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1. INTRODUCTION

In EMP (Electromagnetic Pulse) studies, a typical problem is to determine the response of a scatterer due to a transient source. This problem is commonly attacked first in the frequency domain and then an inverse Fourier transform is employed to obtain the solution in the time domain. Briefly this procedure may be outlined below. Let $G(\bar{r}, t)$ be the response (e.g., current, scattered field, etc.) due to a time-harmonic source $S(\bar{r})e^{-i\omega t}$. Then $G(\bar{r}, t)$ must also be time-harmonic

$$G(\bar{r}, t) = G_o(\bar{r}, \omega)\ e^{-i\omega t} \tag{1.1}$$

and furthermore, $G_o(\bar{r}, \omega)$ is regular in the upper half complex ω-plane (causality condition).

Now, let us consider the response due to an arbitrary transient source $V(t)\ S(\bar{r})$ where

$$V(t) = 0 \quad , \qquad t < 0. \tag{1.2}$$

To this end, we first write $V(t)$ in terms of its Fourier spectrum

$$V(t) = \frac{1}{2\pi} \int_C V_o(\omega)\ e^{-i\omega t}\ d\omega \tag{1.3}$$

where C is a contour shown in Figure 1, and $V_o(\omega)$ is a regular function in the upper half complex ω-plane.[1] Applying the superposition principle, we may readily arrive at the conclusion that the response $R(\bar{r}, t)$ due to the transient source $V(t)\ S(\bar{r})$ is given by

$$R(\bar{r}, t) = \frac{1}{2\pi} \int_C V_o(\omega)\ G_o(\bar{r}, \omega)\ e^{-i\omega t}\ d\omega\ . \tag{1.4}$$

For $t < 0$, we may close the contour C with a semi-infinite circle in the upper ω-plane. Since $V_o(\omega)$ and $G_o(\omega)$ are regular there, it follows immediately

$$R(\bar{r}, t) = 0, \qquad t < 0 \tag{1.5}$$

which is to be expected. Thus, the main problem is to evaluate the integral in Equation (1.4) for $t > 0$. In many cases the integral has to be processed numerically. Not only that it requires extensive machine calculations, but this procedure has to be repeated whenever the source $V_o(\omega)$ is changed. In this connection it is desirable to be able to decompose the part of contribution to $R(\bar{r}, t)$ from the scatterer $G_o(\bar{r}, \omega)$, and that from the source $V_o(\omega)$. When this is accomplished we may predict some general features of the response $R(\bar{r}, t)$ without even specifying the source, and this will simplify many considerations in EMP studies.

Baum[2] recently proposed a singularity expansion method to accomplish the decomposition mentioned above. The approach is to study the natural resonance frequencies of the scatterer under consideration, which is precisely specified by the pole locations of $G_o(\bar{r}, \omega)$ in the lower complex ω-plane. In addition to poles, $G_o(\bar{r}, \omega)$ may have branch singularities. Thus, for $t > 0$, we may deform the contour C in Equation (1.4) to enclose the poles and the branch cuts in the lower ω-plane and express $R(\bar{r}, t)$ in the following form*

$$R(\bar{r}, t) = -i \sum_n V_o(\omega_n) \, [\text{Res. } G_o(\bar{r}, \omega_n)] \, e^{-i\omega_n t} + R_{br}(\bar{r}, t) \tag{1.6}$$

where $\{\omega_n\}$ are the natural resonance frequencies; "Res." stands for "residue of," and $R_{br}(\bar{r}, t)$ is the contribution from the branch cut integrals. A study of Equation (1.6) immediately reveals that this approach is particularly

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*For simplicity we assume $V_o(\omega)$ has no singularities. Otherwise, additional terms shall be included in (1.6).

useful provided that the following two conditions are met:

(A) $R_{br}(\bar{r}, t)$ if not identically zero is negligibly small as compared with the residue contributions;

(B) Only the first few $\{\omega_n\}$ have small imaginary parts.

Under these two conditions, $R(\bar{r}, t)$ is well approximated by the first few residue terms in Equation (1.6), except for an initial period when t is small. We note that each residue term consists of three factors:*

(i) $\exp i\omega_n t$ represents the temporal variation with an oscillation period determined by Re ω_n and an attenuation rate by Im ω_n.

(ii) Res. $G_o(\bar{r}, \omega_n)$ gives the spatial variation.

(iii) $V_o(\omega_n)$ brings in the dependence on the source, which is merely a weighting factor and is independent of $(\bar{r}, t)$.

Thus, in the singularity expansion method, we are able to decompose neatly the temporal variation, the spatial variation, and the source dependence. This not only simplifies many numerical computations, but also provides us a better physical insight of the EMP problem under consideration. Of course, this is true only if the two conditions in (A) and (B) are met. There seems no general way that we can test the satisfaction of these two conditions; we almost have to consider specific examples and learn from experience.

The purpose of this note is to present one of such examples, and hopefully to shed some light on the question whether the resonance frequency approach is a useful one in the EMP study. The geometry that we choose to investigate is a finite cylinder with small radius-to-height ratio, and the cylinder is excited by a usual δ-gap source at its center. Our choice of the cylinder is based on the following two reasons. First, the cylinder is by far one of the most common geometries encountered in the EMP study. Second, there has been a large accumulation of knowledge on the scattering

*Using the terms coined by Baum, Res. $G_o(\bar{r}, \omega_n)$ is the natural mode, and $V_o(\omega_n)$ is the coupling coefficient.

by a cylinder in the frequency domain which is necessary in the determination of $G_o(\bar{r}, \omega)$ in Equation (1.4) and its natural resonance frequencies. Furthermore, the scattering by a cylinder with a transient source has been recently studied by Sassman through numerical techniques,[7] which would provide us a valuable check of our analytical results.

