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Electromagnetic Scattering from Configurations of Thin Wires with Multiple Junctions

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Electromagnetic Scattering from Arbitrary Configurations

of Thin Wires with Multiple Junctions

ABSTRACT

The development of a system of integral equations for an arbitrary configuration of thin wires having multiple junctions is discussed. Particular attention is given to careful treatment of the necessary boundary conditions.

FOREWORD

We should like to express our appreciation to Dr. Carl Baum and Dr. C. D. Taylor for helpful discussions during this period.

1. Introduction

The general formulation for the treatment of an arbitrary configuration of thin wires with a single junction has been developed [1,2]. The formulation is adaptable to many geometries commonly encountered in antenna problems such as Tee-antennas [3], Vee-antennas, L-antennas [4,5], crossed-dipoles [6,7], tripoles [8] and many others. The formulation presented in this report is an extension of the existing theory to configurations having more than one junction. The basic modifications to present theory result from the fact that one can no longer enforce the boundary condition for zero end currents on each wire of a multiple junction structure. The role of these end currents is explicitly shown (this is not to be confused with end current corrections that are now being used to treat thick antennas).

2. Analysis

According to thin-wire scattering theory [3,4], the tangential component of the vector potential and the scalar potential at a point S on a conductor are

$$A_s(S) = \frac{\mu}{4\pi} \int_L dS' \, I(S') \, \hat{S}' \cdot \hat{S} \, G(S,S') \qquad (1)$$

$$\phi(S) = \frac{1}{4\pi\varepsilon} \int_L dS' \, \lambda(S') \, G(S,S') \qquad (2)$$

where G(S,S') is the usual Green's function and I(S') and λ(S') the linear current and charge densities, respectively. For an N-wire system of arbitrarily oriented wires, these potentials on the nth wire can be cast into the form

$$A_{Sn}(S_n) = \frac{\mu}{4\pi} \sum_{m=1}^{N} \int_{L_m} dS'_m (\hat{S}'_m \cdot \hat{S}_n)\, I_m(S'_m)\, G(S_n, S'_m) \tag{3}$$

where

$$G(S_n, S'_m) = \frac{\exp\left[-jk\sqrt{r^2(S_n, S'_m) + a_m^2}\right]}{\sqrt{r^2(S_n, S'_m) + a_m^2}}$$

$I_m(S'_m)$ = total axial current at the point S'_m on the mth wire

$\hat{S}_n$ = unit vector tangential to the nth wire at point S_n

$\hat{S}'_m$ = unit vector tangential to the mth wire at point S'_m

L_m = arc length of the mth wire

$r(S_n, S'_m)$ = linear separation distance from point S'_m on the surface of the mth wire to the point S_n on the surface of the nth wire

The usual assumption of harmonic time dependence, $e^{j\omega t}$, is made but suppressed. For the N-wire system, the scalar potential becomes

$$\phi_n(S_n) = j\frac{\zeta}{4\pi k} \sum_{m=1}^{N} \int_{L_m} dS'_m \frac{d}{dS_m'} \left[I_m(S'_m)\right] G(S_n, S'_m) \tag{4}$$

In writing (4) from (2), the equation of continuity is used and $\zeta = \sqrt{\mu/\varepsilon}$.

Since the wires are assumed to be perfectly conducting, the tangential component of the total electric field must vanish on the surface of each wire; hence, on the nth wire

$$E_{S_n}(S_n) + E^i_{S_n}(S_n) = 0 \qquad (5)$$

where $E^i_{S_n}(S_n)$ is the tangential component of the incident field and $E_{S_n}(S_n)$ is the tangential component of the scattered field. The E field in terms of ϕ and A becomes

$$-E^i_{S_n}(S_n) = -\frac{d}{dS_n}\phi_n(S_n) - j\omega A_{S_n}(S_n) \qquad (6)$$

In order to work with a system of integral equations rather than a system of integro-differential equations, it is convenient to define

$$\Phi_n(S_n) = -j\frac{k^2}{\omega}\int_0^{S_n} dS_n' \, \phi_n(S_n') \qquad (7)$$

(7) in (6) yields

$$\left[\frac{d^2}{dS_n^2} + k^2\right]\Phi_n(S_n) = k^2\left[\Phi_n(S_n) - A_{S_n}(S_n)\right] - j\frac{k^2}{\omega}E^i_{S_n}(S_n) \qquad (8)$$

