p. 21
p. 22
p. 27

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Analytical Properties of the Field Scattered by a Perfectly Conducting, Finite Body

by

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Abstract

Electromagnetic scattering by a perfectly conducting body of finite extent is considered from an integral equation point of view. It is shown that the operator inverse to the integral operator of the magnetic field formulation is an analytic, operator-valued function in the complex frequency plane except at certain points (the natural frequencies) where it has poles. Furthermore, a representation of the inverse operator in terms of the natural frequencies and the nontrivial solutions of the homogeneous integral equation is given. Explicit expressions for the scattered field in terms of exponentially damped sinusoidal oscillations are given for the special case where the incident wave is a delta-function plane wave and the inverse operator has only simple poles.

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I. Introduction

It has been observed, when solving many EMP scattering and interaction problems, that the time dependence of various quantities, such as the current induced on an object in an EMP simulator, seems to be described by one or more exponentially damped sinusoidal oscillations. Resonance phenomena are especially pronounced when the test object consists of one or more long slender conductors; missiles and aircraft are examples of such objects. In the case of a missile, main body resonances have been observed. The current induced on an aircraft when it is subjected to an electromagnetic pulse (EMP) has resonances that can be associated with both the length of the fuselage and the length of the wings.[26]

In this note we are going to consider electromagnetic scattering by a perfectly conducting body of arbitrary shape but of finite extent. It has been conjectured by Dr. C. E. Baum (private communication and [1]) that for such a body the scattered field due to an incident delta-function (in time) plane wave can be described by damped sinusoidal oscillations alone. Based upon previous works it can be shown that this is true for a sphere and possibly other bodies[10]. In fact, many of the properties of the scattered field of a sphere have served as a guideline for us in establishing and conjecturing many of the properties of the scattered field of an arbitrarily shaped body.

The resonant frequency, damping constant and current distribution of some of the natural modes have been calculated for a prolate spheroid and thin wire (see [2] through [9]). However, for an arbitrary incident wave, no general method of calculating the coupling coefficient of each mode seems to exist. A brief review of some previous work on the natural frequencies of bodies is given in section II. It should be noted that for a body of infinite extent, such as an infinitely long, perfectly conducting, circular cylinder, one cannot express the scattered field in terms of damped sinusoidal oscillations alone[11],[12].

The analysis in this note is based on the integral equation derived from the magnetic field formulation. In section III we derive some elementary properties of the operator defined by the kernel of this integral equation. In section IV we first transform this integral equation into a Fredholm integral equation of the second kind where the kernel in the integral equation is of

Hilbert-Schmidt type. This integral equation is then solved by using the Fredholm determinant theory. From this solution we can show that the operator inverse to the integral operator of the magnetic field formulation is an analytic function in the complex frequency plane except at certain points where it has poles. Then, making use of the resolvent of the integral equation, in section V we derive some general properties of the scattered field due to a delta-function incident wave.

From the complex natural frequencies, the nontrivial solutions of the integral equation at the natural frequencies, and the nontrivial solutions of the adjoint integral equation at the natural frequencies, we construct, in section VI, the inverse operator of the integral operator of the magnetic field formulation. Knowing this operator, it is easy to calculate the scattered field due to an arbitrary incident wave.

Explicit expressions for the adjoint operator in the magnetic field formulation are deduced in appendix C. By comparing the integral operators and their adjoints for both the exterior and interior electromagnetic scattering problems, we can see that the homogeneous integral equation of the exterior electromagnetic scattering problem has nontrivial solutions at the resonant frequencies of the interior (cavity) problem.

II. Some Previous Work on Natural Oscillations of Bodies

It is well known (see for example [14]) that the problem of finding the electromagnetic fields inside a cavity resonator can be reduced to the following eigenvalue problem. Find the values of γ and the functions $\underline{E} \not\equiv 0$ that satisfy the differential equations

$$\nabla\times\nabla\times\underline{E} + \gamma^2 \underline{E} = 0, \qquad \nabla\cdot\underline{E} = 0 \tag{2.1}$$

in Ω with the boundary condition $\underline{n}\times\underline{E} = 0$ on S. Here Ω is a simply connected region of finite extent, and $\underline{n}$ is the outward normal of S, the bounding surface of Ω. This eigenvalue problem can be transformed into the problem of finding the eigenvalues and eigenfunctions of a positive, Hermitian, compact operator A. Since A is a compact operator its spectrum is denumerable, and furthermore, since A is a positive Hermitian operator, all its eigenvalues are real and nonnegative. From this it can be shown that the eigenvalues, γ_n, of (2.1) are purely imaginary. It can also be shown that the set of eigenfunctions $\{\underline{E}_n\}$ of (2.1) forms a complete set in Ω. We can then make a Fourier series expansion, with respect to $\{\underline{E}_n\}$, of an arbitrary field due to some sources in Ω. It is then easy to show that the total electromagnetic field in the cavity has two types of singularities in the complex frequency plane. The first type of singularity is due to the singularities of the source. The second type of singularity is a simple pole at the resonant frequencies of the cavity.

The problem of finding the values of γ for which (2.1) has nontrivial solutions when the domain Ω is not bounded is much more complicated. Rellich[15] has considered the eigenvalue problem of finding those $\lambda \neq 0$ and $f \neq 0$ such that

$$\nabla^2 f + \lambda f = 0 \tag{2.2}$$

in Ω and f = 0 on S, when Ω', the complement of Ω, is of finite extent and S is of finite extent. It can be shown that for any nontrivial solution of (2.2) with $\lambda \neq 0$ there exists an R > 0 such that

$$\int_{\Omega_R} |f|^2 \, dV > MR \tag{2.3}$$

where M is a positive constant and Ω_R is the intersection of Ω and a sphere with its center in Ω' and radius R. The problem of finding the spectrum of ∇^2 when both Ω and S are of infinite extent has been treated in [15] through [17]. Equation (2.3) indicates that it is difficult to consider the eigenvalue problem (2.1) for infinite domains. In order to avoid this difficulty, we will here investigate in detail the properties of the solution of an integral equation satisfied by the induced currents on the surface of the scattering body. From this solution it is easy to calculate the scattered field outside the body.

Before we discuss the general case, we will briefly look at the problem of scattering by a perfectly conducting sphere. Making use of the method of separation of variables, this problem can be solved analytically (see for example [10]). The solution of this problem is of great help in understanding scattering from bodies of arbitrary shape. Therefore, scattering of an incident plane wave by a perfectly conducting sphere has been treated in great detail in Interaction Note 88 ([1]). The scattered field due to an incident delta-function plane wave is an analytic function in the complex frequency plane except at certain points where it has simple poles. The location of these poles, γ'_{nm} and γ''_{nm}, are given by

$$\begin{aligned} h_n(ia\gamma'_{nm}) &= 0 \\ h_n(ia\gamma''_{nm}) + ia\gamma''_{nm} h_n(ia\gamma''_{nm}) &= 0 \end{aligned} \qquad (2.4)$$

where

$$h'_n(x) = \frac{dh_n}{dx}(x)$$

and the $h_n(x) = h_n^{(1)}(x)$ are spherical Hankel functions.

The problem of determining the natural* oscillations of a few nonspherical bodies has been treated by many authors. Pocklington[2] calculated the natural oscillations of a thin wire bent into a circular ring by using a differential-

*The natural oscillations are also called free oscillations in the literature.

integral equation (the Pocklington equation). This equation was used by Oseen[4] to calculate the natural oscillations of a straight, thin wire. Hallén[5] also calculated the fundamental natural oscillation of a straight, thin wire by reducing the differential-integral equation derived by Pocklington to an integral equation (the Hallén integral equation). The natural oscillations of a prolate spheroid were first treated by Abrahams[2], who succeeded in finding the wavelength and logarithmic decrement of the fundamental oscillation in the limiting case of a very thin spheroid. By solving the Helmholtz equation in spheroidal coordinates, Page and Adams[7] calculated the wavelength and logarithmic decrement of the fundamental natural oscillation of a prolate spheroid with arbitrary eccentricity. Later, Page[9] calculated the first three axially symmetric natural oscillations of a prolate spheroid.

The analysis in this note is based on the integral equation derived from the magnetic field formulation,

$$\frac{1}{2}\,\underline{j}(\underline{r}) - \int_S \underline{n}(\underline{r})\times[\nabla G(\underline{r},\underline{r}')\times\underline{j}(\underline{r}')]dS' = \underline{j}^{inc}(\underline{r}), \qquad (2.5)$$

where $\underline{j}(\underline{r})$ is the surface current density on the perfectly conducting surface S, $\underline{n}(\underline{r})$ is the outward normal of S, $\underline{j}^{inc} = \underline{n}\times\underline{H}^{inc}$, $\underline{H}^{inc}$ is the magnetic field of the incident wave and G is the free space Green's function,

$$G(\underline{r},\underline{r}') = (4\pi|\underline{r} - \underline{r}'|)^{-1}\, e^{-\gamma|\underline{r}-\underline{r}'|}.$$

Although all our results are based on this formulation, the results are independent of the formulation used when solving the electromagnetic scattering problem. This follows from the uniqueness theorem for real frequencies and the principle of analytic continuation.