The organization of this note is as follows. In Section 2, the expression for the induced current on the cylinder due to a unit-step source is derived. Since the current in the frequency domain is well-known, this step involves only the taking of an inverse Fourier transform with respect to the frequency ω. The next three sections deal with the analytical evaluation of this inverse transform. In Section 6, the final result of the current due to a step source is discussed and compared with the numerical computations reported by Sassman. Finally, a short conclusion is presented in Section 7.

2. CURRENT ON A THIN CYLINDER

In this section, we will derive the current expression on a thin solid* cylindrical antenna due to a transient source. The geometry of the antenna is shown in Figure 2, and its radius-to-height ratio is assumed to be small

$$\frac{a}{h} \ll 1 \ . \tag{2.1}$$

The antenna is excited by a voltage source at a small gap around $z = 0$. First, let the source be a time-harmonic, unit-amplitude, slice-generator, which gives rise to an incident electric field

$$E_z^{(i)}(\bar{r}, t) = \delta(z)\, e^{-i\omega t} \ , \text{ at } \rho = a+ \ . \tag{2.2}$$

The induced current due to the source in Equation (2.2) has been the subject of intensive research in the past. When the condition in Equation (2.1) is satisfied, an approximate analytical expression can be found by the Wiener-Hopf technique (Appendix A) and it reads

$$I(z, \omega) = I^{(\infty)}(z, \omega) + \frac{R\, U(h, \omega)}{1 - R\, U(2h, \omega)} \left[U(z - h, \omega) + U(z + h, \omega)\right] \ . \tag{2.3}$$

The terms appearing in Equation (2.3) are explained below. $I^{(\infty)}(z, \omega)$ is the current on an infinitely long antenna due to the source in Equation (2.2), and is given by

$$I^{(\infty)}(z, \omega) = \begin{cases} \dfrac{\pi}{Z_o} \dfrac{(-1)}{\ell n \dfrac{\Gamma ka}{\sqrt{2}}} + i \dfrac{2ka}{Z_o} \ell n\, k|z| \ , & \dfrac{|z|}{a} \to 0 \\ U(z, \omega) \ , & |z| \gg a \end{cases} \tag{2.4}$$

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*As shown in Reference 3, there is little difference between a solid and a tubular antenna as long as it is thin.

where $k = \omega/c$, $Z_o = 120\pi$, and $\Gamma = 1.78107$. The universal function $U(z, \omega)$ on a cylinder has several different expressions. The one given by Kunz[4] reads

$$U(z, \omega) = \frac{2\pi\, e^{ik|z|}}{Z_o\, \ell n\, [2ik|z|/(\Gamma ka)^2]} , \tag{2.5}$$

and the one given by Shen et al.[5] reads

$$U(z, \omega) = \frac{i\, e^{ik|z|}}{Z_o}\, \ell n \left[1 - \frac{2\pi i}{\ell n \left[\dfrac{k|z| + \sqrt{(kz)^2 + 0.3}}{\Gamma (ka)^2} \right] + i\, \dfrac{3}{2}\pi} \right] . \tag{2.6}$$

Clearly, Equation (2.5) and Equation (2.6) agree when $|z|/a \to \infty$. Although Equation (2.6) may give more accurate results for small $|z|/a$, we use Equation (2.5) in the following manipulation. This is because of the fact that the branch singularities at $\omega = \pm(c/z)\sqrt{0.3}$ in Equation (2.6) were introduced by Shen et al. in an arbitrary manner, and these singularities may affect the evaluation of the current in the time domain (i.e. in the integration with respect to ω). The reflection coefficient $R(\omega)$ appeared in Equation (2.3) is that of a traveling wave at the end of a thin cylinder (solid or hollow), and is given by approximately

$$R(\omega) = \frac{Z_o}{2\pi}\, [\ell n(\Gamma ka)^2 - i\pi] . \tag{2.7}$$

It may be remarked that the expressions given in Equation (2.4) through Equation (2.7) represent the most dominant terms in series expansions for small (ka), and their higher order terms may be found in the literature.[1]

Next, let us consider the induced current on a cylinder due to a transient source, namely a step source (a DC source switched on at t=0) which gives rise to an incident electric field

$$E_z^{(i)}(\bar{r}, t) = \begin{cases} 0 , & t < 0 \\ \delta(z) , & t > 0 \text{ and } \rho = a+ . \end{cases} \tag{2.8}$$

Using the Fourier transform as outlined in the introduction, it is a simple matter to verify that the induced current is given by

$$I(z, t) = \frac{i}{2\pi} \int_C \frac{1}{\omega} I(z, \omega) e^{-i\omega t} d\omega \tag{2.9}$$

where $I(z, \omega)$ is given in Equation (2.3) and the contour C is shown in Figure 1. Equation (2.9) may be compared with Equation (1.4). The remaining step is to evaluate $I(z, t)$ by deforming the contour C into the lower half ω-plane and concentrating on the singularities of $I(z, \omega)$. We will separate $I(z, t)$ in Equation (2.9) into two parts

$$I(z, t) = I^{(\infty)}(z, t) + I^{(s)}(z, t) \tag{2.10}$$

where $I^{(\infty)}(z, t)$, the part from the integration over the first term in Equation (2.3), may be identified as the induced current on an <u>infinitely</u> long antenna due to a step source, and $I^{(s)}(z, t)$, the part from the second term in Equation (2.3), may be identified as the contribution of the multiple scattering between the two ends of a finite antenna. They will be evaluated in the next three sections.

3. EVALUATION OF $I^{(\infty)}(z, t)$

First, we will evaluate $I^{(\infty)}(z, t)$, which is the current on an infinitely long cylinder due to a step source. From Equations (2.3), (2.4), (2.5), and (2.9), we have

$$I^{(\infty)}(z, t) = \frac{i}{Z_o} \int_C \frac{e^{-i\omega(t - \frac{|z|}{c})}}{\omega \ln [iq / \omega]} d\omega \qquad (3.1)$$

where $q = 2 \left(\frac{c}{a} \right) \left(\frac{z}{a} \right) \frac{1}{\Gamma^2}$ = a real constant.

