Interaction Notes

Note 111

TIME-HARMONIC ANALYSIS OF THE INDUCED CURRENT ON A THIN CYLINDER ABOVE A FINITELY CONDUCTING HALF-SPACE

by

JOHN N. BOMBARDT, JR.
Harry Diamond Laboratories
Washington, D.C. 20438

6 June 1972

TIME HARMONIC ANALYSIS OF THE INDUCED CURRENT ON A THIN CYLINDER ABOVE A FINITELY CONDUCTING HALF-SPACE

ABSTRACT

A thin cylinder above a finitely conducting half-space is illuminated by a monochromatic plane wave. The electric vector of the incident plane wave is parallel to the axis of the cylinder and the Poynting vector of this incident plane wave intersects the surface of the finitely conducting half-space at an oblique angle. Green's functions are employed in a formulation which requires the application of boundary conditions in two coordinate systems: a cylindrical coordinate system for the thin cylinder and a Cartesian coordinate system for the finitely conducting half-space. Formal expressions are derived for the total current on the thin cylinder and the electric field at the surface of the half-space under the condition that the height of the thin cylinder above the half-space is much greater than its radius. The height of the thin cylinder above the half-space, however, is unrestricted with respect to wavelength.

CONTENTS

1. INTRODUCTION

The intent of this memorandum is to analyze the scattering properties of a cylindrical conductor above a finitely conducting half-space. The boundary-value problem is simplified by treating a cylindrical conductor of infinite length and considering only parallel polarization such that the only component of the electric field is parallel to the axis of the cylindrical conductor. Related problems have, indeed, been addressed in the literature dating back to 1926[1]. Much of the earlier work analyzes the transmission or radiation characteristics of a wire above ground under the assumption that the wavelength is much longer than the height. This long-wavelength assumption is not valid for many wavelengths of interest in problems concerning the interaction of an electromagnetic pulse (EMP) with long, overhead transmission lines.

The objectives of the present study are: (a) to develop a useful frequency domain characterization of induced shield current on a long cable above a finitely conducting earth due to EMP or other RF excitation, and (b) to apply a Green's function approach to the idealized boundary-value problem in order to employ conventional assumptions made in electromagnetic scattering theory and linear antenna theory. The accomplishment of the former objective should provide a new tool for use in EMP vulnerability and hardening programs. The accomplishment of the latter objective should help bring the present study and earlier efforts into the proper perspective.

---

[1]Carson, J.R., "Wave Propagation in Overhead Wires with Ground Return," Bell Syst. Tech. Jour., 5, October 1926.

The format of this study is a didactic one in the sense that the actual problem of interest is addressed through a stepwise procedure. That is, several preliminary applications of Green's theorem are treated in order of increasing mathematical complexity. Each of these preliminary applications adds to the mathematical machinery available to analyze the plane wave scattering from a thin cylinder above a finitely conducting half-space. Ideally, this procedure should lead to a clear understanding of the mathematical approximations employed in this study and an intuitive feeling for the utility of the Green's function method in analyzing other related boundary-value problems involving the presence of a finitely conducting half-space.

Most modern textbooks on electromagnetic theory or mathematical physics treat the scalar form of Green's theorem and Green's functions. Only a brief description of fundamental definitions will be given here. Many electromagnetic boundary-value problems involve solving the scalar, homogeneous Helmholtz equation subject to arbitrary Dirichlet or Neumann boundary conditions on a closed boundary surface S:

$$\nabla^2\Psi(\bar{r}) + k^2\Psi(\bar{r}) = 0$$

where $\bar{r}$ is within or on S. The Green's function for such a boundary-value problem, $G(\bar{r}|\bar{r}')$, is the solution of the inhomogeneous Helmholtz equation for a unit point source at $\bar{r}'$, i.e.

$$\nabla^2 G(\bar{r}|\bar{r}') + k^2 G(\bar{r}|\bar{r}') = -4\pi\delta(\bar{r}-\bar{r}'),$$

which satisfies homogeneous Dirichlet or Neumann boundary conditions on the surface S. Green's functions will be used in the present study to obtain solutions of the homogeneous Helmholtz equation which are subject to inhomogeneous boundary conditions.

The reciprocity principle and Green's theorem lead to the usual relationship between $\Psi(\overline{r})$ and $G(\overline{r}|\overline{r}')$:

$$\Psi(r) = \frac{1}{4\pi} \oint \left[G(\overline{r}|\overline{r}') \frac{\partial \Psi(\overline{r}')}{\partial n} - \Psi(\overline{r}') \frac{G(\overline{r}|\overline{r}')}{\partial n} \right]_{\overline{r}' \text{on } S} dS$$

where $\overline{r}$ is within or on S and

$$\frac{\partial}{\partial n} = \hat{n} \cdot \overline{\nabla}'.$$

The unit vector $\hat{n}$ is normal to the surface S and is directed from the interior of S, i.e. $\hat{n}$ points away from the region where $\psi(\overline{r})$ is to be measured. Lastly, recall that the closed boundary surface, S, need not be simply connected but can be made up of more than one simply connected surface.

2. INDUCED CURRENT ON A CYLINDER IN FREE SPACE

A conducting circular cylinder of infinite length is exposed to an incident plane wave with the electric vector parallel to the axis of the cylinder as shown in figure 1. The total electric field, $E_z(\overline{r})$, is the sum of the incident electric field, $E_z^i(\overline{r})$, and the scattered electric field:

$$E_z(\overline{r}) = E_z^i(\overline{r}) + \frac{1}{4\pi} \oint \left[G_o(\overline{r}|\overline{r}') \frac{\partial E_z(\overline{r}')}{\partial n} - E_z(\overline{r}') \frac{\partial G_o(r|r')}{\partial n} \right]_{\overline{r}' \text{on } S} dS$$

where $G_o(\overline{r}|\overline{r}')$ is the two-dimensional free-space Green's function, i.e.

$$G_o(\overline{r}|\overline{r}') = i\pi H_o^{(1)}(k\ |\overline{r} - \overline{r}'|\ ), \qquad (1)$$

and

$$E_z^i(\overline{r}) = E_o \exp(-ikx) \qquad (2)$$

Time dependence of the form $\exp(-i\omega t)$ is suppressed throughout. The surface S includes the cylindrical surface at infinity and the coaxial surface of the perfectly conducting cylinder such that the region of interest is

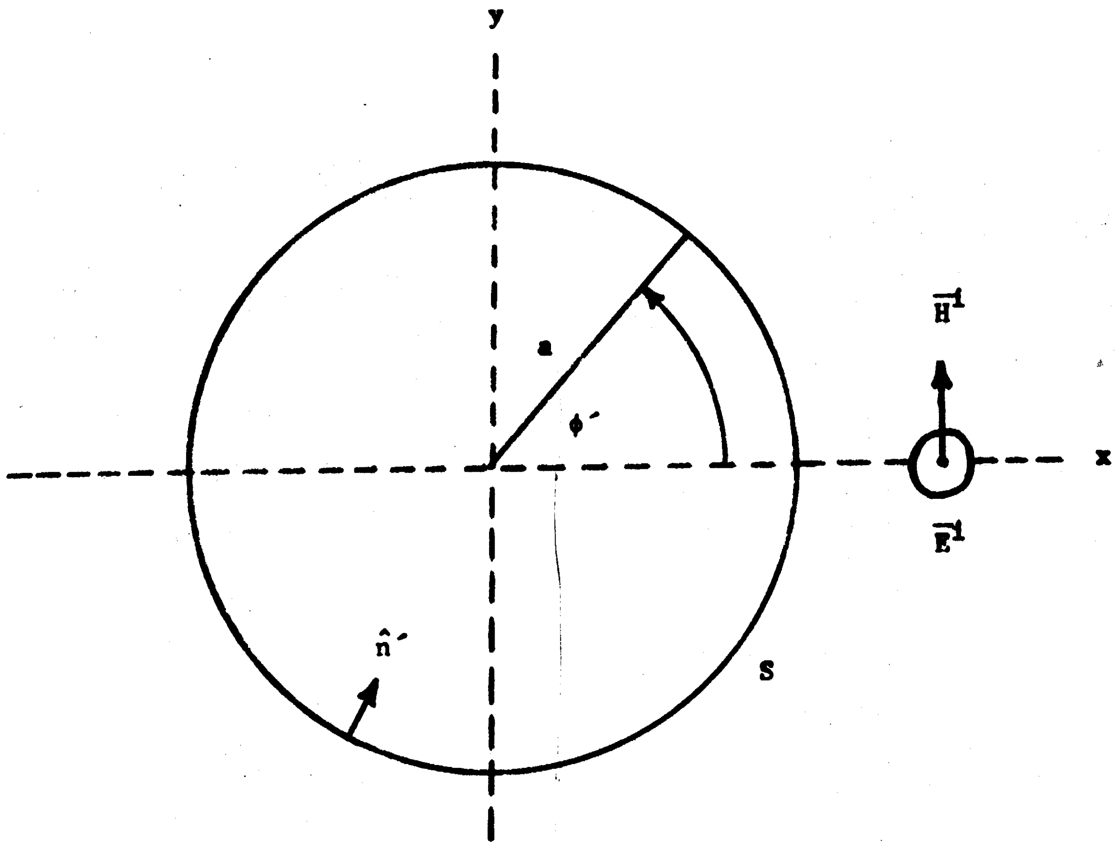

Figure 1. A circular cylinder illuminated by an incident plane wave.