For example, the natural frequencies of the axisymmetric modes of a perfectly conducting disk with radius a are given by those γ for which the following integral equation has nontrivial solutions[25]

$$f(u) + \int_0^a K(u,v)f(v)dv = 0, \qquad (2.6)$$

where

$$K(u,v) = \pi^{-1}\{(u - v)^{-1} \sin[\gamma(u - v)] - (u + v)^{-1} \sin[\gamma(u + v)]\}.$$

As a second example, in the case of a thin wire it may be easier to use an integral equation based on the electric field formulation of the electromagnetic scattering problem.

III. Some Elementary Properties of the Integral Equation Derived From the Magnetic Field Formulation

In this section we will deduce some elementary properties of the integral equation derived from the magnetic field formulation. Especially, we will show that the operator defined by the kernel of this integral equation is a bounded operator.

Let Ω be a simply connected region bounded by a surface S, and let Ω be of finite extent, i.e., the diameter of Ω, $D(\Omega)$, is finite,

$$D(\Omega) = \sup|\underline{r} - \underline{r}'| < \infty \tag{3.1}$$

where $\underline{r}$ and $\underline{r}'$ belong to Ω. Next, we assume that we can introduce an orthogonal coordinate system (ξ_1,ξ_2) on S such that there exists a mapping, $S \to S_\xi$, between S and some region S_ξ in the Euclidean (ξ_1,ξ_2)-plane. We also assume that this mapping is one-to-one except at a finite number of points, P_j, on S. Moreover, let the body be such that the Gaussian curvatures exist everywhere on S. All our results remain valid with the somewhat weaker condition that the Gaussian curvatures of S exist except on a finite number of arcs, C_j, each arc being of finite length. However, we find that this weaker condition introduces an uninteresting complication in the proof of our results.

Denote the unit vectors of the coordinate system on S by $\hat{\xi}_1$ and $\hat{\xi}_2$, these two vectors being defined everywhere on S except possibly at P_j. We then have the orthogonality relations

$$\hat{\xi}_i \cdot \hat{\xi}_j = \delta_{ij} \tag{3.2}$$

where δ_{ij} is the Kronecker symbol,

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j. \end{cases}$$

Suppose now that we have an incident electromagnetic wave with all its sources located in the complement of Ω. When S is a perfect conductor it then follows that the electromagnetic fields are zero in Ω. Making use of the magnetic field formulation, and the assumption that S is a perfect conductor, we arrive at the following integral equation for the induced surface current density, $\underline{j}$, on the exterior side of S

$$(\tfrac{1}{2}\,\underline{\underline{I}} - \underline{\underline{L}})\cdot\underline{j} = \underline{j}^{inc}. \tag{3.3}$$

Here $\underline{\underline{I}}$ is the identity operator, $\underline{\underline{L}}$ is an integral operator defined by

$$\underline{\underline{L}}\cdot\underline{j} = \int_S \underline{n}\times(\nabla G\times\underline{j})\,dS, \tag{3.4}$$

$\underline{j}^{inc} = \underline{n}\times\underline{H}^{inc}$, $\underline{H}^{inc}$ is the magnetic field of the incident wave, $\underline{n}$ is the outward unit normal to S, G is the free-space Green's function

$$G(\underline{r},\underline{r}') = (4\pi|\underline{r} - \underline{r}'|)^{-1}\, e^{-\gamma|\underline{r}-\underline{r}'|}, \tag{3.5}$$

$\gamma = -ik$ is the propagation constant of the incident wave, and ∇ operates on the first argument of $G(\underline{r},\underline{r}')$.

Let us now define a Hilbert space $\underline{\mathcal{H}}$, the elements, $\underline{j}$ in $\underline{\mathcal{H}}$, being given by

$$\underline{j} = j_1\hat{\xi}_1 + j_2\hat{\xi}_2, \qquad j_i \in L^2(S) \tag{3.6}$$

where $L^2(S)$ is the Hilbert space of all square integrable functions with support S. Moreover, we have

$$\|\underline{j}\|^2 = \langle\underline{j},\underline{j}\rangle = \int_S (j_1 j_1^* + j_2 j_2^*)\,dS < \infty.$$

Next, we will show that $\underline{\underline{L}}$ is a bounded operator in $\underline{\mathcal{H}}$. We have

$$\underline{\underline{L}}\cdot\underline{j} = \underline{\underline{L}}'\cdot\underline{j} + \underline{\underline{L}}''\cdot\underline{j} \tag{3.7}$$

where

$$\underline{\underline{L}}'\cdot\underline{j} = \int_S \underline{n}\times(\nabla G'\times\underline{j})\,dS$$

$$\underline{\underline{L}}''\cdot\underline{j} = \int_S \underline{n}\times(\nabla G''\times\underline{j})\,dS$$

$$G'(\underline{r},\underline{r}') = (4\pi|\underline{r} - \underline{r}'|)^{-1}$$

and

$$G''(\underline{r},\underline{r}') = G(\underline{r},\underline{r}') - G'(\underline{r},\underline{r}').$$

The functions $G''(\underline{r},\underline{r}')$ and $\nabla G''(\underline{r},\underline{r}')$ are bounded when $\underline{r}$ and $\underline{r}'$ belong to S and γ is finite. Thus, the operator $\underline{\underline{L}}''$ is a bounded operator in $\mathcal{K}$. The triangle inequality gives

$$\|\underline{\underline{L}}\cdot\underline{j}\| \leq \|\underline{\underline{L}}'\cdot\underline{j}\| + \|\underline{\underline{L}}''\cdot\underline{j}\| \leq (\|\underline{\underline{L}}'\| + \|\underline{\underline{L}}''\|)\|\underline{j}\|. \tag{3.8}$$

where $\|\underline{\underline{L}}\| = \sup\{\|\underline{\underline{L}}\cdot\underline{j}\|/\|\underline{j}\|\}$ and $\|\underline{\underline{L}}''\|$ is finite. Moreover, we have

$$\|\underline{\underline{L}}'\cdot\underline{j}\|^2 = \langle\underline{\underline{L}}'\cdot\underline{j},\underline{\underline{L}}'\cdot\underline{j}\rangle = \int_{S\times S\times S}\{\underline{n}(\underline{r})\times[\nabla G'(\underline{r},\underline{r}')\times\underline{j}(\underline{r}')]\}$$

$$\cdot\{\underline{n}(\underline{r})\times[\nabla G'(\underline{r},\underline{r}'')\times\underline{j}^*(\underline{r}'')]\}dSdS'dS'' \tag{3.9}$$

In appendix A we have shown that

$$|\underline{n}(\underline{r})\times[\nabla G'(\underline{r},\underline{r}')\times\underline{j}(\underline{r}')]| \leq [C_1|\underline{r} - \underline{r}'|^{-1} + C_2(\underline{r},\underline{r}')]|\underline{j}(\underline{r}')| \tag{3.10}$$

where C_1 is a positive constant and $C_2(\underline{r},\underline{r}')$ is positive and bounded when $\underline{r}$ and $\underline{r}'$ belong to S. Thus, we have

$$\|\underline{\underline{L}}'\cdot\underline{j}\|^2 \leq \int_{S\times S\times S}[C_1|\underline{r} - \underline{r}'|^{-1} + C_2(\underline{r},\underline{r}')]$$

$$[C_1|\underline{r} - \underline{r}''|^{-1} + C_2(\underline{r},\underline{r}'')]|\underline{j}(\underline{r}')||\underline{j}(\underline{r}'')|dSdS'dS''$$

$$= \int_S dS[\int_S C_2(\underline{r},\underline{r}')|\underline{j}(\underline{r}')|dS']^2$$

$$+ 2\int_S dS\int_S C_1|\underline{r} - \underline{r}'|^{-1}|\underline{j}(\underline{r}')|dS'\int_S C_2(\underline{r},\underline{r}'')|\underline{j}(\underline{r}'')|dS''$$

$$+ C_1^2\int_S|\underline{j}(\underline{r}')|dS'\int_S|\underline{j}(\underline{r}'')|dS''\int_S|\underline{r} - \underline{r}'|^{-1}|\underline{r} - \underline{r}''|^{-1}\,dS. \tag{3.11}$$

It is easy to show that[18]

$$\int_S |\underline{r} - \underline{r}'|^{-1} |\underline{r} - \underline{r}''|^{-1} \, dS \le C_3 \ln|\underline{r}' - \underline{r}''| + C_4 \tag{3.12}$$

where C_3 and C_4 are constants. Because S is of finite extent it follows from the Schwartz inequality and (3.11) and (3.12) that there exists a finite constant, M, such that

$$\|\underline{\underline{L}}'\cdot\underline{j}\|^2 \le M\cdot\|\underline{j}\|^2. \tag{3.13}$$

Combining (3.5), (3.8) and (3.13) we see that $\underline{\underline{L}}$ is a bounded, analytic operator-valued function of γ.