Equation (3.1) is valid when $|z| >> a$. Clearly,

$$I^{(\infty)}(z, t) = 0 \quad , \ t < \frac{|z|}{c} \ . \qquad (3.2)$$

For $t > |z|/c$. we may deform the contour C into the lower complex ω-plane. The only singularity of the integrand is that there is a pair of branch points at $\omega = 0$ and $\omega = \infty$. Let us introduce a branch cut in the lower ω-plane along its imaginary axis shown in Figure 1, and deform the contour C into P_1 and P_2. Note that

$$\omega = \begin{cases} |\omega| \, e^{i \frac{3\pi}{2}} & \text{on} \quad P_1 \\ |\omega| \, e^{-i \frac{\pi}{2}} & \text{on} \quad P_2 \ . \end{cases} \qquad (3.3)$$

The deformation of the contour results in

$$I^{(\infty)}(z, t) = \frac{2\pi}{Z_o} \int_0^\infty \frac{e^{-|\omega|(t - \frac{|z|}{c})} \, d|\omega|}{|\omega| \left[\ln \frac{q}{|\omega|} - i\pi\right] \left[\ln \frac{q}{|\omega|} + i\pi\right]} \ . \qquad (3.4)$$

A change of variable with

$$|\omega|\left(t - \frac{|z|}{c}\right) = v \qquad (3.5)$$

yields

$$I^{(\infty)}(z, t) = \frac{2\pi}{Z_o} \int_0^\infty \frac{e^{-v}}{v\left[\ln \frac{q(t - |z|/c)}{v} - i\pi\right]\left[\ln \frac{q(t-|z|/c)}{v} + i\pi\right]} \, dv \, . \qquad (3.6)$$

The above integral has to be evaluated numerically. However, for the late time behavior with

$$\frac{c}{a}\left(t - \frac{|z|}{c}\right) \gg 1 \qquad (3.7)$$

the integral in Equation (3.6) can be estimated approximately as below. Under the condition in Equation (3.7), the main contribution of the integral comes from the portion between $v = 0$ and $v = v_o$, a small constant. A good choice is $v_o = 1/\Gamma = (1/1.78107)$.[5] With this approximation, Equation (3.6) becomes

$$I^{(\infty)}(z, t) \sim \frac{\pi}{Z_o} \int_0^{1/\Gamma} \frac{1}{v\left[\ln \frac{q(t - |z|/c)}{v}\right]^2} \, dv \, . \qquad (3.8)$$

When the integral in Equation (3.8) is carried out, we have the final result

$$I^{(\infty)}(z, t) \sim \frac{\pi}{Z_o} \frac{1}{\ln\left[\frac{2(ct-|z|)}{\Gamma a} \left|\frac{z}{a}\right|\right]} \, , \quad \frac{ct - |z|}{a} \gg 1 \, . \qquad (3.9)$$

This expression may be checked with the one obtained by Latham and Lee[8] through a different method. Their expression reads

$$I_{LL}^{(\infty)}(z, t) \sim \frac{\pi}{Z_o} \frac{1}{\ell n[\frac{2(ct - |z|)}{\Gamma a}]}, \quad \frac{ct - |z|}{a} >> 1. \qquad (3.10)$$

Note that Equation (3.9) and Equation (3.10) agree for large $(ct - |z|)/a$.

4. NATURAL RESONANCE FREQUENCIES

The more interesting part of the current is $I^{(s)}(z, t)$ obtained from the multiple scattering between the two ends of the finite cylinder. It follows from Equation (2.3) and Equation (2.9) that

$$I^{(s)}(z, t) = \frac{i}{2\pi} \int_C \frac{R\,U(\omega, h)[U(z - h, \omega) + U(z + h, \omega)]}{\omega[1 - R\,U(2h, \omega)]} \cdot e^{-i\omega t}\, d\omega \, . \tag{4.1}$$

There are two types of singularities in the integrand in the lower ω-plane. In addition to the branch cut running from $\omega = 0$ to $\omega = \infty$ along the negative imaginary ω-axis, there are simple poles (Figure 1). The location of the poles is determined by the following equation:

$$1 - R\,U(2h, \omega) = 0 \, . \tag{4.2}$$

The solutions of Equation (4.2), denoted by $\{\omega_n\}$, are the natural resonance frequencies of a thin cylinder. We will now consider the solution in some detail. Explicitly, Equation (4.2) reads as

$$1 + e^{i2kh} \left[\frac{2\ln(\Gamma ka) - i\pi}{2\ln(\Gamma ka) - \ln(4ikh)} \right] = 0 \tag{4.3}$$

where $k = \omega/c$. Let us concentrate on the solutions of ω such that

$$\frac{\omega}{c} a << 1 \, . \tag{4.4}$$

Under the condition of Equation (4.4), the equation in Equation (4.3) can be approximated by

$$1 + e^{i2kh} \left[1 + \frac{\ln(4kh) - i\pi/2}{2\ln(\Gamma ka)} \right] = 0 \, . \tag{4.5}$$

Obviously, Equation (4.5) admits a string of solutions of the form

$$kh = \frac{n\pi + \Delta_n}{2} \quad , \quad n = \pm 1, \pm 3, \pm 5, --- \tag{4.6}$$

where Δ_n is to be determined. Substituting Equation (4.6) into Equation (4.5) yields the result

$$\Delta_n \sim i \frac{\ln(2n\pi) - i(\pi/2)}{\ln(\Gamma \frac{a}{h} \frac{n\pi}{2})^2} . \tag{4.7}$$