between these coaxial surfaces. The integration over the cylindrical surface at infinity vanishes. Consequently, the total electric field at a point in space becomes

$$E_z(\rho,\phi) = E_o\exp(-ik\rho\cos\phi) + \frac{1}{4\pi}\int_0^{2\pi} [G_o(\rho,\phi|a,\phi') \frac{\partial E_z(\rho',\phi')}{\partial(-\rho')}\Big|_{\rho'=a}$$

$$- E_z(a,\phi') \frac{\partial G_o(\rho,\phi|\rho',\phi')}{\partial(-\rho')}\Big|_{\rho'=a}]a d\phi'$$

where

$$E_z(a,\phi') = 0$$

so that

$$E_z(\rho,\phi) = E_o\exp(-ik\rho\cos\phi) - \frac{1}{4\pi}\int_0^{2\pi} G_o(\rho,\phi|a,\phi') \frac{\partial E_z(\rho',\phi')}{\partial\rho'}\Big|_{\rho'=a} a d\phi'.$$

The normal derivative of the electric field at the cylinder's surface is proportional to the surface current in that

$$\frac{\partial E_z(\rho',\phi')}{\partial\rho'}\Bigg|_{\rho'=a} = - i\omega\mu_o H_\phi(a,\phi') = - i\omega\mu_o K_z(\phi') \quad (3)$$

where $K_z(\phi')$ is the surface current distribution. Therefore the total electric field at an arbitrary point in space is given by

$$E_z(\rho,\phi) = E_o\exp(-ik\rho\cos\phi) + \frac{i\omega\mu_o a}{4\pi}\int_0^{2\pi} G_o(\rho,\phi|a,\phi')\, K_z(\phi')\, d\phi'. \quad (4)$$

An integral equation for the surface current is obtained when the boundary condition on the electric field at the surface of the cylinder is imposed:

$$\frac{\omega\mu_o a}{4}\int_0^{2\pi} K_z(\phi')\, H_o^{(1)}\, [2ka|\sin((\phi-\phi')/2)|]\, d\phi' = E_o\exp(-ika\cos\phi) \quad (5)$$

where the kernel of the integral operator can be written as

$$H_0^{(1)}[2ka|\sin((\phi-\phi')/2)|] = \sum_{\ell=o}^{\infty} \varepsilon_\ell J_\ell(ka)H_\ell^{(1)}(ka)\cos\ell(\phi-\phi')$$

for

$$\varepsilon_\ell = \begin{cases} 1 & , \ell = 0 \\ 2 & , \ell = 1, 2, 3, \ldots \end{cases} \tag{6}$$

The integral equation can be solved in a straightforward manner by expanding the surface current in a Fourier series,

$$K_z(\phi') = \sum_{m=o}^{\infty} K_m \cos m\phi'. \tag{7}$$

Using the well-known expansion

$$\exp(-ika\cos\phi) = \sum_{m=o}^{\infty} \varepsilon_m(-i)^m J_m(ka)\cos m\phi,$$

substituting the series expansion for the kernel of the integral operator in equation (5), employing equation (7) in equation (5) and integrating leads to

$$\frac{\omega\mu_o a\pi}{2} \sum_{m=o}^{\infty} K_m J_m(ka)H_m^{(1)}(ka)\cos m\phi = E_o \sum_{m=o}^{\infty} \varepsilon_m(-i)^m J_m(ka)\cos m\phi.$$

Equating coefficients of the same eigenfunction in the last result yields

$$K_m = \frac{2}{\omega\mu_o a\pi} E_o \frac{\varepsilon_m(-i)^m}{H_m^{(1)}(ka)} \tag{8}$$

so that

$$K_z(\phi) = \frac{2}{\pi ka} \frac{E_o}{\sqrt{\mu_o/\varepsilon_o}} \sum_{m=o}^{\infty} \frac{\varepsilon_m(-i)^m}{H_m^{(1)}(ka)} \cos m\phi. \tag{9}$$

From equation (4), the scattered field from the cylinder, $E_z^s(\overline{r})$, is given by

$$E_z^s(\rho,\phi) = \frac{i\omega\mu_o a}{4\pi} \int_o^{2\pi} G_o(\rho,\phi|a,\phi') \, K_z(\phi') d\phi'. \qquad (10)$$

If the cylinder circumference is very small compared to the wavelength of the incident radiation, the surface current is uniform:

$$K_z(\phi') \longrightarrow \frac{2}{\omega\mu_o a\pi} \frac{E_o}{H_o^{(1)}(ka)} \qquad (11)$$

as $ka \to 0$. Consequently, the scattered electric field specified in equation (10) becomes

$$E_z^s(\rho,\phi) \longrightarrow - \frac{E_o}{2\pi H_o^{(1)}(ka)} \int_o^{2\pi} H_o^{(1)}[k(a^2 + \rho^2 - 2a\rho\cos(\phi-\phi'))^{1/2}]d\phi'$$

and, thus,

$$E_z^s(\rho,\phi) \longrightarrow - E_o \frac{J_o(ka)}{H_o^{(1)}(ka)} H_o^{(1)}(k\rho) \qquad (12)$$

as $ka \to 0$. An azimuthally uniform surface current implies an azimuthally uniform scattered electric field and vice versa. Although this is somewhat self-evident, it is an important factor in constructing self-consistant formulations for more complex boundary-value problems that involve narrow cylinders.

3. INDUCED CURRENT ON A CYLINDER ABOVE A PERFECTLY CONDUCTING PLANE

Two coordinate systems are displayed in Figure 2: a coordinate system associated with the reflecting half-space, (X,Y), and a coordinate system associated with the perfectly conducting cylinder, (x,y). The origin of the (X,Y) system is at the point of reflection of the incident plane wave in the absence of a cylinder. The origin of the (x,y) system is on the