We now go on to investigate $\underline{\underline{L}}$ as $\text{Re}\{\gamma\} \to +\infty$. Define a subsurface, $S_\delta(\underline{r})$, of S around an arbitrary point $\underline{r} \in S$. We then have

$$\underline{\underline{L}}\cdot\underline{j} = \underline{\underline{L}}_\delta\cdot\underline{j} + \underline{\underline{L}}_r\cdot\underline{j} \tag{3.14}$$

where

$$\underline{\underline{L}}_\delta\cdot\underline{j} = \int_{S_\delta} \underline{n}\times(\nabla G\times\underline{j})\,dS$$

$$\underline{\underline{L}}_r\cdot\underline{j} = \int_{S-S_\delta} \underline{n}\times(\nabla G\times\underline{j})\,dS.$$

From (3.11) and (3.12) it follows that for any given $\delta > 0$ there exists S_δ such that

$$\|\underline{\underline{L}}_\delta\cdot\underline{j}\|^2 < \delta^2\|\underline{j}\|^2/2. \tag{3.15}$$

It follows from (3.5), by choosing $\text{Re}\{\gamma\}$ greater than some number $N(\delta)$, that

$$\|\underline{\underline{L}}_r\cdot\underline{j}\|^2 < \delta^2\|\underline{j}\|^2/2. \tag{3.16}$$

Thus, the operator $\frac{1}{2}\underline{\underline{I}} - \underline{\underline{L}} \to \frac{1}{2}\underline{\underline{I}}$ as $\text{Re}\{\gamma\} \to +\infty$ when operating on elements belonging to $\underline{\mathcal{H}}$.

Finally in this section, we will investigate the inverse operator $(\frac{1}{2}\underline{\underline{I}} - \underline{\underline{L}})^{-1}$ as $\mathrm{Re}\{\gamma\} \to +\infty$. We start with the Neumann series

$$(\underline{\underline{I}} - 2\underline{\underline{L}})^{-1} = \underline{\underline{I}} + \sum_{k=1}^{\infty} 2^k \underline{\underline{L}}^k \tag{3.17}$$

which converges for $\mathrm{Re}\{\gamma\} > N(\frac{1}{2})$. Here $\underline{\underline{L}}^1 = \underline{\underline{L}}$ and $\underline{\underline{L}}^k$, $k \geq 2$, is defined by

$$\underline{\underline{L}}^k \cdot \underline{j} = \underline{\underline{L}} \cdot (\underline{\underline{L}}^{k-1} \cdot \underline{j}). \tag{3.18}$$

The triangle inequality gives

$$\|(\underline{\underline{I}} - 2\underline{\underline{L}})^{-1}\| \leq \|\underline{\underline{I}}\| + \sum_{k=1}^{\infty} 2^k \delta^k = \|\underline{\underline{I}}\| + 2\delta(1 - 2\delta)^{-1} \to \|\underline{\underline{I}}\| \quad \text{as} \quad \mathrm{Re}\{\gamma\} \to +\infty. \tag{3.19}$$

From this it follows that

$$(\frac{1}{2}\underline{\underline{I}} - \underline{\underline{L}})^{-1} \to 2\underline{\underline{I}} \quad \text{as} \quad \mathrm{Re}\{\gamma\} \to +\infty. \tag{3.20}$$

In the next section we will go on to determine the analytical properties in the complex frequency plane of the solution of (3.3).

IV. Solution of the Integral Equation From the Fredholm Theory of Hilbert-Schmidt Operators

In this section we will construct a solution of the integral equation (3.3),

$$(\frac{1}{2}\,\underline{\underline{I}} - \underline{\underline{L}})\cdot\underline{j} = \underline{j}^{inc},$$

by solving the integral equation

$$(\frac{1}{4}\,\underline{\underline{I}} - \underline{\underline{L}}^2)\cdot\underline{j} = (\frac{1}{2}\,\underline{\underline{I}} + \underline{\underline{L}})\cdot\underline{j}^{inc} \equiv \underline{f}^{inc}. \tag{4.1}$$

Obviously, any solution of (3.3) also satisfies (4.1). Moreover, in appendix C we have shown that if γ is such that the inverse operator $(\frac{1}{2}\,\underline{\underline{I}} - \underline{\underline{L}})^{-1}$ exists, then the inverse operator $(\frac{1}{2}\,\underline{\underline{I}} + \underline{\underline{L}})^{-1}$ also exists, and vice versa. Thus, the solution of (4.1) coincides with the solution of (3.3) for all those values of γ for which $(\frac{1}{2}\,\underline{\underline{I}} - \underline{\underline{L}})^{-1}$ exists.

Making use of the unit vectors $\hat{\xi}_1$ and $\hat{\xi}_2$ of the coordinate system ξ_1 and ξ_2, we can represent the kernel, $\underline{\underline{K}}(\underline{r},\underline{r}';\gamma)$, of the integral expression

$$(\underline{\underline{L}}\cdot\underline{j})(\underline{r}) \equiv \int_S \underline{\underline{K}}(\underline{r},\underline{r}';\gamma)\cdot\underline{j}(\underline{r}')dS' \tag{4.2}$$

by a 2×2 matrix having elements $K_{i\ell}(\underline{r},\underline{r}';\gamma)$, $i,\ell = 1,2$. Using matrix multiplication, the kernel of the integral expression

$$(\underline{\underline{L}}^2\cdot\underline{j})(\underline{r}) = \int_S \underline{\underline{K}}_2(\underline{r},\underline{r}';\gamma)\cdot\underline{j}(\underline{r}')dS' \tag{4.3}$$

is given by

$$\underline{\underline{K}}_2(\underline{r},\underline{r}';\gamma) = \int_S \underline{\underline{K}}(\underline{r},\underline{r}'';\gamma)\cdot\underline{\underline{K}}(\underline{r}'',\underline{r}';\gamma)dS''. \tag{4.4}$$

The kernel $\underline{\underline{K}}_2(\underline{r},\underline{r}';\gamma)$ is finite except at $\underline{r} = \underline{r}'$ where we have asymptotically

$$\underline{\underline{K}}_2(\underline{r},\underline{r}';\gamma) \sim \underline{\underline{C}}\,\ln|\underline{r} - \underline{r}'| \tag{4.5}$$

(c.f. (3.12)). Thus, the Hilbert-Schmidt norm of $\underline{\underline{L}}^2$, $\|\underline{\underline{L}}^2\|$, defined by

$$||\underline{\underline{L}}^2||^2 = \int_{S\times S} ||\underline{\underline{K}}_2(\underline{r},\underline{r}';\gamma)||^2 \, dSdS', \tag{4.6}$$

is finite for any finite γ since $D(\Omega)$ is finite. Moreover, since each element in $\underline{\underline{K}}(\underline{r},\underline{r}';\gamma)$ is an analytic function of γ it follows that $\underline{\underline{K}}_2(\underline{r},\underline{r}';\gamma)$ is an analytic matrix-valued function of γ. Thus, $\underline{\underline{L}}^2$ is an analytic Hilbert-Schmidt operator-valued function of γ. Notice that the Hilbert-Schmidt norm of $\underline{\underline{L}}$ does not exist since

$$\underline{\underline{K}}(\underline{r},\underline{r}';\gamma) \sim \underline{\underline{C}}'|\underline{r} - \underline{r}'|^{-1} \qquad \text{as } \underline{r}' \to \underline{r}$$

(c.f. (A.1) in appendix A).