Thus, the solution for a string of natural resonance frequencies takes the following approximated form

$$\omega_n = \frac{n\pi c}{2h} \left[1 + \frac{1 + i \frac{2}{\pi} \ln(2n\pi)}{4n\ln(\Gamma \frac{a}{h} \frac{|n|\pi}{2})} \right] , \tag{4.8}$$

$$n = \pm 1, \pm 3, \pm 5, ---$$

which is valid under the condition of Equation (4.4), or more explicitly

$$m = \frac{|n|\pi}{2}\left(\frac{a}{h} \right) << 1 . \tag{4.9}$$

For $m \sim 1$ or larger, no convenient expressions can be found for $\{\omega_n\}$. Fortunately, those higher order resonance frequencies have large imaginary parts, and usually contribute very little to the late-time behavior of the current. The real and imaginary parts of $\{\omega_n\}$ in Equation (4.8) can be separated as below. Remember that, for negative n,

$$\ln 2n\pi = \ln 2|n|\pi + i\pi . \tag{4.10}$$

It follows from Equation (4.8) that

$$\mathrm{Re}\ \omega_n = \frac{n\pi c}{2h} \left\{ 1 - \frac{1}{4|n|[\ln(h/a) - \ln 2.8|n|]} \right\} \tag{4.11a}$$

$$\text{Im}\ \omega_n = -\ \frac{c}{4h}\ \frac{\ell n 2|n|\pi}{[\ell n(h/a) - \ell n\ 2.8|n|]} \qquad (4.11b)$$

for $n = \pm 1, \pm 3, \pm 5, - - -$.

Note the relation that

$$\text{Re}\ \omega_{-n} = -\ \text{Re}\ \omega_n \qquad (4.12a)$$

$$\text{Im}\ \omega_{-n} = \text{Im}\ \omega_n \qquad (4.12b)$$

which is to be expected. A sketch of $\{\omega_n\}$ in the complex ω-plane is given in Figure 1. At this point, it is interesting to point out that the solutions in (4.11) are not the only natural frequencies of a cylinder.[2] Rather, they represent the string of natural frequencies which is closest to the real ω-axis (implying least damping), and may be considered as the principal natural frequencies.

It is also interesting to mention that the principal natural frequencies of a thin cylinder have been studied quite extensively in the literature. We will now compare our result in Equation (4.8) with some of the classical ones. For the convenience of comparison, let us introduce a parameter used by Schelkunoff,[6] known as "logarithmic decrement,"

$$\delta_n = \frac{2\pi|\text{Im}\ \omega_n|}{|\text{Re}\ \ \omega_n|} = \frac{\pi}{Q_n} \qquad (4.13)$$

where Q_n is the usual quality factor. From Equation (4.11), we may obtain the expression for δ_n ,

$$\delta_n = \frac{\ell n\ 2|n|\pi}{|n|[\ell n\ (\frac{h}{a}) - \ell n\ 2.8\ |n| - \frac{1}{4|n|}]} . \qquad (4.14)$$

The first two δ_n as derived from Equation (4.14) are tabulated in Table I where we have also given the expressions obtained by other authors. All demoninators are in agreement in the asymptotic sense (i.e. for large h/a). However, our numerical values in the numerators are somewhat different from all the others. A possible explanation for this is that all the other authors, except Hallén, derived their results based on spheroids or biconical antennas, and inaccuracy may have resulted because of the approximation of the cylinders by those geometries. In the case of Hallén, he used the following expression for the universal function

$$U(z, \omega) = \frac{2\pi e^{ik|z|}}{Z_o \; \ell n[2ik|z|/\Gamma(ka)^2]} \quad \text{(Hallén)} . \tag{4.15}$$

When compared with the one used in the present paper, we find that Hallén's expression in Equation (4.15) differs from our expression in Equation (2.5) by a factor $\Gamma = 1.78107$ in the argument of the logarithmic function.* It has been shown by Kunz[4] that Equation (2.5) is more accurate than Equation (4.15).

*It may be noted that, if Equation (4.15) is used in Equation (4.2), we may recover exactly the numerical values in the numerators of δ_1 and δ_3 given by Hallén ($\ell n\ 2\pi = 1.84$ and $\ell n\ 2\pi\Gamma = 2.43$).

Table I. LOGARITHMIC DECREMENTS

Abraham:

$$\delta_1 = \frac{2.43}{\ell n\left(\frac{h}{a}\right) + 0.69} \qquad \delta_3 = \frac{1.17}{\ell n\left(\frac{h}{a}\right) + 0.69}$$

Brillouin:

$$\delta_1 = \frac{1.22}{\ell n\left(\frac{h}{a}\right) + 0.69} \qquad \delta_3 = \frac{0.47}{\ell n\left(\frac{h}{a}\right) + 0.69}$$

Hallén:

$$\delta_1 = \frac{2.43}{\ell n\left(\frac{h}{a}\right) + 0.69} \qquad \delta_3 = \frac{1.17}{\ell n\left(\frac{h}{a}\right) + 0.69}$$

Page:

$$\delta_1 = \frac{2.43}{\ell n\left(\frac{h}{a}\right) - 0.52} \qquad \delta_3 = \frac{1.24}{\ell n\left(\frac{h}{a}\right) + 0.94}$$

Schelkunoff:

$$\delta_1 = \frac{2.43}{\ell n\left(\frac{h}{a}\right)} \qquad \delta_3 = \frac{1.17}{\ell n\left(\frac{h}{a}\right) - 0.2}$$

Lee:

$$\delta_1 = \frac{1.84}{\ell n\left(\frac{h}{a}\right) - 1.28} \qquad \delta_3 = \frac{0.98}{\ell n\left(\frac{h}{a}\right) - 2.21}$$

5. EVALUATION OF $I^{(s)}(z, t)$

With its poles determined in the previous section, we will now consider the integral for $I^{(s)}(z, t)$ in Equation (4.1) under the late-time condition $(ct/h) >> 1$. Deforming the contour C in Equation (4.1) into the lower complex ω-plane (Figure 1), there are two types of contributions to $I^{(s)}(z, t)$: one is that of the branch cut integral denoted by $I_{br}^{(s)}(z, t)$, and the other is residues at the poles at $\omega = \{\omega_n\}$ denoted by $I_{po}^{(s)}(z, t)$.