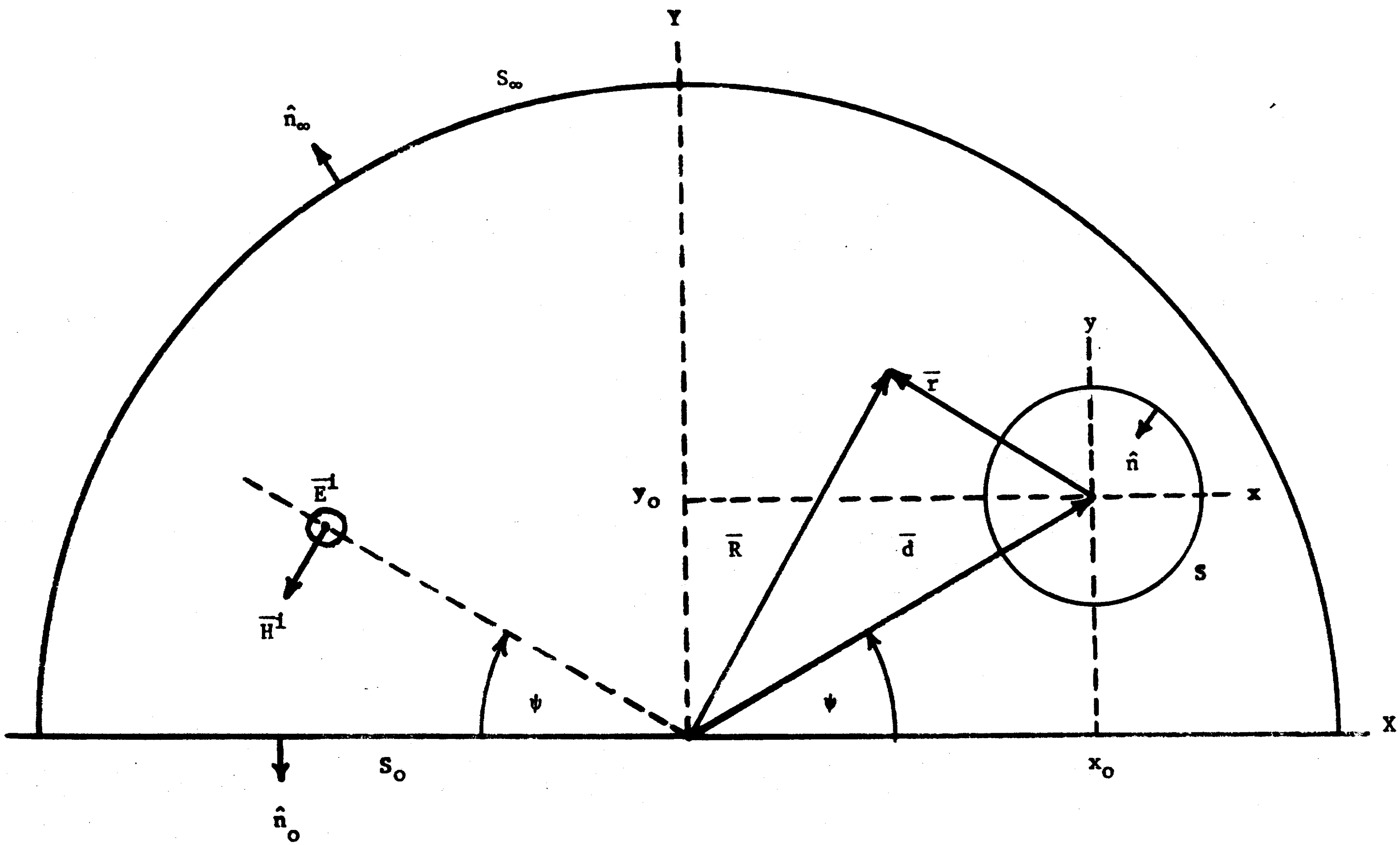

Figure 2. A circular cylinder above a reflecting half-space subjected to an incident plane wave.

axis of the cylinder and located at

$$x = 0 \Rightarrow X = x_o = y_o / \tan\psi$$

and

$$y = 0 \Rightarrow Y = y_o$$

where ψ is the angle of incidence of the incident plane wave with respect to the plane at $Y = 0$. Here, the plane surface at $Y = 0$ is taken to be perfectly conducting. Thus, the region of interest lies between the closed surface S, the surface of the perfectly conducting cylinder, and the closed surface made up of S_∞, the semi-cylindrical surface at infinity with its axis at $X = 0$ and $Y = 0$, and S_o, the plane surface at $Y = 0$.

The total electric field in the absence of the perfectly conducting cylinder, E_z^o, is given by

$$E_z^o(\bar{R}) = E_o[\exp(ik(X\cos\psi - Y\sin\psi)) - \exp(ik(X\cos\psi + Y\sin\psi))] \quad (13)$$

in the (X, Y) system or

$$E_z^o(\bar{r}) = \mathcal{E}_o\{\exp(+ik(x\cos\psi - y\sin\psi)) - \exp(+ik[x\cos\psi + (y+2y_o)\sin\psi])\} \quad (14)$$

in the (x,y) system where

$$\mathcal{E}_o = E_o \exp(+ik(x_o\cos\psi - y_o\sin\psi)) \equiv \text{the incident plane wave evaluated at } (x_o, y_o). \quad (15)$$

Green's function for a perfectly conducting plane can also be expressed in either the (X, Y) system or the (x,y) system: that is,

$$G(\bar{R}|\bar{R}') = i\pi \{ H_o^{(1)}[k((X - X')^2 + (Y - Y')^2)^{1/2}] - H_o^{(1)}[k((X - X')^2 + (Y + Y')^2)^{1/2}]\} \quad (16)$$

or

$$G(\bar{r}|\bar{r}') = i\pi \{ H_0^{(1)}[k((x - x')^2 + (y - y')^2)^{1/2}] \quad (17)$$

$$- H_0^{(1)}[k((x - x')^2 + (y + y' + 2y_0)^2)^{1/2}] \}.$$

Since the plane at $Y = 0$ is taken to be perfectly conducting, the boundary-value problem can be formulated entirely in the (x,y) system using equations (14), (15), and (17).

The total electric field can be represented as the sum of the field in the absence of a perfectly conducting cylinder and the field due to the presence of the cylinder over the perfectly conducting plane. The latter field is related to Green's function for a perfectly conducting plane via the appropriate surface integrals over S_∞, S_0, and S. The surface integral over S_∞ vanishes because of the behavior of the Green's function and electromagnetic fields at infinity. The surface integral over S_0 vanishes because both Green's function and the total electric field vanish identically at $Y = 0$. Therefore, the total electric field at an arbitrary field point becomes

$$E_z(\bar{r}) = E_z^0(\bar{r}) + \frac{1}{4\pi} \oint \left[G(\bar{r}|\bar{r}') \frac{\partial E_z(\bar{r}')}{\partial n} - E_z(\bar{r}') \frac{\partial G(\bar{r}|\bar{r}')}{\partial n}\right]_{\bar{r}' \text{on } S} dS$$

where

$$E_z(\bar{r}' \text{ on } S) = 0$$

so that

$$E_z(\bar{r}) = E_z^0(\bar{r}) + \frac{1}{4\pi} \oint \left[G(\bar{r}|\bar{r}') \frac{\partial E_z(\bar{r}')}{\partial n}\right]_{\bar{r}' \text{ on } S} dS. \quad (18)$$

This result can be expressed in cylindrical coordinates as

$$E_z(\rho,\phi) = E_z^0(\rho,\phi) + \frac{ika}{4\pi} \sqrt{\frac{\mu_0}{\varepsilon_0}} \int_0^{2\pi} G(\rho,\phi|\rho' = a,\phi')K_z(\phi')d\phi' \quad (19)$$

where $K_z(\phi')$ is the surface current defined in equation (3).

The surface current is rigorously defined by imposing the boundary condition at $\rho = a$, i.e.

$$E_z(a,\phi) = 0.$$

The integral equation for the surface current is, then,

$$\frac{ika}{4\pi}\sqrt{\frac{\mu_o}{\varepsilon_o}}\int_0^{2\pi} G(a,\phi|a,\phi')K_z(\phi')d\phi' = -E_z^o(a,\phi) \tag{20}$$

where

$$G(a,\phi|a,\phi') = i\pi\ \{\ H_o^{(1)}[ka((\cos\phi - \cos\phi')^2 + (\sin\phi - \sin\phi')^2)^{1/2}] - H_o^{(1)}[k(a^2(\cos\phi - \cos\phi')^2 + (a\sin\phi + a\sin\phi' + 2y_o)^2)^{1/2}]\ \} \tag{21}$$

and

$$E_z^o(\rho,\phi) = E_o[\exp(+ik\rho\cos(\phi+\psi)) - \exp(+2iky_o\sin\psi)\ \exp(ik\rho\cos(\phi-\psi))] \tag{22}$$

An exact analytical solution of equation (20) for the surface current does not appear to be feasible. However, the related problem concerning scattering by two parallel circular cylinders has been treated by numerical methods and has been reported in the literature[2,3].

An analytically tractable situation occurs when the height above ground, y_o, is much greater than the cylinder radius, a, even though y_o and a are not restricted with respect to wavelength. The kernel of the integral operator in equation (20) simplifies for $y_o \gg a$:

$$G(a,\phi|a,\phi') \xrightarrow[y_o \gg a]{} i\pi\ \{\ H_o^{(1)}[2ka|\sin((\phi-\phi')/2)|] - H_o^{(1)}(2ky_o)\ \}$$

---

[2] Row, R. V., "Theoretical and Experimental Study of Electromagnetic Scattering by Two Identical Conducting Cylinders," Journal of Applied Physics, Vol. 26, Number 6, June, 1955.

