INTERACTION NOTES

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The Current Induced in a Horizontal Conductor Near the Earth's Surface by a Monopole Antenna*

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Abstract

The current induced in a radial wire on or near the surface of the ground is obtained as a solution to the differential equation for a transmission line with a distributed driving source. The source of the fields driving the transmission line in this note is a vertical monopole antenna driven against the ground. At the air-earth interface, the monopole antenna produces a radial electric field which is used as the distributed source term in the transmission line analysis. Solutions are presented for arbitrary terminations of the conductor and for four special cases.

antennas, transmission lines, ground effects

* Extracted from Electromagnetic Field Distortions and Currents in and Near Buried Cables and Bunkers, AFWL-TR-65-39, of same date.

I STATEMENT OF THE PROBLEM

It is desired to determine the current induced in a radial conductor near the surface of the gound in the vicinity of a vertical monopole antenna. The configuration of the antenna and conductor is illustrated in Fig. 1. It is assumed that the monopole antenna is electrically small and that the antenna height is small compared to the distance between the antenna and the near-end of the radial conductor. It is also assumed that the vertical height, or depth of burial if the conductor is below the surface,of the conductor is much less than one wavelength (in the appropriate medium) at the frequencies of interest. The frequency range of interest is that for which the soil behaves as a good conductor. That is, we are concerned with frequencies such that

$$\frac{\sigma}{\omega \epsilon} \gg 1, \qquad (1)$$

where σ is the conductivity of the soil, $\omega = 2\pi f$ and ϵ is the permittivity of the soil.

Because the soil behaves as a good conductor, we may obtain the principal components of the fields of the monopole in the air by treating the ground as a perfect conductor. We thus use the vertical electric field and azimuthal magnetic field of a monopole driven against a perfectly conducting ground as the driving field to obtain the current in the soil. To obtain the radial component of the electric field in the soil, however,

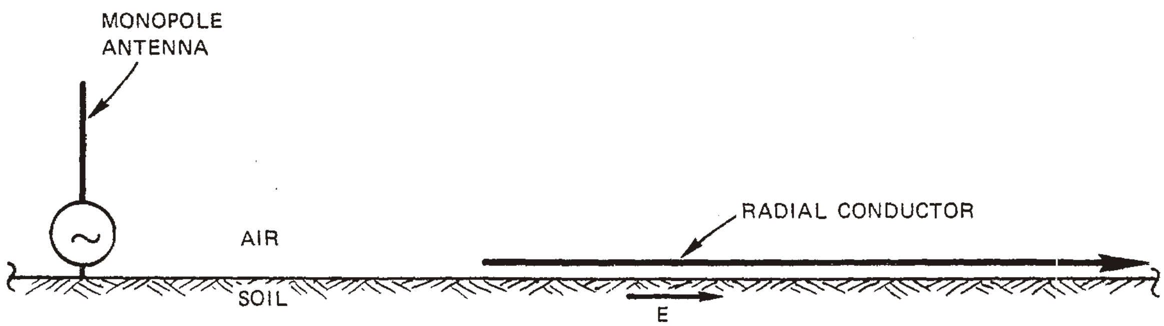

FIGURE 1 RADIAL NEAR-SURFACE CONDUCTOR IN THE FIELD OF A MONOPOLE ANTENNA

we recognize that the soil has finite conductivity and make use of the surface impedance of the soil and the soil current density (or azimuthal magnetic field). Having obtained the radial electric field at the surface of the soil, we treat the conductor as one conductor of transmission line with the soil as the other conductor and the radial electric field in the soil as a distributed voltage source driving the transmission line.