The formal solution to (8) is [2]

$$\Phi_n(S_n) = C_n \cos kS_n + D_n \sin kS_n$$

$$+ k\int_0^{S_n} dS_n' \left[\Phi_n(S_n') - A_{S_n}(S_n')\right] \sin k(S_n - S_n')$$

$$-j\frac{k}{\omega}\int_0^{S_n} dS_n' \, E^i_{S_n}(S_n') \sin k(S_n - S_n') \qquad (9)$$

To derive the desired form of the integral equations, it is convenient to define [1]

$$F_1(S_n) = k\int_0^{S_n} dS_n' \; \Phi_n(S_n') \sin k(S_n-S_n') \qquad (10)$$

Substitution of (7) and (4) into (10) yields

$$F_1(S_n) = k\int_0^{S_n} dS_n' \frac{\mu}{4\pi} \sum_{m=1}^{N} \int_0^{S_n'} d\zeta_n \int_{L_m} dS_m' \frac{d}{dS_m'}\left[I_m(S_m')\right] G(\zeta_n, S_m') \qquad (11)$$

An integration by parts on the S_m' integration leads to

$$F_1(S_n)= \frac{\mu k}{4\pi} \sum_{m=1}^{N} \int_0^{S_n} dS_n' \int_0^{S_n'} d\zeta_n \left[I_m(L_m^u)G(\zeta_n,L_m^u)-I_m(L_m^1)G(\zeta_n,L_m^1)\right] \sin k(S_n-S_n')$$

$$- \frac{\mu k}{4\pi} \sum_{m=1}^{N} \int_0^{S_n} dS_n' \int_0^{S_n'} d\zeta_n \int_{L_m} dS_m' \; I_m(S_m') \; \frac{\partial G(\zeta_n, S_m')}{\partial S_m'} \sin k(S_n-S_n') \qquad (12)$$

If the order of integration is changed [1], the S_n' integration can be performed in (12) and

$$F_1(S_n)= \Phi_n(S_n) + \frac{\mu}{4\pi} \sum_{m=1}^{N} \int_0^{S_n} dS_n' \int_{L_m} dS_m' \; I_m(S_m') \; \frac{\partial G(S_n', S_m')}{\partial S_m'} \cos k(S_n-S_n')$$

$$- \frac{\mu}{4\pi} \sum_{m=1}^{N} \int_0^{S_n} dS_n' \left[I_m(L_m^u)G(S_n',L_m^u)-I_m(L_m^1)G(S_n',L_m^1)\right] \cos k(S_n-S_n') \qquad (13)$$

and, from (7) and (4)

$$\Phi_n(S_n) = \frac{\mu}{4\pi} \sum_{m=1}^{N} \int_0^{S_n} dS_n' \left[I_m(L_m^u) G(S_n', L_m^u) - I_m(L_m^l) G(S_n', L_m^l) \right]$$

$$- \frac{\mu}{4\pi} \sum_{m=1}^{N} \int_0^{S_n} dS_n' \int_{L_m} dS_m' \; I_m(S_m') \frac{\partial G(S_n', S_m')}{\partial S_m'} \qquad (14)$$

where L_m^u and L_m^l are the upper and lower limits of the S_m' integration, respectively.

The function $F_2(S_n)$ is defined as

$$F_2(S_n) = k \int_0^{S_n} dS_n' \; A_{S_n}(S_n') \sin k(S_n - S_n') \qquad (15)$$

Substitution of (3) into (15) and integration by parts leads to

$$F_2(S_n) = \frac{\mu}{4\pi} \sum_{m=1}^{N} \int_{L_m} dS_m' (\hat{S}_m' \cdot \hat{S}_n) I_m(S_m') G(S_n, S_m')$$

$$- \frac{\mu}{4\pi} \sum_{m=1}^{N} \int_{L_m} dS_m' (\hat{O}_n \cdot \hat{S}_m') I_m(S_m') G(0, S_m') \cos k \, S_n$$

$$- \frac{\mu}{4\pi} \sum_{m=1}^{N} \int_{L_m} dS_m' \int_0^{S_n} dS_n' \; I_m(S_m') \left[(\hat{S}_m' \cdot \hat{S}_n') \frac{\partial G(S_n', S_m')}{\partial S_n'} + G(S_n', S_m') \frac{\partial (\hat{S}_n' \cdot \hat{S}_m')}{\partial S_n'} \right] \cdot$$

$$\cos k \, (S_n - S_n') \qquad (16)$$

and $\hat{0}_n$ is the unit vector tangent to the nth wire at $S_n = 0$.