[3] Olaofe, G. O., "Scattering by Two Cylinders," Radio Science, Volume 5, Number 11, November, 1970

and equation (20) becomes, for $Z_o = \sqrt{\mu_o/\varepsilon_o}$,

$$- \frac{ka}{4} Z_o \int_0^{2\pi} \{ H_0^{(1)}[2ka|\sin((\phi-\phi')/2)|] - H_0^{(1)}(2ky_o)\}K_z(\phi')d\phi' = - E_z^o(a,\phi) \tag{23}$$

which can be solved in the same way as equation (5) using Galerkin's method. The surface current can be expanded in a Fourier series as

$$K_z(\phi) = \frac{I}{2\pi a} + \sum_{n=1}^{\infty} (A_n \cos n\phi + B_n \sin n\phi) \tag{24}$$

where the total current is, simply,

$$a \int_0^{2\pi} K_z(\phi')d\phi' = I. \tag{25}$$

Expanding the kernel of the integral operator in equation (23) and substituting the Fourier series expansion of the surface current leads to

$$\frac{ka}{4} Z_o \int_0^{2\pi} \{[H_0^{(1)}(2ky_o) - J_o(ka)H_0^{(1)}(ka)]$$

$$- 2[\sum_{m=1}^{\infty} J_m(ka)H_m^{(1)}(ka)\cos m\phi \cos m\phi'$$

$$+ \sum_{m=1}^{\infty} J_m(ka)H_m^{(1)}(ka)\sin m\phi \sin m\phi'] \}[I/2\pi a$$

$$+ \sum_{n=1}^{\infty} (A_n \cos n\phi' + B_n \sin n\phi')]d\phi' = - E_z^o(a,\phi)$$

where

$$E_z^o(a,\phi) = E_o (1 - \exp(+2iky_o \sin\psi))$$

$$+ 2 E_o(1 - \exp(+2iky_o \sin\psi)) \sum_{m=1}^{\infty} i^m J_m(ka)\cos m\phi \cos m\psi$$

$$- 2 E_o(1 + \exp(+2iky_o \sin\psi)) \sum_{m=1}^{\infty} i^m J_m(ka)\sin m\phi \sin m\psi.$$

Performing the indicated integrations and, subsequently, equating coefficients of the same eigenfunction yields

$$I = \frac{4E_o}{kZ_o} \frac{(1 - \exp(+2iky_o\sin\psi))}{[J_o(ka)H_o^{(1)}(ka) - H_o^{(1)}(2ky_o)]} \tag{26}$$

$$A_m = \frac{4E_o}{kZ_oa\pi} (1 - \exp(+2iky_o\sin\psi)) \frac{i^m\cos m\psi}{H_m^{(1)}(ka)} , \tag{27}$$

$$B_m = \frac{-4E_o}{kZ_oa\pi} (1 + \exp(+2iky_o\sin\psi)) \frac{i^m\sin m\psi}{H_m^{(1)}(ka)} \tag{28}$$

so that

$$K_z(\phi) = \frac{2}{kZ_oa\pi} \frac{[E_o(1 - \exp(+2iky_o\sin\psi))]}{[J_o(ka)H_o^{(1)}(ka) - H_o^{(1)}(2ky_o)]} + \frac{4E_o}{kZ_oa\pi} \sum_{m=1}^{\infty} \frac{i^m\cos m(\phi+\psi)}{H_m^{(1)}(ka)} - \frac{4E_o}{kZ_oa\pi} \exp(+2iky_o\sin\psi) \sum_{m=1}^{\infty} \frac{i^m\cos m(\phi-\psi)}{H_m^{(1)}(ka)} . \tag{29}$$

Conventional transmission line theory invokes the assumptions that a and y_o are both small compared to the wavelength in addition to the restriction that $y_o >> a$. Employing the small argument expansions for the Bessel and Hankel functions that appear in equation (26), the total current becomes

$$I = \frac{2}{k} \frac{E_z^i\sin(ky_o\sin\psi)}{Z_c} \tag{30}$$

where

$$E_z^i = 2E_o \exp(+ikx_o\cos\psi) \tag{31}$$

and

$$Z_c = \frac{Z_o}{\pi} \ln\left(\frac{2y_o}{a}\right) . \tag{32}$$

The current in equation (30) is the short-circuit current for a two-wire transmission line excited by an incident plane wave with an electric vector parallel to the line[4]. The Poynting vector of this incident plane wave makes an angle ψ with the normal to the plane of the transmission line. The amplitude of the incident electric field along the axis of the two-wire line is E_z^i and the characteristic impedance of the two-wire line is Z_c. Apparently, then, the Green's function formalism reduces properly to well-known transmission line theory in the limit of long wavelengths.

4. TOTAL FIELD ABOVE A FINITELY CONDUCTING HALF-SPACE

Suppose the cylinder, denoted by the surface S, is not present in figure 2 and that the half-space for $Y \leq 0$ is a conducting dielectric with electrical parameters ε_g, σ_g, and μ_o. The total electric field above the conducting dielectric is given by

$$E_z(\bar{R}) = E_z^o(\bar{R}) - \frac{1}{4\pi} \int \left[E_z(\bar{R}') \frac{\partial G(\bar{R}|\bar{R}')}{\partial n_o} \right]_{\bar{R}' \text{ on } S_o} dS_o$$

since Green's function vanishes when $\bar{R}'$ is on S_o. In the (X,Y) coordinate system this total field becomes

$$E_z(X, Y) = E_z^o(X, Y) + \frac{1}{4\pi} \int_{-\infty}^{+\infty} E_z(X', 0) \left. \frac{G(X,Y|X',Y')}{\partial Y'} \right|_{Y'=0} dX'. \qquad (33)$$

The total field above the finitely conducting half-space is represented in equation (33) as the sum of the total field above a perfectly conducting

---

[4]Harrison, C. W., Jr., "Receiving Characteristics of Two-Wire Lines Excited by Uniform and Non-Uniform Electric Fields." Interaction Note 15, Sandia Corporation Monograph SC-R-64-164, Albuquerque, New Mexico, May 1964.

plane and a term due to the presence of a non-zero field on the surface of the conducting dielectric. In the absence of a scatterer above the conducting dielectric, the field distribution on the surface Y = 0 is well-known and provides an example problem for checking mathematical manipulations and procedures.

The zeroth order Hankel function, $H_o^{(1)}[k(u^2+v^2)^{1/2}]$, can be represented by an integral along the real axis[5]:

$$H_o^{(1)}[k(u^2+v^2)^{1/2}] = \frac{1}{\pi} \int_{-\infty}^{+\infty} \exp(+i\xi u) \frac{\exp(+iv(k^2-\xi^2)^{1/2})}{(k^2-\xi^2)^{1/2}} d\xi \qquad (34)$$

where k can be complex and $v \geq 0$. Using this identity, the derivative of the Green's function appearing in equation (33) can be written as

$$\left.\frac{\partial G(X,Y|X',Y')}{\partial Y'}\right|_{Y'=0} = 2 \int_{-\infty}^{+\infty} \exp(+i\xi(X - X') + iY(k^2 - \xi^2)^{1/2}) d\xi$$

so that

$$E_z (X,Y) = E_z^o (X,Y) \qquad (35)$$
$$+ \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp (+i\xi X + iY(k^2-\xi^2)^{1/2}) [\int_{-\infty}^{+\infty} \exp(-i\xi X')E_z(X',0)dX']d\xi.$$

At the surface of the conducting dielectric, this result becomes

$$E_z (X,0) = \int_{-\infty}^{+\infty} E_z(X',0) \left[\frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp(+i\xi(X - X'))d\xi\right] dX'$$

where

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp(+i\xi(X - X'))d\xi = \delta (X - X')$$

so that equation (35) degenerates to an identity at Y = 0.

---

[5]Head, J. H., "The Effects of the Air-Earth Interface on the Propagation Constants of a Buried Insulated Conductor," Interaction Note 50, Kaman Nuclear Report KN-785-70-6(R), Colorado Springs, Colorado, 19 February 1970, p.17.