II RADIAL ELECTRIC FIELD

The radial electric field is the voltage drop per unit radial distance resulting from the flow of current in the ground. The radial field strength at the surface can thus be computed from the radial earth current and the surface impedance per unit radial length of the earth. The radial current is directly related to the azimuthal magnetic field by

$$I_r = 2\pi r H_\Phi \tag{2}$$

and the impedance per unit length in the radial direction is readily obtainable from the theory of the surface impedance of skin effect:

$$Z_s = \frac{1+j}{2\pi r \sigma \delta} \tag{3}$$

where σ is the soil conductivity and δ is the skin depth defined by

$$\delta = \frac{1}{\sqrt{\pi f \mu \sigma}} \quad . \tag{4}$$

The radial electric-field strength at the interface is thus

$$E = I_r Z_s$$

$$= - \frac{(1+j)}{\sigma \delta} H_\Phi$$

or

$$E = - (1+j) \sqrt{\frac{\pi f \mu}{\sigma}} H_\Phi \quad . \qquad (5)$$

Thus the radial electric field lags the azimuthal magnetic field by 135° in phase and differs in magnitude by a factor involving the square root of the frequency.

The magnetic field at a distance r from a short monopole of height h is

$$H_\Phi = \frac{I_0 h}{4\pi} \left(\frac{jk}{r} + \frac{1}{r^2} \right) e^{-jkr} \sin\theta \qquad (6)$$

where I_0 is the base current of the monopole antenna. For the electrically short monopole antenna,

$$I_0 = j\omega C_a V_a , \qquad (7)$$

where C_a is the antenna capacitance and V_a is the driving voltage.

III TRANSMISSION LINE DRIVEN BY A DISTRIBUTED SOURCE

The analysis of the coupling between the conductor and ground may be pursued along lines similar to classical transmission-line analysis. The conductor is coupled to the ground through the conductor-to-ground capacitance and ground resistance, while current flow along the conductor is impeded by the conductor resistance and inductance. However, instead of being driven at one end by a lumped source as is conventional in transmission-line problems, the conductor is driven along the line by a distributed source as illustrated in the circuit model of Fig. 2. This distributed driving source is the radial electric field in the ground in the vicinity of the conductor.

From Fig. 2, the following partial differential equations relating the voltage and current along the line to the driving field and impedance of the conductor are obtained:

$$\frac{dI}{dx} = - YV \qquad (8)$$

$$\frac{dV}{dx} = - E -(Z)I \quad . \qquad (9)$$

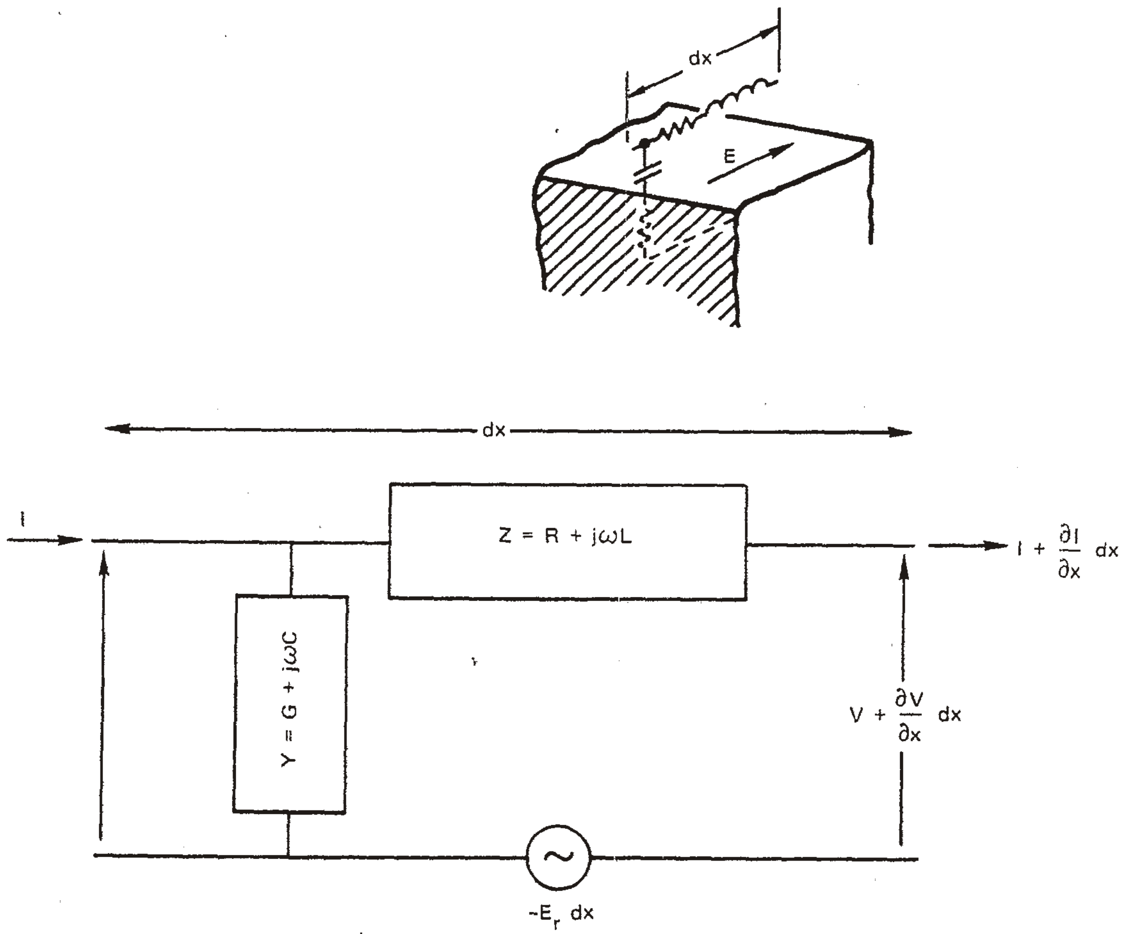

FIGURE 2 TRANSMISSION-LINE MODEL FOR RADIAL CONDUCTOR

By differentiating the first equation, solving for dV/dx, and substituting this value in the second equation, we obtain

$$\frac{d^2I}{dx^2} - \gamma^2 I = YE \qquad (10)$$

where

γ^2 = ZY, the propagation factor squared

Z = R + jωL, the wire impedance per unit length

Y = G + jωC, the wire admittance per unit length

and

$$E = (1-j)\sqrt{\frac{\pi f \mu}{\sigma}} \frac{\omega h C_a V_a}{4\pi} \left[\frac{jk}{x+a} + \frac{1}{(x+a)^2} \right] e^{-jk(x+a)} \qquad (11)$$

is the driving field, the radial field at the air/earth interface.* The differential equation to be solved is thus

$$\frac{d^2I}{dx^2} - \gamma^2 I = A \left[\frac{jk}{x+a} + \frac{1}{(x+a)^2} \right] e^{-jk(x+a)} \qquad (12)$$

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*It has been pointed out by W. R. Graham and D. Marston at AFWL that the above expression for the radial electric field will be in error very close to the base of the antenna (i.e., within one skin depth of the antenna base). Very near the antenna base, the current density in the soil will be approximately spherically distributed in the ground instead of parallel to the surface and exponentially decaying with depth as assumed in deriving Eq. (11).

where

$$A = (1-j)\ Y\sqrt{\frac{\pi f \mu}{\sigma}}\ \frac{\omega h C_a V_a}{4\pi} . \tag{13}$$

The solution of this equation is[2]

$$I(x) = K_1 e^{\gamma x} + K_2 e^{-\gamma x} + \frac{A}{2\gamma}\left\{ e^{\gamma x}\int e^{-\gamma x}\left[\frac{jk}{x+a} + \frac{1}{(x+a)^2}\right] e^{-jk(x+a)} dx \right.$$

$$\left. - e^{-\gamma x}\int e^{\gamma x}\left[\frac{jk}{x+a} + \frac{1}{(x+a)^2}\right] e^{-jk(x+a)} dx \right\} . \tag{14}$$