First, let us discuss the branch cut integral $I_{br}^{(s)}(z, t)$. Using further simplified expressions for $U(z, \omega)$ in Equation (2.5) and $R(\omega)$ in Equation (2.7),

$$U(z, \omega) = \frac{-2\pi e^{ik|z|}}{Z_o \ell n(\Gamma ka)^2} \tag{5.1}$$

$$R(\omega) = \frac{Z_o}{2\pi} \ell n \; (\Gamma ka)^2 \tag{5.2}$$

we have from Equation (4.1) that

$$I_{br}^{(s)}(z, t) = \frac{-i}{Z_o} \int_{branch} \frac{\cos \frac{\omega|z|}{c}}{\cos \frac{\omega h}{c}} \; \frac{e^{-i\omega \, (t - \frac{h}{c})}}{\omega \ell n \, (\frac{iq}{\omega})} \, d\omega \tag{5.3}$$

where q is defined in Equation (3.1). Combining the integrals over P_1 and P_2 (Figure 1) and introducing a new variable

$$u = i\omega \; (t - \frac{h}{c}). \tag{5.4}$$

The integral in Equation (5.3) can be written as

$$I_{br}^{(s)}(z, t) = \frac{-2\pi}{Z_o} \int_0^\infty \frac{\cosh \left(\frac{uz}{ct - h} \right)}{\cosh \left(\frac{uh}{ct - h} \right)} \; \frac{e^{-u}}{u[\ell n \frac{iq(t - h/c)}{u}]^2} \, du. \tag{5.5}$$

Following the same argument used in association with the evaluation of Equation (3.6), we may approximate Equation (5.5) by

$$I_{br}^{(s)}(z, t) \sim \frac{-2\pi}{Z_o} \int_0^{1/\Gamma} \left[1 - \frac{1}{2} \frac{(h^2 - z^2)u^2}{(ct - h)^2} \right]$$

$$\cdot \frac{e^{-u}}{u[\ell n \frac{q(t - h/c)}{u}]^2} du \sim \frac{-2\pi}{Z_o} \frac{1}{\ell n[\frac{2(ct - h)}{\Gamma a} \left|\frac{z}{a}\right|]}$$

$$+ 0\left(\frac{(h^2 - z^2)}{c^2 t^2}\right) \tag{5.6}$$

which is valid under condition (ct/h) >> 1.

Next consider the contribution to Equation (4.1) due to the principal poles at $\omega = \{\omega_n\}$ specified in Equation (4.8). Using the expressions in Equation (5.1) and Equation (5.2), the integrand of Equation (4.1) may be approximately written as

$$\frac{-i}{Z_o} \frac{\cos \frac{\omega z}{c}}{\cos \frac{\omega h}{c}} \frac{e^{-i\omega t}}{\omega \ell n(iq/\omega)} . \tag{5.7}$$

Its residue at $\omega = \omega_n$ is approximately given by

$$\frac{i}{30n\pi} \frac{\cos(\frac{n\pi}{2} \frac{z}{h})}{\ell n(4\ h|z|/\Gamma^2 a^2 n\pi)} e^{-i\omega_n t} . \tag{5.8}$$

Summing the residue contributions due to the principal poles, we have

$$I_{po}^{(s)}(z, t) \sim \sum_n \frac{i}{30n\pi} \frac{\cos\ (\frac{n\pi}{2} \frac{z}{h})}{\ell n(4\ h|z|/\Gamma^2 a^2 n\pi)} e^{-i\omega_n t} \tag{5.9}$$

where $\{\omega_n\}$'s are given in Equation (4.8). As mentioned in the paragraph following Equation (4.12), the $\{\omega_n\}$ found in Equation (4.8) are the principal poles. Thus, in addition to the terms in (5.9), we may add residue contributions from other poles. However, due to the fact that those additional poles are relatively far away from the real ω-axis, corresponding to larger damping constants, they are less important in the late-time behavior. Hence, we have not included them in Equation (5.9).

6. LATE-TIME BEHAVIOR OF I(z, t)

Summarizing the results in the previous three sections, the late-time current due to a step source in Equation (2.8) on a Delta-gap excited cylindrical antenna is approximately given by

$$I(z,\ t) \sim \frac{2\pi}{Z_o} \left\{ \frac{1}{\ln\left[\frac{2(ct - |z|)}{\Gamma a} \left|\frac{z}{a}\right|\right]} - \frac{1}{\ln\left[\frac{2(ct - h)}{\Gamma a} \left|\frac{z}{a}\right|\right]} \right\}$$

$$+ \sum_n \frac{i}{30n\pi} \frac{\cos\left(\frac{n\pi}{2} \frac{z}{h}\right)}{\ln(4ih|z|/\Gamma^2 a^2 n\pi)} e^{-i\omega_n t} . \qquad (6.1)$$

The first term in Equation (6.1) can be further simplified to give the result

$$I(z,\ t) \sim \sum_n \frac{i}{30n\pi} \frac{\cos\left(\frac{n\pi}{2} \frac{z}{h}\right)}{\ln(4\ h|z|/\Gamma^2 a^2 n\pi)} e^{-i\omega_n t}$$

$$+ \frac{2\pi}{Z_o} \frac{|z| - h}{ct\left[\ln \frac{2ct\,|z|}{\Gamma a^2}\right]^2} \qquad (6.2)$$

The final result in Equation (6.2) is approximately valid under the following conditions:

(i) thin cylinder, (h/a) >> 1;

(ii) away from the feed, |z|/a >> 1;

(iii) late-time, (ct/h) >> 1.