In order to make use of the Fredholm determinant theory for the solution of (4.1) we proceed as follows. Making use of the unit vectors $\hat{\xi}_1$ and $\hat{\xi}_2$ the integral equation (4.1) can be written as

$$\frac{1}{4} j_i(\underline{r}) - \sum_{\ell=1}^{2} \int_S B_{i\ell}(\underline{r},\underline{r}';\gamma) j_\ell(\underline{r}')dS' = f_i(\underline{r}), \qquad i = 1,2 \tag{4.7}$$

where

$$B_{i\ell}(\underline{r},\underline{r}';\gamma) = \sum_{k=1}^{2} \int_S K_{ik}(\underline{r},\underline{r}'';\gamma) K_{k\ell}(\underline{r}'',\underline{r}';\gamma)dS''.$$

Let us also define a surface S_1 by

$$S_1 = \{\underline{r} : \underline{r} = \underline{r}_o + \underline{r}',\ \underline{r}' \in S \text{ and } \underline{r}_o = \hat{t} \max_{\underline{r}'' \in S} [\hat{t}\cdot\underline{r}'' + d]\}$$

where $\hat{t}$ is an arbitrary unit vector and $d > 0$. Obviously, S and S_1 are two nonintersecting surfaces. Moreover, define

$$\eta(\underline{r}) = \begin{cases} j_1(\underline{r}), & \underline{r} \in S \\ j_2(\underline{r} - \underline{r}_o), & \underline{r} \in S_1 \end{cases} \tag{4.8}$$

$$\Gamma(\underline{r},\underline{r}';\gamma) = \begin{cases} 4B_{11}(\underline{r},\underline{r}';\gamma), & \underline{r} \in S \text{ and } \underline{r}' \in S \\ 4B_{12}(\underline{r},\underline{r}' - \underline{r}_o;\gamma), & \underline{r} \in S \text{ and } \underline{r}' \in S_1 \\ 4B_{21}(\underline{r} - \underline{r}_o,\underline{r}';\gamma), & \underline{r} \in S_1 \text{ and } \underline{r}' \in S \\ 4B_{22}(\underline{r} - \underline{r}_o,\underline{r}' - \underline{r}_o;\gamma), & \underline{r} \in S_1 \text{ and } \underline{r}' \in S_1 \end{cases} \tag{4.9}$$

$$\phi(\underline{r}) = \begin{cases} 4f_1(\underline{r}), & \underline{r} \in S \\ 4f_2(\underline{r} - \underline{r}_o), & \underline{r} \in S_1 \end{cases} \tag{4.10}$$

Equation (4.1) can then be transformed to the following scalar integral equation

$$\eta(\underline{r}) - \int_{S_2} \Gamma(\underline{r},\underline{r}';\gamma)\eta(\underline{r}')dS' = \phi(\underline{r}), \qquad \underline{r} \in S_2 \tag{4.11}$$

where $S_2 = S \cup S_1$.

From (4.6) it follows immediately that

$$\int_{S_2 \times S_2} |\Gamma(\underline{r},\underline{r}';\gamma)|^2 \, dSdS' < \infty \tag{4.12}$$

which means that we can apply the Fredholm determinant theory when solving the integral equation (4.11). It should be noted here that $B_{ik}(\underline{r},\underline{r}';\gamma)$ does not exist for $\underline{r} = \underline{r}'$. This implies that $\Gamma(\underline{r},\underline{r}';\gamma)$ does not exist on the subset S_3 of $S_2 \times S_2$ where $\underline{r} = \underline{r}'$,

$$S_3 = \{(\underline{r},\underline{r}') : \underline{r} = \underline{r}' \text{ and } (\underline{r},\underline{r}') \in S_2 \times S_2\}.$$

However, we can arbitrarily set $\Gamma(\underline{r},\underline{r}';\gamma) = 0$ on S_3 without affecting the value of

$$\int_{S_2} \Gamma(\underline{r},\underline{r}';\gamma)\eta(\underline{r}')dS' \tag{4.13}$$

when η belongs to the Hilbert space of all square integrable functions with support S_2.

The integral equation (4.11) can now be solved by means of the Fredholm determinant theory[18]. We have

$$\eta(\underline{r}) = \phi(\underline{r}) + \int_{S_2} \Lambda(\underline{r},\underline{r}';\gamma)\phi(\underline{r}')dS' \tag{4.14}$$

where

$$\Lambda(\underline{r},\underline{r}';\gamma) = \Delta(\underline{r},\underline{r}';\gamma)/d(\gamma) \tag{4.15}$$

$$\Delta(\underline{r},\underline{r}';\gamma) = \sum_{m=0}^{\infty} (-1)^m \Delta_m(\underline{r},\underline{r}';\gamma)/m! \tag{4.16}$$

$$d(\gamma) = \sum_{m=0}^{\infty} (-1)^m d_m(\gamma)/m! \tag{4.17}$$

Here, $d_m(\gamma)$ and $\Delta_m(\underline{r},\underline{r}';\gamma)$ can be determined from the recursion formulas

$$d_o(\gamma) = 1 \tag{4.18}$$

$$\Delta_o(\underline{r},\underline{r}';\gamma) = \Gamma(\underline{r},\underline{r}';\gamma) \tag{4.19}$$

$$d_{m+1}(\gamma) = \int_{S_2} \Delta_m(\underline{r},\underline{r};\gamma)dS \tag{4.20}$$

$$\Delta_m(\underline{r},\underline{r}';\gamma) = d_m(\gamma)\Gamma(\underline{r},\underline{r}';\gamma) - m\int_{S_2} \Gamma(\underline{r},\underline{r}'';\gamma)\Delta_{m-1}(\underline{r}'',\underline{r}';\gamma)dS'' \tag{4.21}$$

Since $B_{ij}(\underline{r},\underline{r}';\gamma)$ are analytic functions of γ it follows trivially that $\Delta_m(\underline{r},\underline{r}';\gamma)$ and $d_m(\gamma)$ also are analytic functions of γ. Moreover, since the series expressions (4.16) and (4.17) converge for all values of γ (c.f. [18] and [19]) it follows that $\Lambda(\underline{r},\underline{r}';\gamma)$ and $d(\gamma)$ are analytic functions of γ. From (4.15) it then follows that $\Lambda(\underline{r},\underline{r}';\gamma)$ is an analytic function of γ except at $\gamma = \gamma_n$, where $d(\gamma_n) = 0$. Since $d(\gamma)$ is an analytic function of γ there can only be a finite number of zeros of $d(\gamma)$ in any finite region of the complex γ-plane. It also follows that $\Lambda(\underline{r},\underline{r}';\gamma)$ has a pole at γ_n. The order of the pole is given by the order of the zero of $d(\gamma)$, provided that $\Delta(\underline{r},\underline{r}';\gamma_n) \not\equiv 0$. If $\Delta(\underline{r},\underline{r}';\gamma_n) \equiv 0$, the order of the pole is given by the difference of the order of the zero of $d(\gamma)$ and the order of the zero of $\Lambda(\underline{r},\underline{r}';\gamma)$. It can also be shown that the homogeneous integral equation

$$\eta(\underline{r}) - \int_{S_2} \Gamma(\underline{r},\underline{r}';\gamma)\eta(\underline{r}')dS = 0 \qquad (4.22)$$

has nontrivial solutions for $\gamma = \gamma_n$ (see [19]).

Thus, we have the following solution of the integral equation (4.1)

$$\underline{j} = 4\underline{f}^{inc} + 4d^{-1}\,\underline{\underline{D}}\cdot\underline{f}^{inc} \qquad (4.23)$$

where $\underline{\underline{D}}$ is an integral operator defined by

$$(\underline{\underline{D}}\cdot\underline{f})(\underline{r}) \equiv \int_S \underline{\underline{\Delta}}(\underline{r},\underline{r}';\gamma)\cdot\underline{f}(\underline{r}')dS'. \qquad (4.24)$$

The kernel of the integral expression (4.24), $\underline{\underline{\Delta}}(\underline{r},\underline{r}';\gamma)$, is a 2×2 matrix with elements

$$\begin{aligned}
\Delta_{11}(\underline{r},\underline{r}';\gamma) &= \Lambda(\underline{r},\underline{r}';\gamma)\\
\Delta_{12}(\underline{r},\underline{r}';\gamma) &= \Lambda(\underline{r},\underline{r}'+\underline{r}_o;\gamma)\\
\Delta_{21}(\underline{r},\underline{r}';\gamma) &= \Lambda(\underline{r}+\underline{r}_o,\underline{r}';\gamma)\\
\Delta_{22}(\underline{r},\underline{r}';\gamma) &= \Lambda(\underline{r}+\underline{r}_o,\underline{r}'+\underline{r}_o;\gamma)
\end{aligned} \qquad (4.25)$$

To sum up, we have shown that $(\frac{1}{2}\underline{\underline{I}} - \underline{\underline{L}})^{-1}$ is an analytic operator-valued function of γ except at certain values, γ_n, where it has poles. The singularities in the induced current on the surface of the body, $\underline{j}$, and thus also the scattered electromagnetic field, are due partly to these poles and partly to the singularities of the incident field. The first type of singularity might be called a body pole*, since it is completely determined by the scattering body, and the second type, a waveform singularity. A simple method of representing $(\frac{1}{2}\underline{\underline{I}} - \underline{\underline{L}})^{-1}$ in terms of the body poles will be given in section VI.

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*By analogy to network theory one could also call this type of singularity a natural pole.