Because of the similarity of the two integrals, it will be necessary to solve only one of them; the second integral can then be evaluated by replacing γ by $-\gamma$ in the solution for the first. Hence, letting

$$\mathcal{I}_1 = \int e^{-\gamma x}\left[\frac{jk}{x+a} + \frac{1}{(x+a)^2}\right] e^{-jk(x+a)} dx$$

$$= e^{-jka}\left[jk\int \frac{e^{-(jk+\gamma)x}}{x+a} dx + \int \frac{e^{-(jk+\gamma)x}}{(x+a)^2} dx \right] \tag{15}$$

we can make use of the integral formulas[3]

$$\int \frac{e^{qx}}{x+a} dx = e^{-qa}\left\{ \log(x+a) + \sum_{n=1}^{\infty} \frac{[q(x+a)]^n}{n \cdot n!} \right\} \tag{16}$$

and

$$\int \frac{e^{qx}}{(x+a)^2} dx = -\frac{e^{qx}}{x+a} + qe^{-qa}\left\{ \log(x+a) + \sum_{n=1}^{\infty} \frac{[q(x+a)]^n}{n \cdot n!} \right\} \tag{17}$$

so that

$$\mathcal{L}_1 = e^{-jka}\left\{ jk\int \frac{e^{-(jk+\gamma)x}}{x+a}\,dx - \frac{e^{-(jk+\gamma)x}}{x+a} - (jk+\gamma)\int \frac{e^{-(jk+\gamma)x}}{x+a}\,dx \right\} . \quad (18)$$

After rearranging,

$$\mathcal{L}_1 = e^{\gamma a}\left\{ - \frac{e^{-(\gamma+jk)(x+a)}}{x+a} - \gamma \int \frac{e^{-(jk+\gamma)(x+a)}}{x+a}\,dx \right\} . \quad (19)$$

Thus

$$F(x) = \frac{A}{2\gamma}\left\{ e^{\gamma x}\mathcal{L}_1 - e^{-\gamma x}\mathcal{L}_2 \right\}$$

$$= -\frac{A}{2}\left\{ e^{\gamma(x+a)}\int \frac{e^{-(jk+\gamma)(x+a)}}{x+a}\,dx + e^{-\gamma(x+a)}\int \frac{e^{-(jk-\gamma)(x+a)}}{x+a}\,dx \right\} . \quad (20)$$

After performing the integrations and rearranging,

$$F(x) = -A\left\{ \cosh \gamma(x+a) \log (x+a) + \frac{e^{\gamma(x+a)}}{2}\sum_{n=1}^{\infty} \frac{\left[-(jk+\gamma)(x+a)\right]^n}{n \cdot n!} + \frac{e^{-\gamma(x+a)}}{2}\sum_{n=1}^{\infty} \frac{\left[-(jk-\gamma)(x+a)\right]^n}{n \cdot n!} \right\} . \quad (21)$$

The wire current is then

$$I(x) = K_1 e^{\gamma x} + K_2\, e^{-\gamma x} + F(x) \quad (22)$$

where K_1 and K_2 are arbitrary constants determined by the terminations at the end of the wire.

For large values of $(jk\pm\gamma)(x+a)$, the series is difficult to evaluate because the terms become very large before the series begins to converge. For large arguments, therefore, the asymtotic expression for the exponential integral may be used.[4] Thus,

$$\int_{q(x+a)}^{\infty} \frac{e^{-u}}{u} du = -\gamma_e - \log q(x+a) - \sum_{n=1}^{\infty} \frac{[-q(x+a)]^n}{n \cdot n!}$$

$$\approx \frac{e^{-q(x+a)}}{q(x+a)} \sum_{n=0}^{N} \frac{n!}{[-q(x+a)]^n} \qquad (23)$$

from which

$$\sum_{n=1}^{\infty} \frac{[-q(x+a)]^n}{n \cdot n!} = -\gamma_e - \log q(x+a) - \frac{e^{-q(x+a)}}{q(x+a)} \sum_{n=0}^{N} \frac{n!}{[-q(x+a)]^n} \qquad (24)$$

where $\gamma_e = 0.577\ 2157\ldots$, and N is chosen by considering the accuracy desired and the magnitude of the argument $-q(x+a)$.