The fields in and above a finitely conducting half-space due to an incident plane wave are well known for any polarization of the incident electric field. The electric field at the surface $Y = 0$, for the situation of interest, is given by

$$E_z(X,0) = E_o(1 + R_h)\ \exp(+ikX\cos\psi) \tag{36}$$

where R_h is the reflection coefficient for a horizontally polarized electric field. The horizontal reflection coefficient is given by

$$R_h = \frac{\sin\psi - [(\varepsilon_r+i\sigma_g/\omega\varepsilon_o) - \cos^2\psi]^{1/2}}{\sin\psi + [(\varepsilon_r+i\sigma_g/\omega\varepsilon_o) - \cos^2\psi]^{1/2}} \tag{37}$$

where

$$\varepsilon_r = \varepsilon_g/\varepsilon_o. \tag{38}$$

Substituting the surface field specified by equation (36) into equation (35) gives

$$E_z(X,Y) = E_z^o(X,Y)$$
$$+ E_o(1+R_h) \int_{-\infty}^{+\infty} \exp(+i\xi X+iY(k^2-\xi^2)^{1/2}) \left[\frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp(+iX'(k\cos\psi-\xi))\,dX'\right] d\xi$$

wherein the bracketed integration over X' is a Dirac delta function so that

$$E_z(X,Y) = E_o[\exp(+ik(X\cos\psi-Y\sin\psi))- \exp(+ik(X\cos\psi+Y\sin\psi))]$$
$$+ E_o(1+R_h)\ \exp(+ikX\cos\psi+iY(k^2-k^2\cos^2\psi)^{1/2})$$

or

$$E_z(X,Y) = E_o[\exp(+ik(X\cos\psi-Y\sin\psi))+ R_h\ \exp(+ik(X\cos\psi+Y\sin\psi))]. \tag{39}$$

Equation (39) gives the expected result for the total electric field above a finitely conducting half-space. The functional form of the surface field specified in equation (36) permitted the exact analytical evaluation of the integrals appearing in equation (35). When a scatterer is present above the finitely conducting half-space, however, the functional form of the electric field at $Y = 0$ will not, in general, permit an exact analytical evaluation of those integrals.

5. TRANSMITTED FIELD IN A FINITELY CONDUCTING HALF-SPACE

A refracted wave with a complex angle of refraction, α, is shown in figure 3 in conjunction with incident and reflected waves. A scattering obstacle is not present in the region above the finitely conducting half-space such that the electric field for $Y \geq 0$ has already been derived using Green's function and is given by equation (39). The present objective is to derive the field in the conducting dielectric using the field at $Y = Y_g = 0$ and Green's function.

A new coordinate frame of reference is defined in figure 3 such that a field point in the finitely conducting half-space, $\bar{R}_g$, can be specified in the (X, Y_g) coordinate system for convenience. The transmitted electric field, $E_z^g(\bar{R}_g)$, in the conducting dielectric is given by

$$E_z^g(\bar{R}_g) = \frac{-1}{4\pi} \int \left[E_z^g(\bar{R}_g') \frac{\partial G_g(\bar{R}_g|\bar{R}_g')}{\partial n_g} \right]_{\bar{R}_g' \text{ on } S_o} dS_o$$

where

$$G_g(X, Y_g|X', Y_g') = i\pi\{H_o^{(1)} [k_g \sqrt{(X - X')^2 + (Y_g - Y_g')^2}] - H_o^{(1)} [k_g \sqrt{(X - X')^2 + (Y_g + Y_g')^2}]\} \quad (40)$$

and

$$k_g^2 = \mu_o \varepsilon_g \omega^2 + i\mu_o \sigma_g \omega \ . \quad (41)$$

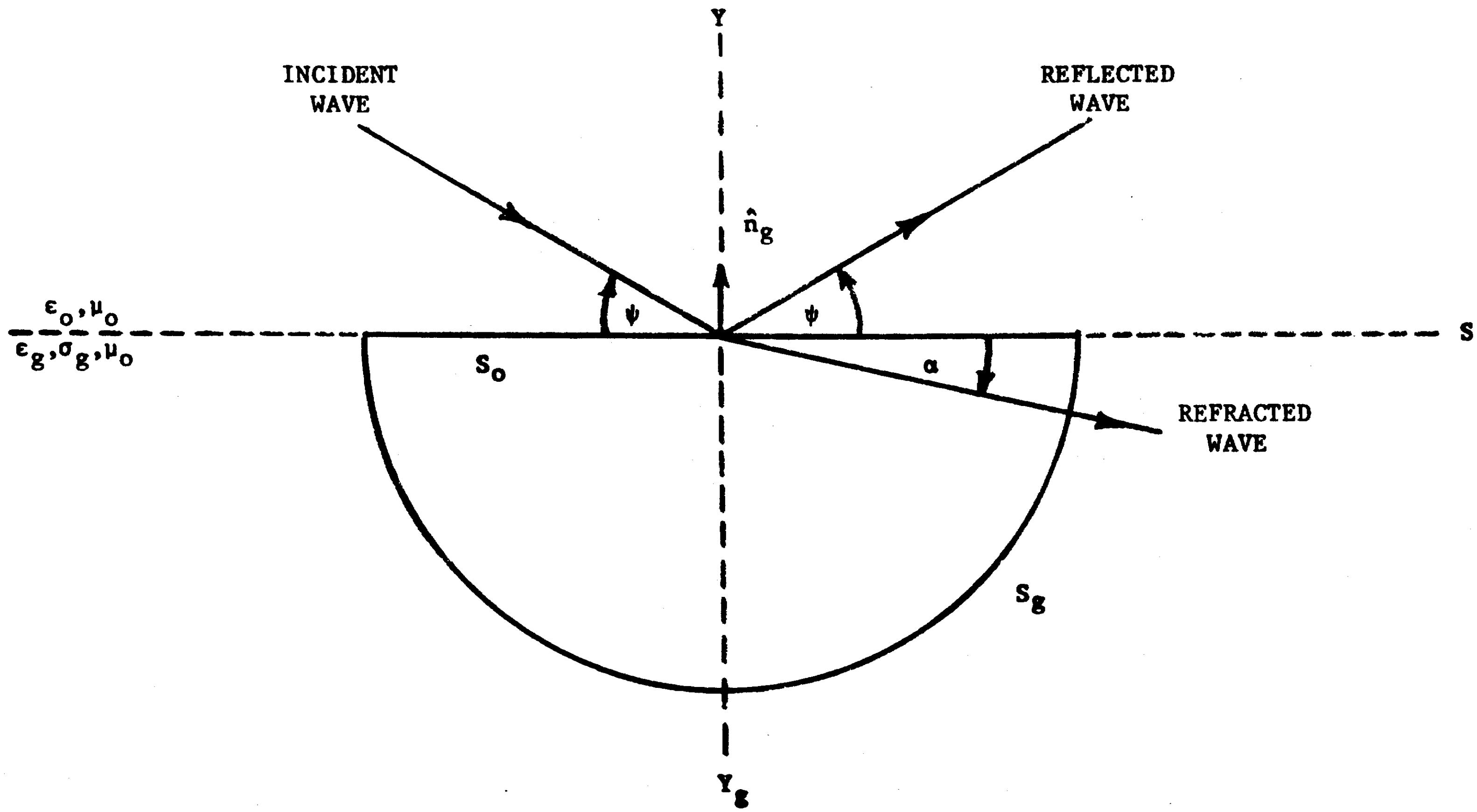

Figure 3. A plane wave incident on a finitely conducting half-space.

In the (X, Y_g) coordinate system, the transmitted field thus becomes

$$E_z^g(X,Y_g) = \frac{1}{4\pi} \int_{-\infty}^{+\infty} E_z^g(X',0) \left. \frac{\partial G_g(X,Y_g|X',Y_g')}{\partial Y_g'} \right|_{Y_g'=0} dX'. \tag{42}$$

The (X,Y_g) coordinate system has been introduced in order to represent the derivative of Green's function in equation (42) as

$$\left. \frac{\partial G_g(X,Y_g|X',Y_g')}{\partial Y_g'} \right|_{Y_g'=0} = 2 \int_{-\infty}^{+\infty} \exp(+i\xi(X-X') + iY_g(k_g^2-\xi^2)^{1/2})\, d\xi$$

so that

$$E_z^g(X,Y_g) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp(+i\xi X+iY_g(k_g^2-\xi^2)^{1/2}) [\int_{-\infty}^{+\infty} \exp(-i\xi X')E_z^g(X',0)\,dX']\,d\xi. \tag{43}$$

Noting that

$$E_z^g(X,0) = E_z(X,0)$$

and using the surface field specified by equation (36) in equation (43) leads to

$$E_z^g(X,Y_g) = E_o(1+R_h) \int_{-\infty}^{+\infty} \exp(+i\xi X+iY_g(k_g^2-\xi^2)^{1/2})\, \delta\, (\xi-k\cos\psi)\,d\xi$$

where the replacement of Y_g by $-Y$ gives

$$E_z^g(X,Y) = E_o(1+R_h)\exp(+i[kX\cos\psi - Y(k_g^2-k^2\cos^2\psi)^{1/2}]).$$

Using Snell's law of refraction,

$$k\cos\psi = k_g\cos\alpha \;, \tag{44}$$

and introducing the transmission coefficient,

$$T_h = 1 + R_h \;, \tag{45}$$

yields the final result for the transmitted field:

$$E_z^g(X,Y) = E_oT_h\exp(+ik_g(X\cos\alpha - Y\sin\alpha)). \tag{46}$$

Equations (46) and (39) are, of course, the classical results for the scattering of a plane wave by a finitely conducting half-space. The mathematical techniques and procedures developed up to this point can now be applied to the boundary-value problem of interest.