The integrals in (13) and (16) and the fact that C_n in (9) equals zero (since $\Phi_n(0) = 0$) may be used to rewrite (9) as

$$\sum_{m=1}^{N} \int_{L_m} dS'_m \, I_m(S'_m) \, \pi \, (S_n, S'_m)$$

$$+ \sum_{m=1}^{N} \int_0^{S_n} dS'_n \left[I_m(L^u_m) G(S'_n, L^u_m) - I_m(L^1_m) G(S'_n, L^1_m) \right] \cos k(S_n - S'_n)$$

$$- C'_n \cos k \, S_n - D'_n \sin k \, S_n = -j \frac{4\pi}{\eta} \int_0^{S_n} dS'_n \, E^i_{S_n}(S'_n) \sin k(S_n - S'_n) \qquad (17)$$

where

$$\pi(S_n, S'_m) = (\hat{S}_n \cdot \hat{S}'_m) G(S_n, S'_m) - \int_0^{S_n} dS'_n \cos k(S_n - S'_n) \Psi(S'_n, S'_m) \qquad (18)$$

and

$$\Psi(S'_n, S'_m) = (\hat{S}_n \cdot \hat{S}'_m) \frac{\partial G(S'_n, S'_m)}{\partial S'_n} + \frac{\partial G(S'_n, S'_m)}{\partial S'_m} + G(S'_n, S'_m) \frac{\partial (\hat{S}'_n \cdot \hat{S}'_m)}{\partial S'_n} \qquad (19)$$

$$C'_n = \sum_{m=1}^{N} \int_{L_m} dS'_m \, (\hat{0}_n \cdot \hat{S}'_m) \, I_m(S'_m) G(0, S'_m)$$

$$D'_n = \frac{4\pi}{\mu} D_n$$

According to (7) and (9) an equation for $\phi_n(S_n)$ similar to (17) can be written

$$\phi_n(S_n) = j\frac{\omega}{k} D_n \cos k S_n + \int_0^{S_n} dS_n' \, E^i_{S_n}(S_n') \cos k(S_n - S_n')$$

$$+ j\omega \int_0^{S_n} dS_n' \left[\Phi_n(S_n') - A_{S_n}(S_n')\right] \cos k(S_n - S_n') \qquad (20)$$

or, in terms of currents on the structure

$$\phi_n(S_n) = j\frac{\omega\mu D_n'}{k4\pi} \cos k S_n - j\frac{\omega}{k}\frac{\mu}{4\pi} C_n' \sin k S_n$$

$$+ j\frac{\omega\mu}{4\pi k} \sum_m \int_0^{S_n} dS_n' \left[I_m(L_m^u) G(S_n', L_m^u) - I_m(L_m^1) G(S_n', L_m^1)\right] \sin k(S_n - S_n')$$

$$- j\frac{\omega\mu}{4\pi k} \sum_m \int_{L_m} dS_m' \, I_m(S_m') \pi_1(S_n, S_m') + \int_0^{S_n} dS_n' \, E_n^i(S_n') \sin k(S_n - S_n') \qquad (21)$$

where

$$\pi_1(S_n, S_m') = \int_0^{S_n} dS_n' \sin k(S_n - S_n') \Psi(S_n', S_m') \qquad (22)$$

3. Boundary Conditions

For a system of N wires with no intersections, there will be 2N undetermined constants from (17). For thin-wire theory there will be 2N boundary conditions of the form

$$I_n(S_n)\Big|_{\text{free ends}} = 0 \qquad n=1,2,---,N \tag{23}$$

For a system of N wires with a single common intersection point, there will again be 2N undetermined constants from (17). In addition there will be a discontinuity in the current on each wire and this effectively introduces another N unknowns. The boundary conditions at the free ends of the wires furnish a set of 2N relations. An application of the Kirchhoff current law at the junction located at l_n on the n<u>th</u> wire, namely

$$\lim_{\delta\to 0} \sum_{n=1}^{N} [I_n(l_n+\delta) - I_n(l_n-\delta)] = 0 \tag{24}$$

provides one additional constraint. Enforcement of the continuity of scalar potential at the junction

$$\phi_1(l_1) = \phi_n(l_n) \qquad n=2,---,N \tag{25}$$

provides the necessary additional N-1 relations in order to obtain a unique solution to the set of equations.

Consider a system of N wires counted in such a way that the first N1 of these intersect at one point and N-N1+1 of the wires intersect at another physical point and that the N1<u>th</u> wire is the electrical connection between the two junctions.