The first few terms in Equation (6.2), due to the residue contributions, attenuate relatively slowly with time. Explicitly, the exponential attenuation constant for the two most dominant terms is given by

$$\mathrm{Im}\,\omega_1 = \mathrm{Im}\,\omega_{-1} = \frac{c}{h}\,\frac{0.46}{\ln(a/h) + 1.03} \tag{6.3}$$

$$\mathrm{Im}\,\omega_3 = \mathrm{Im}\,\omega_{-3} = \frac{c}{h}\,\frac{0.74}{\ln(a/h) + 2.12} \; . \tag{6.4}$$

If we retain only the first two terms in the series in Equation (6.2), i.e. $n = \pm 1$, the current expression becomes

$$I(z,t) \sim \frac{\cos\left(\frac{\pi}{2}\,\frac{z}{h}\right)}{15\pi \ln(4\,h|z|/\Gamma^2 a^2 \pi)} \exp\left[\frac{-0.46}{\ln(h/a) - 1.03}\,\frac{ct}{h}\right] \cdot \sin\left\{\frac{\pi}{2}\left[1 - \frac{0.25}{\ln(h/a) - 1.03}\right]\frac{ct}{h}\right\} + O\left(\frac{h - |z|}{ct[\ln\left(\frac{ct}{a}\right)]^2}\right) \tag{6.5}$$

The first term is due to the most dominant pole contribution while the second term is from that of the branch cut shown in Figure 1. At this point, it is appropriate to mention a recent study made by Baum[2], and Marin and Latham[9] in regard to the presence of branch cut contributions. For the scattering of an <u>incident plane wave</u> by a finite conducting body, they indicated that there is <u>no</u> branch-cut contribution to the time-domain response. Their statement, however, does not contradict the result in (6.2) for the simple reason that the source considered here is an idealized <u>delta-gap generator</u> rather than an incident plane wave.

Now we will make a qualitative comparison between the importance of the dominant residue term in Equation (6.5) and the term due to the branch cut integration. The temporal variations of these two terms are

Residue: $$\exp\left[\frac{-0.46}{\ln(h/a) - 1.03}\left(\frac{ct}{h}\right)\right] \tag{6.6}$$

Branch-Cut: $$\frac{1}{(ct/h)\,[\ln(ct/h) + \ln(h/a)]^2} \tag{6.7}$$

Let us consider the following two time intervals:

$$\text{(i)} \quad \left(\frac{h}{a}\right) \gg \left(\frac{ct}{h}\right) \gg 1 \tag{6.8}$$

$$\text{(ii)} \quad \left(\frac{ct}{h}\right) \gg \left(\frac{h}{a}\right) \gg 1 \quad . \tag{6.9}$$

During the first interval in Equation (6.8), the residue term dominates, while during the second interval in Equation (6.9), the term from the branch cut integration dominates. However, it must be emphasized that in many EMP problems the response in the second interval usually has already become negligibly small, and therefore is of little practical interest.

Next we will use a numerical example to illustrate the accuracy of the simple formula for the natural frequencies in Equation (4.8). Sassman[7] recently calculated the current induced on a finite solid cylinder by a unit step of magnetic field traveling in the broadside direction of cylinder.* He formulated the problem first in terms of an H-integral equation in the frequency domain, and then obtained the transient solution through the inverse Laplace transform. In both steps, he used numerical techniques with the aid of a computer. A result obtained by Sassman are reproduced in Figure 3. The key features of his results may be summarized as below.

(i) The relative current magnitude becomes quite small for $(ct/h) > (h/a)$. Thus, for all practical purposes, it is necessary to concentrate on only the time interval $0 \leq (ct/h) \leq (h/a)$.

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*Note that the excitation used by Sassman is different from ours, but this fact merely changes the weighting factor and does not affect the spatial and the temporal behavior of the current.

(ii) The current oscillates between positive and negative values with respect to time. Except for the first several cycles, the period of oscillation is nearly a constant for a given (h/a). By curve fitting, we have determined the period for three different (h/a) and their values are tabulared in the last column in Table II.

(iii) The magnitude of the current decays nearly exponentially with respect to time. Sassman determined the time constant τ (defined by the time variation $\exp -(t/\tau)$) for each case by curve fitting. His result is tabulated in the last column in Table III.

Table II. PERIOD $T = 2\pi/\mathrm{Re}\,\omega_1$ IN SECONDS

a/h	Schelkunoff	Lee & Leung	Sassman
10^{-3}	$4.14\ \frac{h}{c}$	$4.18\ \frac{h}{c}$	$4.2\ \frac{h}{c}$
10^{-2}	$4.2\ \frac{h}{c}$	$4.3\ \frac{h}{c}$	$4.3\ \frac{h}{c}$
10^{-1}	$4.52\ \frac{h}{c}$	$4.98\ \frac{h}{c}$	$5.1\ \frac{h}{c}$
Formula	$\dfrac{4(h/c)}{1 - \dfrac{0.226}{\ell n(\frac{h}{a}) - 0.31}}$	$\dfrac{4(h/c)}{1 - \dfrac{0.25}{\ell n(\frac{h}{a}) - 1.03}}$	Numerical curve fitting

Table III. TIME CONSTANT $\tau = 1/|\mathrm{Im}\,\omega_1|$ IN SECONDS

a/h	Schelkunoff	Lee & Leung	Sassman
10^{-3}	$11.3\ \frac{h}{c}$	$12.8\ \frac{h}{c}$	$12.5\ \frac{h}{c}$
10^{-2}	$7.5\ \frac{h}{c}$	$7.8\ \frac{h}{c}$	$7.2\ \frac{h}{c}$
10^{-1}	$3.8\ \frac{h}{c}$	$2.8\ \frac{h}{c}$	$5.3\ \frac{h}{c}$
Formula	$\frac{h}{c}\ \frac{\ell n(h/a)}{0.61}$	$\frac{h}{c} \cdot \frac{\ell n(h/a) - 1.03}{0.46}$	Numerical curve fitting