V. Solution of the Integral Equation for a Delta-Function Incident Wave

In this section we will solve (3.3) with the assumption that the incident wave is a delta-function plane wave, i.e.,

$$\underline{H}^{inc} = \underline{I}_o \delta(x - ct) \tag{5.1}$$

where $\underline{I}_o$ is the strength of the incident pulse, $x = \hat{e}\cdot\underline{r}$ and $\hat{e}$ is the direction of propagation of the incident wave. In the frequency domain we then have

$$\underline{j}^{inc} = \underline{j}_o c^{-1} e^{-x\gamma} \tag{5.2}$$

where $\underline{j}_o = \underline{n}\times\underline{I}_o$. The solution of the integral equation

$$(\tfrac{1}{2}\underline{\underline{I}} - \underline{\underline{L}})\cdot\underline{j} = \underline{j}^{inc}$$

is then given by

$$\underline{j} = 2\underline{j}^{inc} + 4\underline{\underline{Q}}\cdot\underline{j}^{inc} \tag{5.3}$$

where $\underline{\underline{Q}}$ is an integral operator,

$$(\underline{\underline{Q}}\cdot\underline{f})(\underline{r}) = \int_S \underline{\underline{R}}(\underline{r},\underline{r}';\gamma)\cdot\underline{f}(\underline{r}')dS, \tag{5.4}$$

and the resolvent, $\underline{\underline{R}}(\underline{r},\underline{r}';\gamma)$, satisfies the integral equation

$$\underline{\underline{R}}(\underline{r},\underline{r}';\gamma) - 2\int_S \underline{\underline{K}}(\underline{r},\underline{r}'';\gamma)\cdot\underline{\underline{R}}(\underline{r}'',\underline{r}';\gamma)dS'' = \underline{\underline{K}}(\underline{r},\underline{r}';\gamma). \tag{5.5}$$

It follows from the analysis in section IV that $\underline{\underline{R}}(\underline{r},\underline{r}';\gamma)$ is an analytic function of γ except at certain values where it has poles. From (3.20) we have

$$\underline{j} \to 2\underline{j}^{inc} \quad \text{as} \quad \mathrm{Re}\{\gamma\} \to +\infty. \tag{5.6}$$

In the time domain we have

$$\underline{J}(\underline{r},t) = (2\pi i)^{-1} c \int_C \underline{j}(\underline{r},\gamma) e^{ct\gamma} \, d\gamma$$

$$= 2\underline{J}^{inc}(\underline{r},t) - 2ic\pi^{-1} \int_C (\underline{\underline{Q}} \cdot \underline{j}^{inc})(\underline{r},\gamma) e^{ct\gamma} \, d\gamma \qquad (5.7)$$

where $\underline{J}^{inc} = \underline{n} \times \underline{H}^{inc}$ and the path of integration, C, is parallel to the imaginary axis and to the right of all singularities of $\underline{j}(\underline{r},\gamma)$. From (5.4) and (5.5) it follows that

$$\underline{J}(\underline{r},t) = 0 \quad \text{for} \quad x > ct.$$

Moreover, for any given t let S_+ be the part of S for which $x - ct > 0$ and S_- the part of S for which $x - ct < 0$. We then have

$$(\underline{\underline{Q}}^+ \cdot \underline{f})(\underline{r}) = \int_{S_+} \underline{\underline{R}}(\underline{r},\underline{r}';\gamma) \cdot \underline{f}(\underline{r}') dS' \qquad (5.8)$$

$$(\underline{\underline{Q}}^- \cdot \underline{f})(\underline{r}) = \int_{S_-} \underline{\underline{R}}(\underline{r},\underline{r}';\gamma) \cdot \underline{f}(\underline{r}') dS' \qquad (5.9)$$

and

$$\underline{\underline{Q}} = \underline{\underline{Q}}^+ + \underline{\underline{Q}}^-. \qquad (5.10)$$

Notice that $S_+ = S$ and $S_- = 0$ for $t < t_o$ and that $S_- = S$ and $S_+ = 0$ for $t > t_1$ where

$$t_o = c^{-1} \min_{r \in S} \{x\} \qquad (5.11)$$

$$t_1 = c^{-1} \max_{r \in S} \{x\}. \qquad (5.12)$$

The pulse first hits the scattering object at $t = t_o$ and has just passed it at $t = t_1$. We have, for arbitrary t

$$\underline{J}(\underline{r},t) = 2\underline{J}^{inc}(\underline{r},t) - \underline{J}^+(\underline{r},t) - \underline{J}^-(\underline{r},t) \qquad (5.13)$$

where

$$\underline{J}^{+}(\underline{r},t) = 2ic\pi^{-1} \int_C (\underline{\underline{Q}}^{+}\cdot\underline{j}^{inc})(\underline{r},\gamma)e^{ct\gamma}\, d\gamma \tag{5.14}$$

and

$$\underline{J}^{-}(\underline{r},t) = 2ic\pi^{-1} \int_C (\underline{\underline{Q}}^{-}\cdot\underline{j}^{inc})(\underline{r},\gamma)e^{ct\gamma}\, d\gamma. \tag{5.15}$$

Interchanging the order of integration in (5.14) we get

$$\underline{J}^{+}(\underline{r},t) = 2ic\pi^{-1} \int_{S_+}\int_C \underline{\underline{R}}(\underline{r},\underline{r}';\gamma)\cdot\underline{j}_o(\underline{r}')e^{-x'\gamma}\, e^{ct\gamma}\, d\gamma dS. \tag{5.16}$$

Since $x - ct > 0$ on S_+ if $t < t_1$ and $S_+ = 0$ if $t > t_1$ it follows that

$$\underline{J}^{+}(\underline{r},t) = 0. \tag{5.17}$$

Moreover, from the convolution theorem of the double-sided Laplace transform it follows that

$$\begin{aligned}\underline{J}^{-}(\underline{r}.t) &= 2ic\pi^{-1} \int_{S_-}\int_C \underline{\underline{R}}(\underline{r},\underline{r}';\gamma)\cdot\underline{j}_o(\underline{r}')e^{-x'\gamma}\, e^{ct\gamma}\, d\gamma dS \\ &= -4\int_{S_-} \underline{\underline{R}}_1(\underline{r},\underline{r}';t - x'/c)\cdot\underline{j}_o(\underline{r}')dS\end{aligned} \tag{5.18}$$

where $\underline{\underline{R}}_1(\underline{r},\underline{r}';t)$ is the inverse Laplace transform of $\underline{\underline{R}}(\underline{r},\underline{r}';\gamma)$.

In the next section we will construct the inverse operator $(\frac{1}{2}\underline{\underline{I}} - \underline{\underline{L}})^{-1}$ from the body poles. The integral (5.15) can then be evaluated by means of residue calculus.

VI. Calculation of the Natural Frequencies and Modes from the Integral Equation

In this section we will show how to construct a solution of the inhomogeneous integral equation

$$(\frac{1}{2}\underline{\underline{I}} - \underline{\underline{L}})\cdot\underline{j} = \underline{j}^{inc} \tag{6.1}$$

from the nontrivial solutions of the homogeneous integral equation

$$(\frac{1}{2}\underline{\underline{I}} - \underline{\underline{L}})\cdot\underline{j} = 0. \tag{6.2}$$

The analysis developed here leads to a representation of $(\frac{1}{2}\underline{\underline{I}} - \underline{\underline{L}})^{-1}$ in terms of nontrivial solutions of (6.2) and nontrivial solutions of the adjoint integral equation

$$(\frac{1}{2}\underline{\underline{I}} - \underline{\underline{L}}^{\dagger})\cdot\underline{h} = 0 \tag{6.3}$$

where $\underline{\underline{L}}^{\dagger}$ is the adjoint operator of $\underline{\underline{L}}$.

In section IV we have shown that the inverse operator, $\underline{\underline{A}}^{-1}(\gamma)$, of $\underline{\underline{A}}(\gamma) \equiv \frac{1}{2}\underline{\underline{I}} - \underline{\underline{L}}$ is an analytic operator-valued function on the complex γ-plane except at a countable number of points, γ_n, where $\underline{\underline{A}}^{-1}(\gamma)$ has poles of some finite order $P = P(n)$. Another way of expressing this is: for $\gamma = \gamma_n$ there exists a function $\underline{j}_n \neq 0$, such that

$$\underline{\underline{A}}(\gamma_n)\cdot\underline{j}_n = 0. \tag{6.4}$$

Let $\Sigma(\gamma)$ be the spectral set of $\underline{\underline{A}}(\gamma)$ so that

$$\Sigma(\gamma) = \sigma(\underline{\underline{A}}) = \{\lambda : \underline{\underline{A}}(\gamma) - \lambda I \text{ has no inverse}\} \tag{6.5}$$

From (6.4) it follows that

$$0 \in \Sigma(\gamma_n). \tag{6.6}$$

In Appendix C we have shown that

$$\sigma(\underline{\underline{A}}^{\dagger}) = \Sigma^{*}(\gamma) \tag{6.7}$$

where

$$\Sigma^{*}(\gamma) = \{\gamma : \gamma^{*} \in \Sigma(\gamma)\}. \tag{6.8}$$

Comparing (6.6) and (6.8) it follows that

$$0 \in \Sigma^{*}(\gamma_n). \tag{6.9}$$

From equations (6.4) through (6.9) it now follows that there exists $\underline{h}_n$ such that

$$\underline{\underline{A}}^{\dagger}(\gamma_n)\cdot\underline{h}_n^{*} = 0 \tag{6.10}$$

and

$$\underline{\underline{A}}^{T}(\gamma_n)\cdot\underline{h}_n = 0. \tag{6.11}$$

where $\underline{\underline{A}}^{T}(\gamma) = \underline{\underline{A}}^{\dagger *}(\gamma)$.

