$$

$$\Lambda_2(\underline{r},\underline{r}';\gamma) = \int_{\Omega\times\Omega} \Lambda_{21}(\underline{r},\underline{r}''';\gamma)R_1(\underline{r}''',\underline{r}'';\gamma)\Lambda_{12}(\underline{r}'',\underline{r}';\gamma)dV'''dV''$$

$$\sigma'(\underline{r}) = \sigma_1(\underline{r}) + \sigma_2(\underline{r}) + \sigma_3(\underline{r})$$

$$\sigma_2(\underline{r}) = \int_{\Omega} \Lambda_{21}(\underline{r},\underline{r}';\gamma)\eta_1(\underline{r}')dV'$$

$$\sigma_3(\underline{r}) = \int_{\Omega\times\Omega} \Lambda_{21}(\underline{r},\underline{r}';\gamma)R_1(\underline{r}',\underline{r}'';\gamma)\eta_1(\underline{r}'')dV'dV''.$$

All the operations we have performed so far are strictly formal and we will now go back and examine the validity of each operation. First, we notice that the kernel $\Lambda_{11}(\underline{r},\underline{r}';\gamma)$ is a square integrable function on the domain $\Omega\times\Omega$. Thus, the integral equation (C.3) can be solved by using the Fredholm determinant theory. Moreover, in the following we assume that $\tilde{\varepsilon}$, $\nabla\tilde{\varepsilon}/\tilde{\varepsilon}$, $\tilde{\mu}$ and $\nabla\tilde{\mu}/\tilde{\mu}$ are analytic functions of $\gamma = s/c$. We will also assume that the incident field is an analytic function of γ implying that η_1 and σ_1 are analytic functions of γ. It then follows from equations (4.3), (4.8), (4.10) of section IV and (C.2) that the kernels Λ_{11}, Λ_{12}, Λ_{21} and Λ_{22} are analytic functions of γ throughout the entire complex γ-plane. Following the procedure used in reference [2] it is easy to show that the resolvent, $R_1(\underline{r},\underline{r}';\gamma)$, is an analytic function of γ except for those values of γ, $\gamma_n^{(1)}$, where there exists a nontrivial solution of the homogeneous integral equation

$$\eta(\underline{r}) - \int_{\Omega} \Lambda_{11}(\underline{r},\underline{r}';\gamma_n^{(1)})\eta(\underline{r}')dV' = 0. \tag{C.6}$$

The resolvent has a pole at $\gamma = \gamma_n^{(1)}$.

Let Δ be a domain in the complex γ-plane such that $R_1(\underline{r},\underline{r}';\gamma)$ is analytic in Δ. (We will discuss later what happens at the poles of $R_1(\underline{r},\underline{r}';\gamma)$). From the integral equation (C.3) it follows that $R_1(\underline{r},\underline{r}';\gamma)$ has a logarithmic singularity at those values of $(\underline{r},\underline{r}')$ where $\Lambda_{11}(\underline{r},\underline{r}';\gamma)$ has a logarithmic singularity. From the Schwartz inequality it then follows that Λ_1 and Λ_2 are square integrable functions on $\Omega\times\Omega$. The integral equation (C.5) can then be solved by using the Fredholm determinant theory. Since σ' is an analytic function of γ on Δ it follows that the only singularities in Δ of the solution of the integral equation (C.5) are poles at $\gamma = \gamma_n'$ where there exists a non-trivial solution of the equation

$$\sigma(\underline{r}) - \int_\Sigma \Lambda'(\underline{r},\underline{r}';\gamma_n')\sigma(\underline{r}')dS' = 0. \qquad \text{(C.7)}$$

It now follows that the operator inverse to the integral operator defining the left hand side of equation (C.5) has two type of singularities. One type is poles at those values of $\gamma = \gamma_n'$ where there exist a nontrivial solution of equation (C.7). The locations of the other type of singularities (if any) coincide with the locations of the poles of the resolvent $R_1(\underline{r},\underline{r}';\gamma)$.

In order to investigate the behavior of the solution of equation (C.5) around the poles of $R_1(\underline{r},\underline{r}';\gamma)$ we will go back to the set of integral equations (C.2). Introducing the resolvent $R_2(\underline{r},\underline{r}';\gamma)$ satisfying the integral equation

$$R_2(\underline{r},\underline{r}';\gamma) - \int_\Sigma \Lambda_{22}(\underline{r},\underline{r}'';\gamma)R_2(\underline{r}'',\underline{r}';\gamma)dS'' = \Lambda_{22}(\underline{r},\underline{r}';\gamma) \qquad \text{(C.8)}$$

we have

$$\sigma(\underline{r}) = \sigma_1(\underline{r}) + \int_\Omega \Lambda_{21}(\underline{r},\underline{r}';\gamma)\eta(\underline{r}')dV' + \int_\Sigma R_2(\underline{r},\underline{r}';\gamma)\sigma_1(\underline{r}')dS'$$

$$+ \int_{\Sigma\times\Omega} R_2(\underline{r},\underline{r}';\gamma)\Lambda_{21}(\underline{r}',\underline{r}'';\gamma)\eta(\underline{r}'')dS'dV''. \qquad \text{(C.9)}$$