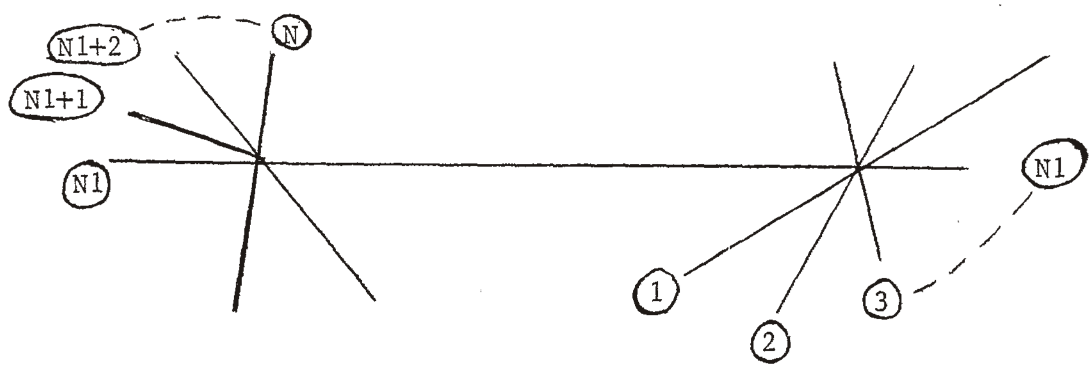

Figure 1. Two Junction Structure

Again, from (17) there will be 2N undetermined constants, and at the right junction there will be N1 current discontinuities. In general, the left junction will have N-N1+1 discontinuities (= the number of intersecting wires at the left intersection); but, for the particular case shown in Figure 1 there will be only N-N1 current discontinuities since wire number (N1+1) does not pass through the junction. For the configuration shown in Figure 1, there are 2N+N1+N-N1=3N unknowns. The free end boundary conditions will number 2N-1 (again due to the fact that one wire terminates at a junction). There will be two Kirchhoff relations, one at each junction. At the right junction there are (N1-1) scalar potential relations, and at the left junction there are (N-N1) scalar potential relations. Thus, the boundary conditions provide

2N-1	(free end conditions)
+ 2	(Kirchhoff law)
+ N1-1	(right scalar potentials)
+ N-N1	(left scalar potentials)
3N	

3N constraint equations and a unique solution will be obtained.

In practice it is convenient to define the coordinates such that the right intersection is located at

$$l_n=0 \qquad n=1,2,---,N1$$

and the left intersection at

$$l_{N1} = - L_{N1}$$

$$l_n=0 \qquad n=N1+1,---,N$$

Thus,

$$D_1=D_n \qquad n=1,2,---,N1 \qquad (26)$$

$$D_{N1+1}=D_n \qquad n=N1+2,---,N \qquad (27)$$

Finally from (21)

$$\phi_{N1}(- L_{N1}) = \phi_n(0) \qquad n=N1+1,---,N \qquad (28)$$

Thus (28) provides at set of integral equations that must be satisfied simultaneously with (17).

One application of multi-junction theory is to model an aircraft in terms of thin-wire approximations as shown in Figure 2.

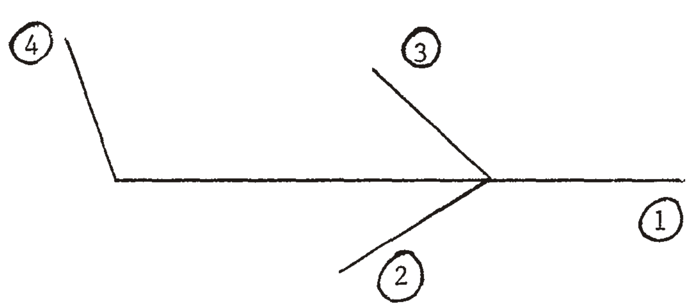

Figure 2

This would be a 4-wire, 2-junction problem with 8 undetermined constants and a single current discontinuity on wire 1 at the fore junction. There are four free end conditions, two Kirchhoff relations, two scalar potential relations on the fore junction, and one scalar potential relation on the aft junction providing a unique solution.

In this particular problem, the end current terms, $I_m(L_m^u)$ and $I_m(L_m^l)$, all vanish identically in (17) due to one of two reasons: 1) either the currents are identically zero, or 2) the Green's functions reduce to the same analytic form at a junction and the Kirchhoff law applied at the junction then causes these terms to vanish. It is true that these terms vanish in (21) as well and for the same reasons. This appears to be a general result in thin-wire scattering theory.

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