Comparing equations (6.4), (6.11), and (C.12) we notice that γ_n can either correspond to an exterior resonance or an interior resonance (cavity mode). In the latter case, γ_n is a purely imaginary number (see for example p. 211 in [14]). Thus, by finding all γ_n such that

$$\underline{\underline{A}}(\gamma_n)\cdot\underline{j}_n = 0, \qquad \underline{j}_n \neq 0$$

we will find both the interior and exterior resonances despite the fact that the integral equation (6.1) originally has been derived for the exterior scattering problem. The fact that the homogeneous integral equation of the exterior scattering problem has nontrivial solutions at the interior resonant frequencies makes the solution of (6.1) difficult at those frequencies[13].

In Appendix F we give a method of differing the exterior resonances from the interior resonances. However, we want to point out here that if γ_n is not purely imaginary, then γ_n is the natural frequency of an exterior mode. We also notice that $\mathrm{Re}\{\gamma_n\} \le 0$ which follows from the physical fact that the currents induced on the surface of the body due to a delta function plane wave can not be exponentially increasing in time (see (6.36)). Let us also normalize $\underline{j}_n$ and $\underline{h}_n$ in the following way

$$\|\underline{j}_n\|^2 = \langle \underline{j}_n, \underline{j}_n \rangle = 1$$

and (6.12)

$$\|\underline{h}_n\|^2 = \langle \underline{h}_n, \underline{h}_n \rangle = 1.$$

Of course this means that both $\underline{j}_n$ and $\underline{h}_n$ belong to $\mathcal{H}$, the Hilbert space defined in section III.

Having the current distribution, $\underline{j}_n$, on S of one mode it is now easy to calculate other field quantities. We have the following expression for the vector potential, $\underline{A}_n$, and the scalar potential, Φ_n, of this mode. We have

$$\underline{A}_n(\underline{r}) = \mu_o \int_S G(\underline{r},\underline{r}';\gamma_n)\underline{j}_n(\underline{r}')dS$$

(6.13)

$$\Phi_n(\underline{r}) = Z_o\gamma_n^{-1} \int_S G(\underline{r},\underline{r}';\gamma_n)\nabla\cdot\underline{j}_n(\underline{r}')dS$$

where

$$G(\underline{r},\underline{r}';\gamma_n) = (4\pi|\underline{r} - \underline{r}'|)^{-1} e^{-\gamma_n|\underline{r}-\underline{r}'|}.$$

For the electric field, $\underline{E}_n$, and magnetic field, $\underline{H}_n$, we have

$$\underline{E}_n = c\gamma_n\underline{A}_n - \nabla\Phi_n$$

and (6.14)

$$\underline{H}_n = \nabla\times\underline{A}_n.$$

For exterior natural modes such that $\mathrm{Re}\{\gamma_n\} < 0$ it follows from (6.13) and (6.14) that all field quantities are exponentially growing in space far away from the body. This will not be of any importance in solving transient electromagnetic problems, because of the causality condition imposed when solving these problems.

Next, we will go on to solve the inhomogeneous integral equation (6.1) when the incident wave is a delta-function (in time) plane wave. We have (c.f. (5.1))

$$\underline{\underline{A}}(\gamma)\cdot\underline{j} = \underline{u}e^{-\gamma x} \tag{6.15}$$

where $\underline{u} = c^{-1}\,\underline{j}_o$. By finding the solution of (6.15) we can construct the inverse operator $\underline{\underline{A}}^{-1}(\gamma)$. It is then easy to obtain the solution of (6.1) for an arbitrary incident field. We have

$$\langle \underline{\underline{A}}(\gamma)\cdot\underline{j},\underline{h}_n\rangle = \langle \underline{j}^{inc},\underline{h}_n\rangle \tag{6.16}$$

and

$$\langle \underline{\underline{A}}(\gamma_n)\cdot\underline{j},\underline{h}_n\rangle = \langle \underline{j},\underline{\underline{A}}^{\dagger}(\gamma_n)\cdot\underline{h}_n\rangle = 0 \tag{6.17}$$

so that

$$\langle [\underline{\underline{A}}(\gamma) - \underline{\underline{A}}(\gamma_n)]\cdot\underline{j},\underline{h}_n\rangle = \langle \underline{j}^{inc},\underline{h}_n\rangle. \tag{6.18}$$

Suppose that γ_n is a pole of order P(n). We can then represent $\underline{j}$ in the following form in a neighborhood of γ_n:

$$\underline{j} = \sum_{p=1}^{P}\sum_{m=1}^{M} C_{npm}(\gamma - \gamma_n)^{-p}\,\underline{j}_{npm} + \underline{j}_n'. \tag{6.19}$$

Here C_{npm} are constants, $C_{nPm} \neq 0$, and $\underline{j}_n'$ is a bounded function. Furthermore, M is some finite degeneracy number, M = M(n,p), such that for fixed n and p, M is the smallest number of independent solutions, $\underline{j}_{npm}$. We will later discuss how to choose these M linearly independent solutions. We then expand $\underline{\underline{A}}(\gamma)$ and $\underline{j}^{inc}$ in a Taylor series around γ_n,

$$\underline{\underline{A}}(\gamma) - \underline{\underline{A}}(\gamma_n) = \sum_{q=1}^{\infty} (\gamma - \gamma_n)^q\,\underline{\underline{B}}_{nq}, \tag{6.20}$$

$$\underline{j}^{inc} = \sum_{q=0}^{\infty} (\gamma - \gamma_n)^q \underline{j}_{nq}^{inc} \tag{6.21}$$

where $\underline{\underline{B}}_{nq}$ is an integral operator such that

$$(\underline{\underline{B}}_{nq}\cdot\underline{f})(\underline{r}) = (-1)^q(4\pi q!)^{-1} \int_S \underline{n}(\underline{r})\times\{\nabla[|\underline{r} - \underline{r}'|^{q-1} e^{-\gamma_n|\underline{r}-\underline{r}'|}]\times\underline{f}(\underline{r}')\}dS' \tag{6.22}$$

and

$$\underline{j}_{nq}^{inc} = (q!)^{-1}(-x)^q e^{-x\gamma_n} \underline{u}. \tag{6.23}$$

Making use of (6.19) through (6.21) we can expand (6.18) in a Laurent series around $\gamma = \gamma_n$. By identifying coefficients in this series expansion we can determine C_{npm}.

Although we have no proof, it is our contention that $\underline{\underline{A}}^{-1}(\gamma)$ has only simple poles when the scattering body is perfectly conducting. In the following we will concentrate on the special but important case where P = 1. The case P = 2 is treated in Appendix E.