Here $\eta(\underline{r})$ satisfies the integral equation

$$\eta(\underline{r}) - \int_\Omega \Lambda''(\underline{r},\underline{r}';\gamma)\eta(\underline{r}')dV' = \eta''(\underline{r}), \qquad \underline{r} \in \Omega \qquad \text{(C.10)}$$

where

$$\Lambda''(\underline{r},\underline{r}';\gamma) = \Lambda_{11}(\underline{r},\underline{r}';\gamma) + \Lambda_3(\underline{r},\underline{r}';\gamma) + \Lambda_4(\underline{r},\underline{r}';\gamma)$$

$$\Lambda_3(\underline{r},\underline{r}';\gamma) = \int_\Sigma \Lambda_{12}(\underline{r},\underline{r}'';\gamma)\Lambda_{21}(\underline{r}'',\underline{r}';\gamma)dS''$$

$$\Lambda_4(\underline{r},\underline{r}';\gamma) = \int_{\Sigma\times\Sigma} \Lambda_{12}(\underline{r},\underline{r}''';\gamma)R_2(\underline{r}''',\underline{r}'';\gamma)\Lambda_{21}(\underline{r}'',\underline{r}';\gamma)dV'''dV''$$

$$\eta''(\underline{r}) = \eta_1(\underline{r}) + \eta_2(\underline{r}) + \eta_3(\underline{r})$$

$$\eta_2(\underline{r}) = \int_\Sigma \Lambda_{12}(\underline{r},\underline{r}';\gamma)\sigma_1(\underline{r}')dS'$$

$$\eta_3(\underline{r}) = \int_{\Sigma\times\Sigma} \Lambda_{12}(\underline{r},\underline{r}';\gamma)R_2(\underline{r}',\underline{r}'';\gamma)\sigma_1(\underline{r}'')dS'dS''.$$

First, we observe that the only singularities of the resolvent $R_2(\underline{r},\underline{r}';\gamma)$ are poles at those values of γ, $\gamma_n^{(2)}$, where there exists a nontrivial solution of the homogeneous integral equation

$$\sigma(\underline{r}) - \int_\Sigma \Lambda_{22}(\underline{r},\underline{r}';\gamma_n^{(2)})\sigma(\underline{r}')dS' = 0. \qquad \text{(C.11)}$$

Assuming that $\gamma_m^{(1)} \neq \gamma_n^{(2)}$ for all possible combinations of m and n it follows immediately by comparing the solutions of equations (C.4), (C.5), (C.9) and (C.10) that the only singularities in the complex γ-plane of the solution of (C.2) are poles.

For the special case where there exist an m and an n such that $\gamma_m^{(1)} = \gamma_n^{(2)}$, we return to the set of integral equations (C.2). Dividing the volume Ω into N_1 equal subvolumes and dividing the surface Σ into N_2 equal subsurfaces we can approximate (C.2) by the following set of equations

$$\phi_i - \sum_{j=1}^{N} \lambda_{ij}\phi_j\delta_j = \phi_{1i}. \qquad \text{(C.12)}$$

Here $N = N_1 + N_2$,

$$\phi_j = \begin{cases} \eta(\underline{r}_j), & 1 \le j \le N_1 \\ \sigma(\underline{r}_j), & N_1 + 1 \le j \le N \end{cases}$$

$$\delta_j = \begin{cases} \|\Omega\|/N_1, & 1 \le j \le N_1 \\ \|\Sigma\|/N_2, & N_1 + 1 \le j \le N \end{cases}$$

$\|\Omega\|$ is the volume of the region Ω and $\|\Sigma\|$ is the area of the surface Σ. The elements λ_{ij} are determined by the kernels Λ_{11}, Λ_{12}, Λ_{21}, Λ_{22}:

$$\lambda_{ij} = \begin{cases} \Lambda_{11}(\underline{r}_i, \underline{r}_j), & \underline{r}_i \in \Omega, & \underline{r}_j \in \Omega \\ \Lambda_{12}(\underline{r}_i, \underline{r}_j), & \underline{r}_i \in \Omega, & \underline{r}_j \in \Sigma \\ \Lambda_{21}(\underline{r}_i, \underline{r}_j), & \underline{r}_i \in \Sigma, & \underline{r}_j \in \Omega \\ \Lambda_{22}(\underline{r}_i, \underline{r}_j), & \underline{r}_i \in \Sigma, & \underline{r}_j \in \Sigma. \end{cases}$$

The solution of (C.2) can be obtained from (C.12) in the proper limit as $N_1 \to \infty$ and $N_2 \to \infty$. This way of obtaining the solution of (C.2) follows closely the Fredholm method of obtaining the solution of an "ordinary" second kind integral equation[18]. From the solution obtained in this way it is now easy to show that the solution of (C.2) is an analytic function of γ except at the poles γ_n where there exists a nontrivial solution of the equation[2]

$$(\underline{\underline{I}} - \underline{\underline{\Gamma}}^3) \cdot \underline{\phi} = 0.$$

This completes our proof that the only singularities are poles of the operator inverse to the integral operator $\underline{\underline{I}} - \underline{\underline{\Gamma}}^3$.

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