To obtain the potential of the wire relative to the undisturbed ground potential, we note that

$$\frac{dI}{dx} = -YV(x) \quad . \qquad (25)$$

Thus the wire voltage V(x) can be written

$$V(x) = -Z_o \left[K_1 e^{\gamma x} - K_2 e^{-\gamma x} \right] - \frac{1}{Y} \frac{dF(x)}{dx} \qquad (26)$$

where $Z_o = \gamma/Y = \sqrt{Z/Y}$ is the characteristic impedance of the line.

Letting

$$\frac{1}{Y}\ \frac{dF(x)}{dx} \ = \ Z_o\ F_v(x) \quad , \tag{27}$$

we have

$$V(x) \ = \ -Z_o \left[K_1\, e^{\gamma x} \ - \ K_2\, e^{-\gamma x} \ + \ F_v(x) \right] \tag{28}$$

where

$$F_v(x) \ = \ -A \left\{ \sinh\ \gamma(x+a)\ \ \log\ (x+a) \ + \ \frac{e^{-jk(x+a)}}{\gamma(x+a)} \right.$$

$$+ \ \frac{e^{\gamma(x+a)}}{2} \sum_{n=1}^{\infty} \frac{\left[-jk+\ \ )(x+a)\right]^n}{n \cdot n!} \tag{29}$$

$$\left. - \ \frac{e^{-\gamma(x+a)}}{2} \sum_{n=1}^{\infty} \frac{\left[-(jk-\gamma)(x+a)\right]^n}{n \cdot n!} \right\} \ .$$

At $x = 0$ the wire is terminated in an impedance Z_1 , so that

$$-Z_1\ \ I(0) \ = \ V(0) \tag{30}$$

or,

$$-Z_1 \left[K_1 \ + \ K_2 \ + \ F(0) \right] \ = \ -Z_o \left[K_1 \ - \ K_2 \ + \ F_v(0) \right] \ . \tag{31}$$

Similarly, at $x = d$, the wire is terminated in an impedance Z_2, so that $Z_2 I(d) = V(d)$ and

$$Z_2 \left[K_1 e^{\gamma d} + K_2 e^{-\gamma d} + F(d) \right] = -Z_o \left[K_1 e^{\gamma d} - K_2 e^{-\gamma d} + F_v(d) \right] . \tag{32}$$

Solving Eqs. (31) and (32) for K_1 and K_2, we get

$$K_1 = \frac{e^{-\gamma d}\left(1 - \frac{Z_0}{Z_2}\right)\left[\frac{Z_0}{Z_1} F_v(0) - F(0)\right] + \left(1 + \frac{Z_0}{Z_1}\right)\left[\frac{Z_0}{Z_2} F_v(d) + F(d)\right]}{\left(1 - \frac{Z_0}{Z_1}\right)\left(1 - \frac{Z_0}{Z_2}\right)e^{-\gamma d} - \left(1 + \frac{Z_0}{Z_1}\right)\left(1 + \frac{Z_0}{Z_2}\right)e^{\gamma d}}$$

$$K_2 = \frac{-e^{\gamma d}\left(1 + \frac{Z_0}{Z_2}\right)\left[\left(\frac{Z_0}{Z_1}\right) F_v(0) - F(0)\right] - \left(1 - \frac{Z_0}{Z_1}\right)\left[\frac{Z_0}{Z_2} F_v(d) + F(d)\right]}{\left(1 - \frac{Z_0}{Z_1}\right)\left(1 - \frac{Z_0}{Z_2}\right)e^{-\gamma d} - \left(1 + \frac{Z_0}{Z_1}\right)\left(1 + \frac{Z_0}{Z_2}\right)e^{\gamma d}} . \tag{33}$$

The general solution is thus obtained when the values of K_1 and K_2 from Eq. (33) are substituted in Eq. (22). Four special cases of interest will now be considered.