It is somewhat amazing that all the above three observations from Sassman's numerical computation check with the predication of the first term given in Equation (6.5). Quantitatively, the oscillation period T and the time constant τ as derived from Equation (6.5) are

$$T = \frac{4}{1 - \frac{0.25}{\ln(h/a) - 1.03}} \left(\frac{h}{c}\right) \text{ sec} \qquad (6.10)$$

$$\tau = \frac{\ln(h/a) - 1.03}{0.46} \left(\frac{h}{c}\right) \text{ sec} . \qquad (6.11)$$

Their numerical values for the three cases considered by Sassman are given in Table II and Table III. For comparison purposes, we have also included the values computed from the formulas given by Schelkunoff.[6] They all agree with each other reasonably well except for one case. For (h/a) = 10, the time constant given by us and that by Schelkunoff is shorter than that given by Sassman. One possible explanation is that, in this case, $\ln(h/a) = 2.3$ is not large enough for the analytical formulas of the type given in Equation (6.11) to be accurate.

7. CONCLUSION

The motivation of this note is to answer, at least partially, whether the singularity expansion method developed by Baum[2] is useful in the EMP studies. The scattering of a unit-step δ-gap voltage source by a finite cylinder with small radius-to-height ratio has been chosen as a test case. We have derived an approximate formula for the natural resonance frequencies in Equation (4.11), examined the late-time behavior of the induced current, and compared its time constant and period with the numerical calculations performed by Sassman[7] (Section 6). The main results and conclusions of these studies are summarized below.

(i) The current expression in the frequency domain has two types of singularities in the lower complex ω-plane, namely, infinitely many simple poles (corresponding to the natural resonance frequencies of the cylinder) and a branch cut (Figure 1). Roughly speaking, the contribution from the poles dominates for time $(ct/h) < (h/a)$, and that from the branch cut for $(ct/h) > (h/a)$. However, in many EMP studies with a transient source, the response usually becomes negligibly small (die - out) before $(ct/h) = (h/a)$. Thus, for all practical purposes, only the response in the time interval $0 < (ct/h) < (h/a)$ is of interest, and that response can be well predicted by the residue contribution except perhaps for (ct/h) small (early-time response).

(ii) With large (h/a), the natural resonance frequencies for a cylinder can be found explicitly and are given in Equation (4.11). Our result differs somewhat from those results obtained by earlier workers, and it is believed to be more accurate.

(iii) During the time interval $1 \ll (ct/h) < (h/a)$, the current can be approximately described by the residue contributions corresponding to the first resonance frequency alone(i.e. Equation (6.5)). The oscillation period, and the time constant for the first resonance frequency are given by Equation (6.10) and Equation (6.11) respectively. These two simple formulas give good results as compared with the numerical computations of Sassman (see Table II and Table III).

From the above discussions, it is clear that the natural resonance frequency approach is indeed a useful method in the EMP studies. For the case of a thin cylinder a simple expression as that in Equation (6.5) predicts all the main features of the late-time behavior of the induced current, thus avoiding the extensive numerical computations necessary in other approaches. For scatterers whose geometries are more complex, it is probably not possible to determine their resonance frequencies by analytical means as we have done here. In those cases, numerical means have to be resorted to, and the development of techniques for that purpose is certainly a worthwhile subject for future EMP researches.

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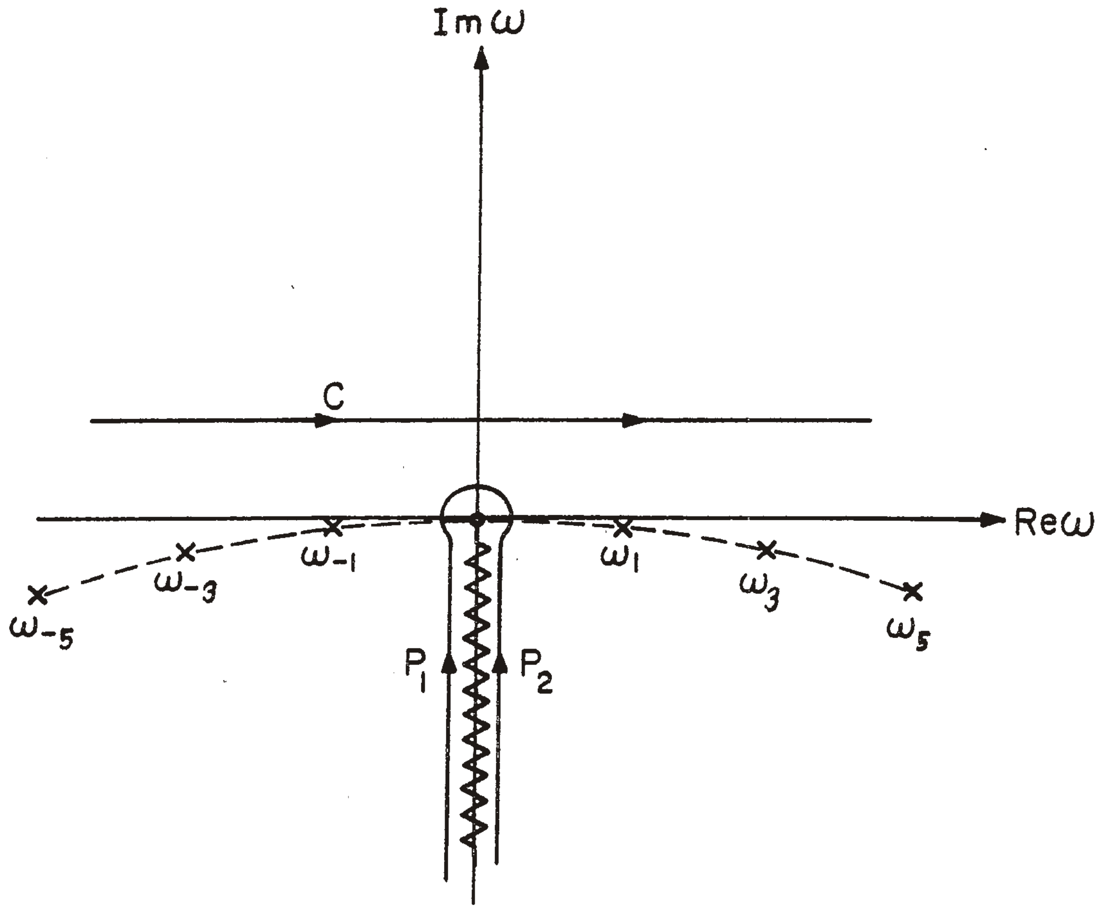