For P = 1 we can drop one index so that

$$C_{n1m} = C_{nm}, \quad \underline{j}_{n1m} = \underline{j}_{nm}, \quad \underline{\underline{B}}_{n1} = \underline{\underline{B}}_n.$$

Because γ_n is a simple pole, we have M linearly independent solutions, $\underline{j}_{nm}$, such that

$$\underline{\underline{A}}(\gamma_n)\cdot\underline{j}_{nm} = 0$$

and

$$\underline{\underline{B}}_n\cdot\underline{j}_{nm} \neq 0, \qquad 1 \le m \le M. \tag{6.24}$$

The adjoint homogeneous integral equation has also M independent solutions $\underline{h}_{nm}$ such that

$$\underline{\underline{A}}^{\dagger}(\gamma_n)\cdot\underline{h}_{nm} = 0$$

and

$$\underline{\underline{B}}_n^{\dagger} \cdot \underline{h}_{nm} \neq 0. \tag{6.25}$$

Because $\underline{j}_{nm}$ and $\underline{h}_{nm}$ are linearly independent and $\underline{\underline{B}}_n \cdot \underline{j}_{nm} \neq 0$ it follows from the Gram-Schmidt procedure that we can choose $\underline{j}_{nm}$ and $\underline{h}_{nm}$ such that

$$\langle \underline{\underline{B}}_n \cdot \underline{j}_{nm_1}, \underline{h}_{nm_2} \rangle \begin{cases} = 0 & \text{for} \quad m_1 \neq m_2 \\ \neq 0 & \text{for} \quad m_1 = m_2 \end{cases} \tag{6.26}$$

Thus, we have

$$C_{nm} = \langle \underline{j}_n^{inc}, \underline{h}_{nm} \rangle / \langle \underline{\underline{B}}_n \cdot \underline{j}_{nm}, \underline{h}_{nm} \rangle \tag{6.27}$$

where

$$\underline{j}_n^{inc} = \underline{u} e^{-x\gamma_n}.$$

For the special case where $\underline{\underline{A}}^{-1}(\gamma)$ has a simple pole at γ_n this leads us to the following dyadic representation of $\underline{\underline{A}}^{-1}(\gamma)$ in a neighborhood of γ_n,

$$\underline{\underline{A}}^{-1}(\gamma) = \sum_m (\gamma - \gamma_n)^{-1} [\langle \underline{\underline{B}}_n \cdot \underline{j}_{nm}, \underline{h}_{nm} \rangle]^{-1} \, \underline{j}_{nm} \, \underline{h}_{nm}^{*} + \underline{\underline{F}}_n(\gamma) \tag{6.28}$$

where $\underline{\underline{F}}_n(\gamma)$ is an analytic operator in a neighborhood of γ_n. The expression (6.28) is, of course, independent of the way we normalize $\underline{j}_{nm}$ and $\underline{h}_{nm}$.

From (6.28) it follows that, anywhere in the γ-plane,

$$\underline{\underline{A}}^{-1}(\gamma) = \sum_{n,m} (\gamma - \gamma_n)^{-1} [\langle \underline{\underline{B}}_n \cdot \underline{j}_{nm}, \underline{h}_{nm} \rangle]^{-1} \, \underline{j}_{nm} \, \underline{h}_{nm}^{*} + \underline{\underline{E}}_1(\gamma) \tag{6.29}$$

where $\underline{\underline{E}}_1(\gamma)$ is an entire, operator-valued function of γ. If it can be shown that $\underline{\underline{A}}^{-1}(\gamma)$ is bounded except for its poles and that $\underline{\underline{A}}^{-1}(0)$ exists then the following more explicit representation of $\underline{\underline{A}}^{-1}(\gamma)$ may be written

$$\underline{\underline{A}}^{-1}(\gamma) = \sum_{n,m} [(\gamma - \gamma_n)^{-1} + \gamma_n^{-1}][\langle \underline{\underline{B}}_n \cdot \underline{j}_{nm}, \underline{h}_{nm} \rangle]^{-1} \, \underline{j}_{nm} \, \underline{h}_{nm}^{*} + \underline{\underline{A}}^{-1}(0). \tag{6.30}$$

Now if $\underline{h}_{nm}$ is the tangential magnetic field on S of an interior mode, and if $\underline{j}^{inc}$ has its sources outside Ω, it follows from Appendix F that

$$\langle \underline{j}_n^{inc}, \underline{h}_{nm} \rangle = 0. \tag{6.31}$$

It then follows from (6.29) and (6.31) that, if the incident field has its sources outside Ω, the solution of (6.15) may be written

$$\underline{j}(\underline{r},\gamma) = \sum_{ext} (\gamma - \gamma_n)^{-1} [\langle \underline{\underline{B}}_n \cdot \underline{j}_{nm}, \underline{h}_{nm} \rangle]^{-1} \langle \underline{j}_n^{inc}, \underline{h}_{nm} \rangle \underline{j}_{nm} + \underline{\underline{E}}_2(\gamma) \cdot \underline{j}^{inc}(\gamma) \tag{6.32}$$

where $\underline{\underline{E}}_2(\gamma)$ is another entire, operator-valued function of γ and $\sum_{ext}$ denotes summation over external modes only. If $\underline{\underline{E}}_2(\gamma)$ is zero the following simple time-domain representation of the current due to a delta function incident wave may be written

$$\underline{J}(\underline{r},t) = U(ct - x_o) \sum_{ext} \langle \underline{n} \times \underline{I}_o e^{\gamma_n (x - x_o)}, \underline{h}_{nm} \rangle [\langle \underline{\underline{B}}_n \cdot \underline{j}_{nm}, \underline{h}_{nm} \rangle]^{-1} \underline{j}_{nm} e^{-\gamma_n (x_o - ct)} \tag{6.33}$$

where $U(x)$ is a unit step function and x_o is the x-coordinate of the first point on the body hit by the incident wave. It was shown in section V that $\underline{J}(\underline{r},t) = 0$ for $x > ct$ so the argument of the unit step function could be taken as $ct - x$. Different representations of $\underline{J}(\underline{r},t)$ have been discussed in [1].

From (6.33) it follows that $\mathrm{Re}\{\gamma_n\} \leq 0$ for the exterior modes since $\underline{J}(\underline{r},t)$ has to be finite when $t \to \infty$. For the interior resonances $\mathrm{Re}\{\gamma_n\} = 0$. From these two facts it is clear that $\underline{\underline{A}}^{-1}(\gamma)$ is analytic in the right half plane.

VII. Concluding Remarks

The operator determining the field scattered by a perfectly conducting body of finite extent is an analytic function in the complex frequency plane except at certain points where the operator has poles of some finite order. Each pole corresponds to a natural frequency of the structure. The location of these poles can be found by finding all those complex frequencies for which the homogeneous integral equation for the surface current has nontrivial solutions. We have also seen that the integral equation has nontrivial solutions at the resonances of the interior (cavity) problem although it was derived for the exterior scattering problem. A method of differing exterior resonances from interior ones was given. For the special case where there are only simple poles, a dyadic series representation of the operator determining the induced current was constructed from the natural frequencies and the nontrivial solutions of the integral equation at those frequencies.

In this note we have conjectured that the inverse operator of the magnetic field formulation only has simple poles. Hopefully, we will, in a future note, be able to shed some light on this question as well as the question of the behavior of the inverse operator for large values of γ in the left semi-plane. Furthermore, in this note we have only discussed electromagnetic scattering by a perfectly conducting body. It is our intention to investigate scattering by imperfectly conducting bodies in a future note. When the permittivity, permeability and conductivity (ε,μ,σ) of the scattering body are entire functions in the complex frequency plane, we expect that the only singularities of the operator determining the induced current are poles. However, the poles might not be simple. Scattering of a delta-function plane wave by an imperfectly conducting body has been discussed in Interaction Note 88. The analysis there is based on a zoning technique where the scattering body is divided into a finite number of small sized volume elements. A set of algebraic equations for the induced current in each volume element of the body is derived. The analytical properties in the complex frequency plane of the solution of this set of equations are also discussed. We intend to approach the problem of scattering by an imperfectly conducting body of finite size by formulating a volume integral equation for the induced current density in the scattering body.

The results reported in this note were attained by using the integral equation derived from the magnetic field formulation of the electromagnetic scattering problem. However, it may be easier to solve certain problems by using other formulations, such as the electric field formulation for thin wires. The practical importance of these results is that they point to a fast method of solving transient electromagnetic scattering problems involving those scatterers for which a few terms in the dyadic series representation of the inverse operator are dominant. Based on the theory presented in this note we are currently undertaking a numerical study of the natural modes of some structures. In this study we will determine the complex resonant frequency and the current distribution of each natural mode for various body shapes. We will also calculate the coupling coefficient of each natural mode to a given incident field. The result of these calculations will be reported in a future note.

Appendix A

The Singularity of the Kernel in the Integral Equation Derived From the Magnetic Field Formulation

In this appendix we will investigate the singularity in the kernel of the integral equation (3.3), i.e., the behavior of $g(\underline{r},\underline{r}')$,

$$g(\underline{r},\underline{r}') = \underline{n}(\underline{r})\times[\nabla|\underline{r} - \underline{r}'|^{-1}\times\underline{f}(\underline{r}')], \tag{A.1}$$

around $\underline{r}' = \underline{r}$ under the assumption that $f(\underline{r}')$ is continuous at $\underline{r}$ (c.f. (3.10)). Let P and P' be two points on S having the position vectors $\underline{r}$ and $\underline{r}'$, respectively (see figure 1). The outward normal to the surface S at P is $\underline{n}(\underline{r})$. Let Σ denote the plane spanned by the vectors $\underline{n}(\underline{r})$ and $\underline{s}(\underline{r},\underline{r}')$, $\underline{s}(\underline{r},\underline{r}') = (\underline{r} - \underline{r}')/|\underline{r} - \underline{r}'|$. The intersection between Σ and S is a curve C. We assume that C is a smooth curve around P so that for example the curvature, κ, of C exists at P.