Case I -- The line is terminated with the same impedance at both ends ($Z_2 = Z_1$). Equations (33) then reduce to

$$K_1 = \frac{e^{-\gamma d}\left(1 - \frac{Z_0}{Z_1}\right)\left[\frac{Z_0}{Z_1} F_v(0) - F(0)\right] + \left(1 + \frac{Z_0}{Z_1}\right)\left[\frac{Z_0}{Z_1} F_v(d) + F(d)\right]}{\left(1 - \frac{Z_0}{Z_1}\right)^2 e^{-\gamma d} - \left(1 + \frac{Z_0}{Z_1}\right)^2 e^{\gamma d}}$$

(34)

$$K_2 = \frac{-e^{\gamma d}\left(1 + \frac{Z_0}{Z_1}\right)\left[\frac{Z_0}{Z_1} F_v(0) - F(0)\right] - \left(1 - \frac{Z_0}{Z_1}\right)\left[\frac{Z_0}{Z_1} F_v(d) + F(d)\right]}{\left(1 - \frac{Z_0}{Z_1}\right)^2 e^{-\gamma d} - \left(1 + \frac{Z_0}{Z_1}\right)^2 e^{\gamma d}} .$$

Case II -- The line is open-circuited at both ends ($Z_1 = Z_2 = \infty$). Equations (33) then reduce to

$$K_1 = \frac{F(0)\, e^{-\gamma d} - F(d)}{2 \sinh \gamma d}$$

(35)

$$K_2 = \frac{F(d) - e^{\gamma d} F(0)}{2 \sinh \gamma d}$$

and Eq. (22) becomes

$$I(x) = \frac{F(0) \sinh \gamma(x-d) - F(d) \sinh \gamma x}{\sinh \gamma d} + F(x) \qquad (36)$$

Case III -- The line is terminated in its characteristic impedance ($Z_1 = Z_2 = Z_0$). Equations (33) reduce to

$$K_1 = -\frac{1}{2}\left[F_v(d) + F(d)\right] e^{-\gamma d}$$

(37)

$$K_2 = \frac{1}{2}\left[F_v(0) - F(0)\right]$$

and Eq. (22) becomes

$$I(x) = F(x) - \frac{1}{2}\left[F_v(d) + F(d)\right] e^{\gamma(x-d)} + \frac{1}{2}\left[F_v(0) - F(0)\right] e^{-\gamma x} . \tag{38}$$

Case III is applicable to buried bare conductors except within about a skin depth of the ends.

Case IV -- The line is terminated in an impedance 1/Y . Equations (33) then become

$$K_1 = \frac{e^{-\gamma d}(1-\gamma)\left[\gamma F_v(0) - F(0)\right] + (1+\gamma)\left[\gamma F_v(d) + F(d)\right]}{(1-\gamma)^2 e^{-\gamma d} - (1+\gamma)^2 e^{\gamma d}}$$

$$K_2 = \frac{-e^{\gamma d}(1+\gamma)\left[\gamma F_v(0) - F(0)\right] - (1-\gamma)\left[\gamma F_v(d) + F(d)\right]}{(1-\gamma)^2 e^{-\gamma d} - (1+\gamma)^2 e^{\gamma d}} . \tag{39}$$

Case IV is applicable to the buried bare conductor to within a few meters of the ends.