6. INDUCED CURRENT ON A THIN CYLINDER ABOVE A FINITELY CONDUCTING HALF-SPACE

6.1 Formulation

The total electric field at a point above the half-space can be represented as a sum of three partial fields: (i) the total field above a perfectly conducting plane in the absence of a scatterer, (ii) a contribution from the presence of a non-zero surface field at Y = 0, and (iii) a field due to the induced current on the scatterer. That is, the total field above the half-space can be expressed as

$$E_z(\bar{r},\bar{R}) = E_z^o(\bar{r} \text{ or } \bar{R}) - \frac{1}{4\pi} \int \left[E_z(\bar{R}') \frac{\partial G(\bar{R}|\bar{R}')}{\partial n_o} \right]_{\bar{R}' \text{ on } S_o} dS_o$$

$$+ \frac{1}{4\pi} \oint \left[G(\bar{r}|\bar{r}') \frac{\partial E_z(r')}{\partial n} \right]_{\bar{r}' \text{on } S} dS$$

where the cylinder coordinates, $\bar{r}$ and $\bar{r}'$, have been mixed with the half-space coordinates, $\bar{R}$ and $\bar{R}'$, for convenience. Thus, this total field becomes

$$E_z(\bar{r},\bar{R}) = E_z^o(\bar{r} \text{ or } \bar{R}) + \frac{1}{4\pi} \int_{-\infty}^{+\infty} E_z(X';0) \left. \frac{\partial G(X,Y|X';Y')}{\partial Y'} \right|_{Y'=0} dX' \qquad (47)$$

$$+ \frac{ika}{4\pi} Z_o \int_0^{2\pi} G(\rho,\phi|\rho' = a,\phi')K_z(\phi')d\phi'$$

where, as before, $K_z(\phi')$ is the surface current distribution on the cylinder.

The transmitted field at a point in the finitely conducting half-space can be represented solely in terms of the field at $Y = 0$. This transmitted field is given by

$$E_z^g(\overline{R}_g) = -\frac{1}{4\pi} \int \left[E_z^g(\overline{R}_g') \frac{\partial G_g(\overline{R}_g|\overline{R}_g')}{\partial n_g} \right]_{\overline{R}_g' \text{ on } S_o} dS_o$$

and, as before,

$$E_z^g(X',0) = E_z(X',0)$$

so that

$$E_z^g(X,Y_g) = \frac{1}{4\pi} \int_{-\infty}^{+\infty} E_z(X',0) \left. \frac{\partial G_g(X,Y_g|X',Y_g')}{\partial Y_g'} \right|_{Y_g'=0} dX' \qquad (48)$$

The surface field at $Y = 0$ and the surface current on the cylinder must be determined through the appropriate boundary conditions. The total electric field must vanish at the surface of the cylinder,

$$E_z(\overline{r},\overline{R}) = 0 \text{ for } \overline{r},\ \overline{R} \text{ on } S, \qquad (49)$$

and the tangential component of the magnetic field must be continuous at $Y = Y_g = 0$,

$$\left. \frac{\partial E_z(\overline{r},\overline{R})}{\partial Y} \right|_{Y=0} = - \left. \frac{\partial E_z^g(\overline{R}_g)}{\partial Y_g} \right|_{Y_g=0} . \qquad (50)$$

Equations (49) and (50) lead to a set of coupled integral equations for $E_z(X',0)$ and $K_z(\phi')$ which, in general, are not subject to exact solutions. In that which follows, approximate solutions are obtained for $E_z(X',0)$ and $K_z(\phi')$ under the conditions that the circumference of the cylinder is much smaller than a wavelength and that the radius of the cylinder is somewhat smaller than its height above the half-space.

6.2 Fields Above and Within the Half-Space

The surface field at Y = 0 is assumed to be of the following form:

$$E_z(X',0) = E_o(1+R_h)\exp(+ikX'\cos\psi) + C\,H_0^{(1)}[k((X'-x_o)^2 + y_o^2)^{1/2}]. \qquad (51)$$

The exponential term in equation (51) is the surface field that would be present in the absence of a scatterer above the half-space, equation (36). The Hankel function in the above equation is the contribution to the surface field due to the presence of the cylinder at ($X = x_o$, $Y = y_o$) and C must be determined via boundary conditions. As will become evident shortly, the Hankel function in equation (51) gives rise to a modification of the image field (due to a cylinder at ($X = x_o$, $Y = -y_o$)) which would be present if the plane at Y = 0 were perfectly conducting.

The partial field above the half-space that originates from a non-zero surface field is denoted as

$$E_z'(X,Y) = \frac{1}{4\pi}\int_{-\infty}^{+\infty} E_z(X',0)\left.\frac{\partial G(X,Y|X',Y')}{\partial Y'}\right|_{Y'=0} dX' \qquad (52)$$

so that

$$E_z'(X,Y) = E_o(1+R_h)\exp(+ik(X\cos\psi + Y\sin\psi))$$

$$+\frac{C}{2\pi}\int_{-\infty}^{+\infty}\exp(+i\xi X + iY(k^2-\xi^2)^{1/2})\,I(\xi)\,d\xi$$

where

$$I(\xi) = \int_{-\infty}^{+\infty}\exp(-i\xi X')\,H_0^{(1)}[k((X'-x_o)^2 + y_o^2)^{1/2}]\,dX'. \qquad (53)$$

The integral $I(\xi)$ can be evaluated using the integral representation of the Hankel function given in equation (34):

$$I(\xi) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{\exp(-i\zeta x_o + iy_o(k^2-\zeta^2)^{1/2})}{(k^2-\zeta^2)^{1/2}} \left[\int_{-\infty}^{+\infty} \exp(+i(\zeta-\xi)X')dX' \right] d\zeta$$

so that recognizing the integral over X´ as a Dirac delta function yields

$$I(\xi) = 2 \frac{\exp(-i\xi x_o + iy_o(k^2-\xi^2)^{1/2})}{(k^2-\xi^2)^{1/2}} . \tag{54}$$

Consequently,

$$E_z'(X,Y) = E_o(1+R_h) \exp(+ik(X\cos\psi + Y\sin\psi)) + C\, H_o^{(1)}[k((X-x_o)^2 + (Y+y_o)^2)^{1/2}] \tag{55}$$

wherein $X = x + x_o$ and $Y = y + y_o$ so that using $x = \rho\cos\phi$ and $y = \rho\sin\phi$ leads to

$$E_z'(\rho,\phi) = E_o(1+R_h) \exp(+2iky_o\sin\psi + ik\rho\cos(\phi-\psi)) + C\, H_o^{(1)} \{ k[\rho^2+(2y_o)^2-2\rho(2y_o)\cos(\phi+\pi/2)]^{1/2}\} . \tag{56}$$

The partial field that originates from the surface current on the cylinder is defined as

$$E_z'' (\rho,\phi) = \frac{ika}{4\pi} Z_o \int_0^{2\pi} G(\rho,\phi|\rho'=a,\phi')\, K_z(\phi')d\phi' \tag{57}$$

which reduces to a more manageable form for the case of a "thin antenna." That is, when the wavelength is long compared to the cylinder circumference, ka<<1, the surface current flows essentially along the axis of the cylinder such that

$$G(\rho,\phi|\rho' = a,\phi') \xrightarrow[ka<<1]{} G(\rho,\phi|0,\phi') . \tag{58}$$

Therefore, equation (57) becomes

$$E_z'' (\rho,\phi) = -\frac{k}{4} Z_o \left(H_o^{(1)}(k\rho) - H_o^{(1)} \{k[\rho^2 + (2y_o)^2 - 2\rho(2y_o)\cos(\phi+\pi/2)]^{1/2}\} \right) I \tag{59}$$

where, as in equation (25),

$$a \int_0^{2\pi} K_z(\phi')d\phi' = I.$$

The total current, I, must also be determined through boundary conditions.