Figure 1. Contours and singularities in comples ω-plane

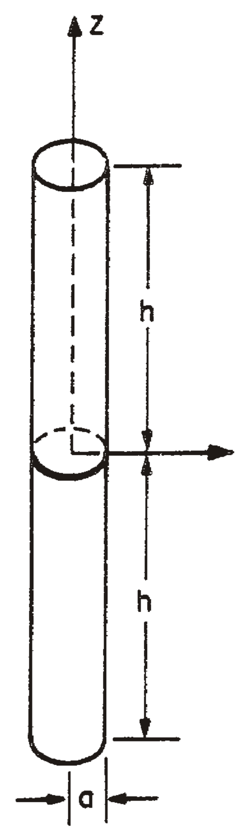

Figure 2. Geometry of a thin cylinder with a δ-gap excitation at its center

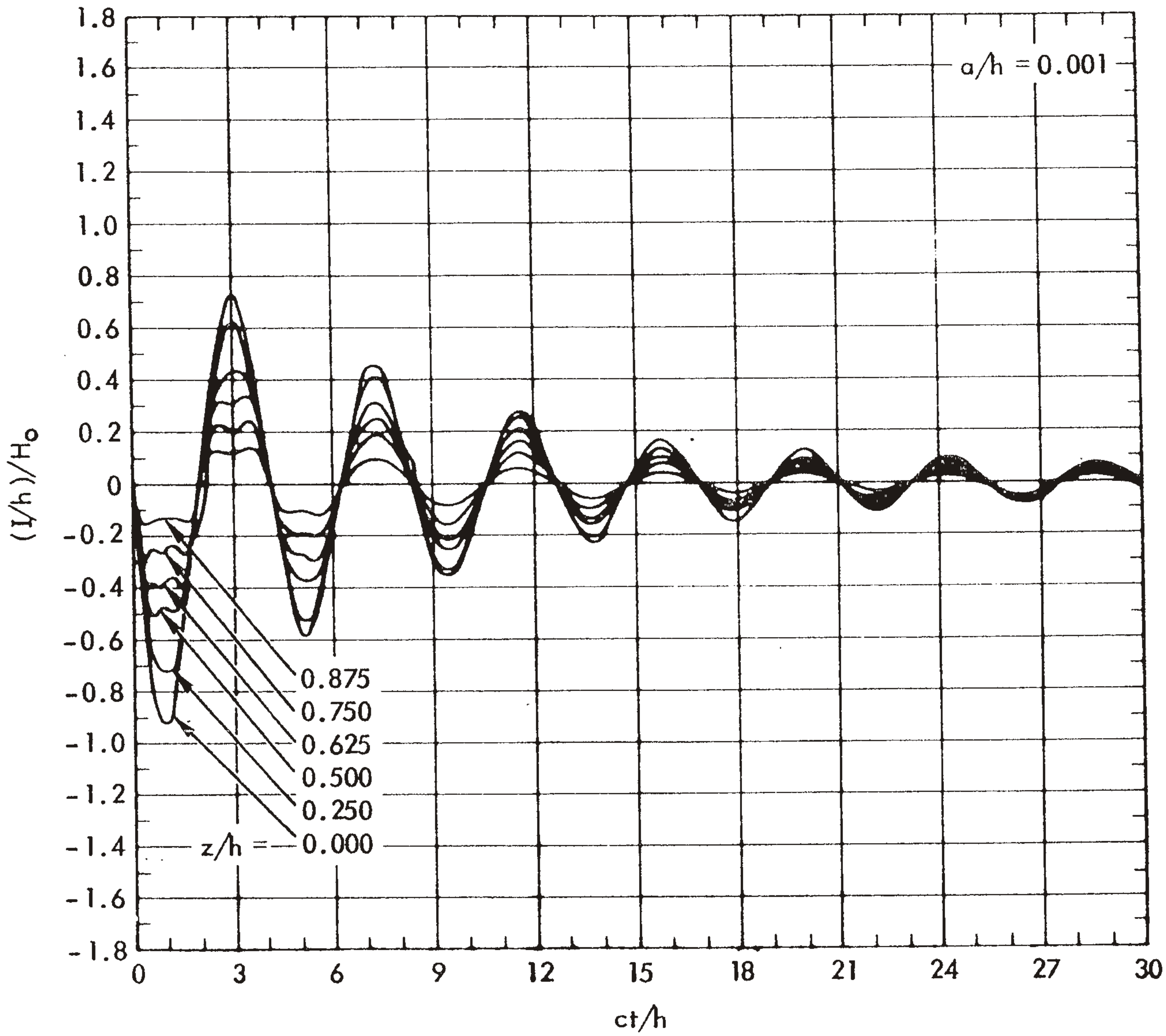

Figure 3. Current on a cylinder induced by a unit step of magnetic field versus ct/h computed by a numerical method (Sassman)