When $(\gamma \pm jk)(x+a)$ is large so that the approximation of Eq. (24) applies and when γd is so large that the denominator of Eqs. (39) can be replaced by $-(1+\gamma)^2 \exp(\gamma d)$, Eqs. (39) approach

$$K_1 = \frac{\gamma F_v(d) + F(d)}{(1+\gamma)\, e^{\gamma d}}$$

$$K_2 = \frac{\gamma F_v(o) - F(o)}{1+\gamma} + \frac{1-\gamma}{(1+\gamma)^2} \frac{\gamma F_v(d) + F(d)}{e^{\gamma d}} . \tag{40}$$

Substituting Eq. (24) in Eqs. (21) and (29),

$$\gamma F_v(d) \pm F(d) = -\frac{A}{2}\left\{ e^{\gamma(d+a)} B(\gamma)(\gamma \pm 1) - e^{-\gamma(d+a)} B(-\gamma)(\gamma \mp 1) \right. \tag{41}$$

$$\left. - \frac{e^{-jk(d+a)}}{d+a} \left[\frac{\gamma \pm 1}{jk+\gamma} S(\gamma, d) - \frac{\gamma \mp 1}{jk-\gamma} S(-\gamma, d) - 1 \right] \right\}$$

and

$$F(x) = -\frac{A}{2}\left\{ e^{\gamma(x+a)} B(\gamma) + e^{-\gamma(x+a)} B(-\gamma) \right. \tag{42}$$

$$\left. - \frac{e^{-jk(x+a)}}{x+a} \left[\frac{S(\gamma, x)}{jk+\gamma} + \frac{S(-\gamma, x)}{jk-\gamma} \right] \right\}$$

where

$$B(\gamma) = \log \frac{1}{jk+\gamma} - 0.577\,2157 \tag{43}$$

and

$$S(\gamma, x) = \sum_{n=o}^{N} \frac{n!}{\left[-(jk+\gamma)(x+a)\right]^n} . \tag{44}$$

Using these values in Eqs. (40) and noting that

$$I(x) = K_1 e^{\gamma x} + K_2 e^{-\gamma x} + F(x)$$

we obtain

$$I(x) \approx \frac{A}{2} \left\{ \frac{e^{-jk(x+a)}}{x+a} \left[\frac{S(\gamma, x)}{jk + \gamma} + \frac{S(-\gamma, x)}{jk - \gamma} \right] \right.$$

$$+ e^{-\gamma x} \frac{e^{-jka}}{a} \left[\frac{\gamma - 1}{\gamma + 1} \frac{S(\gamma, o)}{jk + \gamma} - \frac{S(-\gamma, o)}{jk - \gamma} - 1 \right] \tag{45}$$

$$\left. - e^{-\gamma(d-x)} \frac{e^{-jk(d+a)}}{d+a} \left[\frac{S(\gamma, d)}{jk + \gamma} - \frac{\gamma - 1}{\gamma + 1} \frac{S(-\gamma, d)}{jk - \gamma} - 1 \right] \right\}$$

where terms involving $\exp\left[-\gamma(x+d)\right]$ and $\exp\left[-\gamma(2d + a - x)\right]$ have been neglected. Far from the ends of the wire, $\exp\left[-\gamma(d-x)\right]$ and $\exp(-\gamma x)$ are both small if the real part of the argument is large. Furthermore, when $\gamma(x+a) >> 1$, the value of $S(\gamma, x)$ is closely approximated by the first term. Assuming also that $\gamma >> k$,

$$I(x) \approx - \frac{A}{2} \frac{e^{-jk(x+a)}}{x+a} \frac{2jk}{\gamma 2} \tag{46}$$

far from either end of the wire. This approximation is useful for computing the current induced in buried bare conductors over a few hundred meters from the transmitting antenna. If the other two terms of Eq. (45) are included, the current can also be computed near the ends of the wire.

IV CONCLUSION

A study of the cable current problem has led to the development of a relatively simple analytical theory based on a transmission line model for calculating the current induced in a conductor by electric fields tangential to the surface of the ground. In this transmission line theory it is assumed that the conductor is isolated or that other conductors, if present, are far enough away that they do not appreciably affect the tangential electric fields in the vicinity of the conductor. Currents measured in the cables studied in this note indicate that the transmission line theory is valid for isolated conductors, for arrays of bare conductors in contact with the soil when the conductors are spaced several skin depths (in the soil) apart, and for insulated conductors above or below the surface.

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