Equation (59) can also be written as

$$E_z''(X,Y) = -\frac{k}{4} Z_o(H_0^{(1)}\{k[(X - x_o)^2 + (Y - y_o)^2]^{1/2}\} \tag{60}$$

$$- H_0^{(1)}\{k[(X - x_o)^2 + (Y + y_o)^2]^{1/2}\})I.$$

Collecting results, equations (13), (55) and (60) can be used to express the total field above the half-space in the (X,Y) coordinate system:

$$E_z(X,Y) = E_z^o(X,Y) + E_z'(X,Y) + E_z''(X,Y) .$$

Similarly, equations (22), (56) and (59) can be used to express the total field above the half-space in the (ρ,ϕ) coordinate system:

$$E_z(\rho,\phi) = E_z^o(\rho,\phi) + E_z'(\rho,\phi) + E_z''(\rho,\phi) .$$

The total field above the half-space has been expressed in both coordinate systems in order to facilitate the application of boundary conditions.

Turning attention, now, to the transmitted field in the finitely conducting half-space, equation (51) can be used with equation (48) in order to obtain

$$E_z^g(X,Y_g) = E_o(1+R_h) \exp(+ikX\cos\psi + iY_g(k_g^2-k^2\cos^2\psi)^{1/2}) \tag{61}$$

$$+ \frac{C}{\pi} \int_{-\infty}^{+\infty} \frac{\exp(+i\xi(X-x_o)+iY_g(k_g^2-\xi^2)^{1/2}+iy_o(k^2-\xi^2)^{1/2})}{(k^2 - \xi^2)^{1/2}} d\xi.$$

Note that Snell's law of refraction has not been invoked because of the presence of the scatterer above the half-space. Furthermore, equation (61) reduces properly to equation (51) when $Y_g = 0$.

The electric field above and within the finitely conducting half-space has been expressed in terms of the unknown quantities C and I. These unknown quantities can now be evaluated by imposing the appropriate boundary conditions.

6.3. Boundary Conditions

The boundary condition for the tangential electric field at the surface of the cylinder can be readily imposed using equations (22), (56) and (59);

$$E_z(a,\phi) = E_o \Big[\sum_{m=o}^{\infty} i^m J_m(ka) \cos m(\phi+\psi)$$

$$+ R_h \exp(+2iky_o \sin\psi) \sum_{m=o}^{\infty} i^m J_m(ka) \cos m(\phi-\psi) \Big]$$

$$+ C \sum_{m=o}^{\infty} \varepsilon_m J_m(ka) H_m^{(1)}(2ky_o) \cos m(\phi+\pi/2)$$

$$- I (k/4) Z_o \Big[H_o^{(1)}(ka) - \sum_{m=o}^{\infty} \varepsilon_m J_m(ka) H_m^{(1)}(2ky_o) \cos m(\phi+\pi/2)\Big]$$

$$\equiv 0$$

wherein

$$\exp(+ikq\cos\alpha) = \sum_{m=o}^{\infty} i^m J_m(kq) \cos m\alpha$$

and

$$H_o^{(1)}[k(q^2+p^2-2pq\cos\alpha)^{1/2}] = \sum_{m=o}^{\infty} \varepsilon_m J_m(kq) H_m^{(1)}(kp) \cos m\alpha$$

for $p \geq q$. Integrating over ϕ from 0 to 2π yields

$$J_o(ka) E_z^t(x_o,y_o) + C\, J_o(ka)\, H_o^{(1)}(2ky_o)$$

$$- I\, (k/4)\, Z_o\big[\, H_o^{(1)}(ka) - J_o(ka)\, H_o^{(1)}(2ky_o)\big] = 0$$

for

$$E_z^t(x_o,y_o) = E_o(1 + R_h \exp(+2iky_o\sin\psi)) = \text{total field at}$$

$$(x_o,y_o) \text{ in the absence of a scatterer.} \qquad (62)$$

The "thin antenna" assumption implies

$$J_o(ka) \to 1$$

and

$$H_o^{(1)}(ka) \to 1 + \frac{2i}{\pi} [\gamma + \ln (ka/2)]$$

for ka<<1 where $\gamma = 0.5772$. Employing only the former small argument approximation for the time being, the boundary condition at the surface of the cylinder can be compactly stated as

$$C\, H_o^{(1)}(2ky_o) - I(k/4)\, Z_o[H_o^{(1)}(ka) - H_o^{(1)}(2ky_o)] = - E_z^t(x_o,y_o). \qquad (63)$$

The boundary condition for the tangential magnetic field at $Y = 0$ can be imposed by generating the following derivatives using equations (13), (55), (60) and (61):

$$\left.\frac{\partial E_z(X,Y)}{\partial Y}\right|_{Y=0} = E_o \exp(+ikX\cos\psi)\ (-1 + R_h)\ ik\sin\psi$$

$$+ \frac{i}{\pi} [C + (k/2)Z_oI] \int_{-\infty}^{+\infty} \exp(+i\xi(X-x_o) + iy_o(k^2-\xi^2)^{1/2})\, d\xi$$

and

$$\left.\frac{\partial E_z^g(X,Y)}{\partial Y}\right|_{Y=0} = - E_o \exp(+ikX\cos\psi)\ (1 + R_h)\ i\ (k_g^2-k^2\cos^2\psi)^{1/2}$$

$$- \frac{i}{\pi} C \int_{-\infty}^{+\infty} \left(\frac{k_g^2-\xi^2}{k^2-\xi^2}\right)^{1/2} \exp(+i\xi(X-x_o) + iy_o(k^2-\xi^2)^{1/2})\, d\xi.$$

Consequently,

$$E_o \exp(+ikX\cos\psi) \{[ik\sin\psi - i(k_g^2-k^2\cos^2\psi)^{1/2}] - R_h [ik\sin\psi + i (k_g^2-k^2\cos^2\psi)^{1/2}]\}$$

$$= C \left\{ \frac{i}{\pi} \int_{-\infty}^{+\infty} \left[1 + \left(\frac{k_g^2-\xi^2}{k^2-\xi^2} \right)^{1/2} \right] \exp(+i\xi(X-x_o)+iy_o(k^2-\xi^2)^{1/2}) \, d\xi \right\}$$

$$+ I (k/2) Z_o \left[\frac{i}{\pi} \int_{-\infty}^{+\infty} \exp(+i\xi(X-x_o) + iy_o(k^2-\xi^2)^{1/2}) \, d\xi \right]$$

for all X.

Dividing the last result by X and integrating over X from $-\infty$ to $+\infty$ requires the following integrations:

$$\int_{-\infty}^{+\infty} \frac{\exp(+ikX\cos\psi)}{X} dX = i\pi \quad , k\cos\psi>0$$

and

$$\int_{-\infty}^{+\infty} \frac{\exp(+i\xi X)}{X} dX = i\pi \begin{cases} +1 & , \xi>0 \\ -1 & , \xi<0. \end{cases}$$

These results then lead to

$$E_o\{[ik\sin\psi - i(k_g^2-k^2\cos^2\psi)^{1/2}] - R_h[ik\sin\psi + i(k_g^2-k^2\cos^2\psi)^{1/2}]\}$$

$$= C \left\{ \frac{i}{\pi} \int_{o}^{\infty} \left[1 + \left(\frac{k_g^2-\xi^2}{k^2-\xi^2} \right)^{1/2} \right] \exp(-i\xi x_o + iy_o(k^2-\xi^2)^{1/2}) d\xi \right.$$

$$\left. - \frac{i}{\pi} \int_{-\infty}^{o} \left[1 + \left(\frac{k_g^2-\xi'^2}{k^2-\xi'^2} \right)^{1/2} \right] \exp(-i\xi' x_o + iy_o(k^2-\xi'^2)^{1/2}) d\xi' \right\}$$

$$+ I (k/2) Z_o \left[\frac{i}{\pi} \int_{o}^{\infty} \exp(-i\xi x_o + iy_o(k^2-\xi^2)^{1/2}) d\xi \right.$$

$$\left. - \frac{i}{\pi} \int_{-\infty}^{o} \exp(-i\xi' x_o + iy_o(k^2-\xi'^2)^{1/2}) \, d\xi' \right]$$

and, defining $\xi' = -\xi$, this becomes

$$E_o\{[ik\sin\psi - i(k_g^2-k^2\cos^2\psi)^{1/2}] - R_h[ik\sin\psi+i(k_g^2-k^2\cos^2\psi)^{1/2}]\}$$

$$= C\left\{\frac{2}{\pi}\int_0^\infty \left[1+\left(\frac{k_g^2-\xi^2}{k^2-\xi^2}\right)^{1/2}\right]\sin(\xi x_o)\ \exp(+iy_o(k^2-\xi^2)^{1/2})\ d\xi\right\}$$

$$+ I\ (k/4)\ Z_o\left[\frac{4}{\pi}\int_0^\infty \sin(\xi x_o)\ \exp(+iy_o(k^2-\xi^2)^{1/2})\ d\xi\right].$$

Finally, the last result can be expressed more succinctly by defining

$$F(x_o,y_o,k,k_g) = \frac{2}{\pi}\int_0^\infty \left[1+\left(\frac{k_g^2-\xi^2}{k^2-\xi^2}\right)^{1/2}\right]\sin(\xi x_o)\ \exp(+iy_o(k^2-\xi^2)^{1/2})d\xi \quad (64)$$

with

$$f(\psi,k,k_g) = E_o\{[ik\sin\psi - i(k_g^2-k^2\cos^2\psi)^{1/2}] - R_h[ik\sin\psi+i(k_g^2-k^2\cos^2\psi)^{1/2}]\} \quad (65)$$

and, therefore,

$$C\ F(x_o,y_o,k,k_g) + I\ (k/4)\ Z_o\ F(x_o,y_o,k,k) = f(\psi,k,k_g) \quad (66)$$

for $k\cos\psi>0$.

6.4 Formal Solutions

Application of boundary conditions has led to a system of two algebraic equations, equations (63) and (66), in the two unknown quantities C and I. A straightforward application of Cramer's rule provides formal solutions for C and I:

$$C = \frac{f(\psi,k,k_g)[H_o^{(1)}(ka)-H_o^{(1)}(2ky_o)] - F(x_o,y_o,k,k)E_z^t(x_o,y_o)}{F(x_o,y_o,k,k_g)[H_o^{(1)}(ka)-H_o^{(1)}(2ky_o)] + F(x_o,y_o,k,k)H_o^{(1)}(2ky_o)} \quad (67)$$

and

$$I = \frac{4}{kZ_o}\ \frac{F(x_o,y_o,k,k_g)E_z^t(x_o,y_o) + f(\psi,k,k_g)H_o^{(1)}(2ky_o)}{F(x_o,y_o,k,k_g)[H_o^{(1)}(ka)-H_o^{(1)}(2ky_o)] + F(x_o,y_o,k,k)H_o^{(1)}(2ky_o)}. \quad (68)$$

In order to numerically evaluate I and C, the integral $F(x_o,y_o,k,k_g)$ must be evaluated for arguments of interest. A form of $F(x_o,y_o,k,k_g)$ that is more amenable to numerical evaluation is derived in the Appendix of this work.

Two limiting forms of the total current in equation (68) yield the expected results. First, consider the case where $y_o \to \infty$ such that $H_o^{(1)}(2ky_o) \to 0$ and, therefore,

$$I \to \frac{4}{kZ_o} \frac{E_z^t(x_o,y_o)}{H_o^{(1)}(ka)} \quad \text{as } y_o \to \infty. \tag{69}$$

Far above the half-space, then, the induced current is essentially that on a thin cylinder in free space, equation (11), due to the incident and ground-reflected plane waves which constitute the total field at $(x_o y_o)$, $E_z^t(x_o,y_o)$. Secondly, if $|k_g| \to \infty$ then $F(x_o,y_o,k,k_g) \to \infty$ and $f(\psi,k,k_g)$ remains finite so that

$$I \to \frac{4}{kZ_o} \frac{E_z^t(x_o,y_o)}{[H_o^{(1)}(ka)-H_o^{(1)}(2ky_o)]} \quad \text{as } |k_g| \to \infty. \tag{70}$$

This result is essentially the same as equation (26) when ka is required to be small in that result.

7. CONCLUSION

Formal expressions have been obtained for the total induced current on a thin cylinder above a finitely conducting half-space and the electric field at the surface of that half-space. The assumptions made regarding the cylinder radius, a, the height of the cylinder above the half-space, y_o, and the free-space propagation constant, k, are as follows:

$ka \ll 1$,

the "thin antenna" assumption, and

$y_o >> a.$

An assumption was not made restricting y_o with respect to wavelength and neither conduction currents nor displacement currents were assumed to be dominant in the finitely conducting half-space.

The formal expressions for the total current and the surface field involve integrals that contain several parameters: the angle of incidence, y_o,k, and the complex propagation constant of the finitely conducting half-space. These integrals, in general, do not appear to be amenable to analytical evaluation and, consequently, have been expressed in a form suitable for numerical evaluation. Complex propagation constants for the half-space which are large in magnitude, compared to the free-space propagation constant, in conjunction with small angles of incidence may render these integrals analytically tractable.

Suppose a cylinder is imbedded in the finitely conducting half-space, rather than located above it, and a plane wave is obliquely incident on this half-space. A Green's function formulation analogous to that developed in the present study should also be applicable to this problem.

Future efforts in this problem area will be devoted to numerical evaluation of the total current in equation (68) for particular cases of interest.

APPENDIX

Transformation of $F(x_o,y_o,k,k_g)$ for Numerical Evaluation

The integral $F(x_o,y_o,k,k_g)$ given in equation (64) has a singular integrand at $\xi = k$, which lies on the path of integration. This integral, however, can be split into two integrals and each of these can be transformed into integrals with non-singular integrands. To this end, define

$$F(x_o, y_o,k,k_g) = F_1(x_o,y_o,k,k_g) + F_2(x_o,y_o,k,k_g) \qquad \text{(A-1)}$$

where

$$F_1(x_o,y_o,k,k_g) = \frac{2}{\pi}\int_o^k \left[1 + \left(\frac{k_g^2-\xi^2}{k^2-\xi^2}\right)^{1/2}\right]$$

$$\times \sin(\xi x_o) \exp(+iy_o(k^2-\xi^2)^{1/2}) \, d\xi \qquad \text{(A-2)}$$

with

$$F_2(x_o,y_o,k,k_g) = \frac{2}{\pi}\int_k^\infty \left[1 + \left(\frac{k_g-\xi^2}{k^2-\xi^2}\right)^{1/2}\right]$$

$$\times \sin(\xi x_o) \exp(+iy_o(k^2-\xi^2)^{1/2}) \, d\xi. \qquad \text{(A-3)}$$

The integral $F_1(x_o,y_o,k,k_g)$ can be transformed with the following change of integration variable:

$$\xi = k \sin\theta \ \ni \ \xi = k \Rightarrow \theta = \pi/2 \ \& \ \xi = 0 \Rightarrow \theta = 0$$

with

$$d\xi = k \cos \theta d\theta.$$

Consequently,

$$F_1(x_o,y_o,k,k_g) = \frac{2}{\pi}\int_o^{\pi/2} [k\cos\theta + (k_g^2-k^2\sin^2\theta)^{1/2}]$$

$$\times \sin(kx_o\sin\theta) \exp(+iky_o\cos\theta) \, d\theta \qquad \text{(A-4)}$$

so that F_1 has no singularities along the path of integration and should be integrable numerically. Similarly, the integral $F_2(x_o,y_o,k,k_g)$ can be transformed via

$$\xi^2 = u^2 + k^2 \ni \xi = k \Rightarrow u = o \;\&\; \xi \rightarrow \infty \Rightarrow u \rightarrow \infty$$

with

$$d\xi = udu \,/\, \sqrt{u^2+k^2}$$

and, therefore,

$$F_2(x_o,y_o,k,k_g) = \frac{2}{\pi} \int_o^\infty \{u + \sqrt{u^2-(k_g^2-k^2)^2}\,\}$$

$$\times \frac{\sin(x_o \sqrt{u^2+k^2}\,)}{\sqrt{u^2+k^2}} \exp(-uy_o)\; du. \qquad \text{(A-5)}$$