We have

$$\underline{g}(\underline{r},\underline{r}') = [\underline{n}(\underline{r})\cdot\underline{f}(\underline{r}')]\nabla|\underline{r} - \underline{r}'|^{-1} - [\underline{n}(\underline{r})\cdot\nabla|\underline{r} - \underline{r}'|^{-1}]\underline{f}(\underline{r}'). \tag{A.2}$$

Since C is a smooth curve between P and P' we have

$$\underline{n}(\underline{r})\cdot\underline{f}(\underline{r}') = \kappa|\underline{r} - \underline{r}'|f_1(\underline{r},\underline{r}') + 0(|\underline{r} - \underline{r}'|^2) \tag{A.3}$$

where

$$f_1(\underline{r},\underline{r}') = |[\underline{n}(\underline{r})\times\underline{s}(\underline{r},\underline{r}')]\cdot\underline{f}(\underline{r}')| = \theta'|\underline{f}(\underline{r}')| \tag{A.4}$$

and $0 < \theta' < 1$. Moreover,

$$\nabla|\underline{r} - \underline{r}'|^{-1} = - \underline{s}(\underline{r},\underline{r}')/|\underline{r} - \underline{r}'|^2 \tag{A.5}$$

so that

$$\underline{n}(\underline{r})\cdot\nabla|\underline{r} - \underline{r}'|^{-1} = - [\underline{n}(\underline{r})\cdot\underline{s}(\underline{r},\underline{r}')]/|\underline{r} - \underline{r}'|^2 = - \kappa|\underline{r} - \underline{r}'|^{-1} + C'(\underline{r},\underline{r}') \tag{A.6}$$

where $C'(\underline{r},\underline{r}')$ is bounded around $\underline{r} = \underline{r}'$. It now follows that

$$|\underline{n}\times[\nabla|\underline{r} - \underline{r}'|^{-1}\times\underline{f}(\underline{r})]| = [C_1|\underline{r} - \underline{r}'|^{-1} + C_2(\underline{r},\underline{r}')]|\underline{f}(\underline{r}')| \qquad (A.7)$$

where

$$C_1 = \kappa(1 + \theta), \qquad - 1 \leq \theta \leq 1 \qquad (A.8)$$

and $C_2(\underline{r},\underline{r}')$ is bounded.

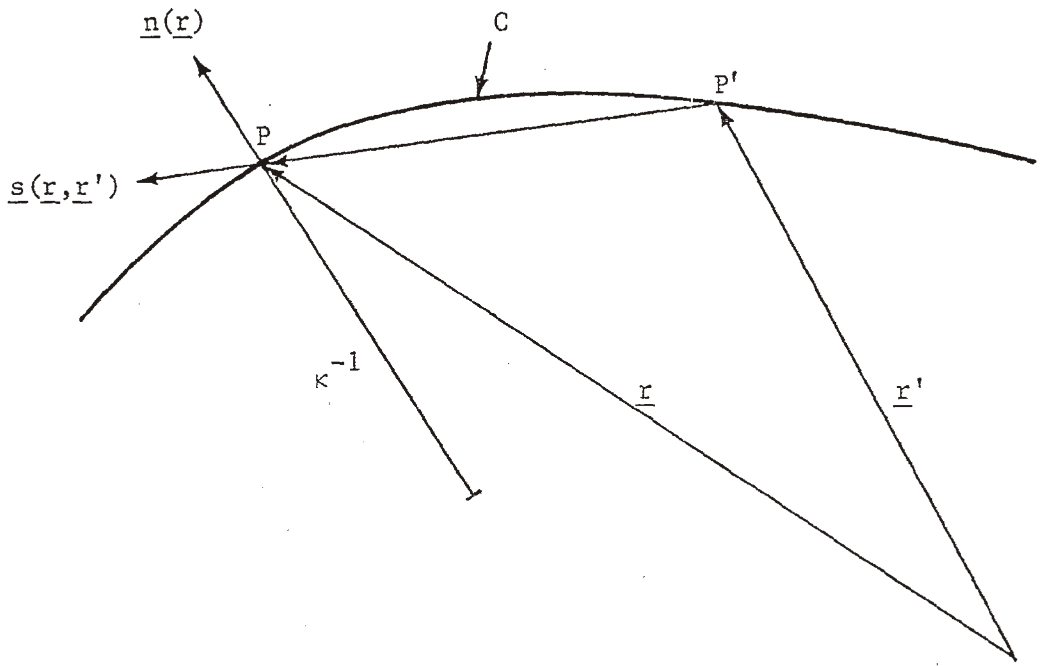

Figure 1. The local geometry of the surface.

Appendix B

Selected Topics From the Theory of Linear Operators in Hilbert Space

In this appendix we will first introduce the notation of Hilbert space and give some examples of different Hilbert spaces. We will then introduce the concept of linear operators and give some examples. Of course, we only intend to give a brief review of some of the concepts from the theory of linear operators in Hilbert space that we have used in various parts of this note. For a detailed study of this subject we refer the interested reader to [20] through [23].

A. Definition of an Abstract Hilbert Space

We start with the postulates defining a Hilbert space. A set of elements f,g,h,... which possesses the properties (I), (II) and (III) listed below will be called an (abstract) Hilbert space and will be denoted by $\mathcal{H}$.

I. $\mathcal{H}$ is a linear space, that is, the operations of addition of elements belonging to $\mathcal{H}$ and multiplication by complex numbers, $\lambda, \mu, \ldots$, of an element in $\mathcal{H}$ follows the rules

1. $f + g = g + f$,
2. $f + (g + h) = (f + g) + h$,
3. there exists an element 0 in $\mathcal{H}$ having the property $f + 0 = f$ for all f in $\mathcal{H}$,
4. for every f in $\mathcal{H}$ there exists an element $-f$ in $\mathcal{H}$ such that $f + (-f) = 0$,
5. $(\lambda + \mu)f = \lambda f + \mu f$
6. $(\lambda\mu)f = \lambda(\mu f)$
7. $1 \cdot f = f$.

II. $\mathcal{H}$ is a metric space whose metric is derived from the scalar product. The scalar product of two elements, f and g in $\mathcal{H}$, is a complex number denoted by $\langle f,g \rangle$ and has the following properties

1. $\langle \lambda f,g \rangle = \lambda \langle f,g \rangle$,
2. $\langle g,f \rangle = \langle f,g \rangle^*$,
3. $\langle f + g,h \rangle = \langle f,h \rangle + \langle g,h \rangle$,

4. $\langle f,f\rangle > 0$ if $f \neq 0$,

5. $\langle f,f\rangle = 0$ if and only if $f = 0$,

6. the norm of f, denoted by $\|f\|$, is defined by $\|f\| = \langle f,f\rangle^{\frac{1}{2}}$,

7. the distance between the elements f and g is $\|f - g\|$.

III. $\mathcal{H}$ is a complete space in the sence that if a sequence of elements $\{f_i\}$, f_i belongs to $\mathcal{H}$, satisfies the Cauchy condition

$$\|f_i - f_k\| \to 0 \quad \text{as} \quad i,k \to \infty$$

then there exists an element f of $\mathcal{H}$ such that

$$\|f - f_i\| \to 0 \quad \text{as } i \to \infty.$$

B. Examples of Hilbert Spaces

We will now give some examples of spaces that satisfy the properties I-III listed above. One such space is $\mathcal{H}_o$, the space consisting of all sequences of complex numbers $(a_1,a_2,\ldots)$ such that

$$\sum_{k=1}^{\infty} |a_k|^2$$

exists. It follows trivially that $\mathcal{H}_o$ is a linear space. Moreover, let f denote the sequency $(x_1,x_2,\ldots)$ and g the sequence $(y_1,y_2,\ldots)$. The scalar product, $\langle f,g\rangle$, is then defined by

$$\langle f,g\rangle = \sum_{i=1}^{\infty} x_i\, y_i^* \tag{B.1}$$

and

$$\|f\| = \{\sum_{i=1}^{\infty} |x_i|^2\}^{\frac{1}{2}}.$$

From the Schwartz inequality and (B.1) it follows that $\langle f,g\rangle$ is a finite number when f and g belongs to $\mathcal{H}$.

Another Hilbert space is $L^2(I)$ where I is some finite interval (a,b), and the elements, f, in $L^2(I)$ are all complex-valued functions, f(x), defined on I and such that

$$\int_a^b |f(x)|^2 \rho(x)dx < \infty$$

where $\rho(x)$ is a positive density function. The scalar product is defined by

$$\langle f,g \rangle = \int_a^b f(x)g^*(x)\rho(x)dx$$

so that

$$\|f\|^2 = \int_a^b |f(x)|^2 \rho(x)dx.$$

We leave it to the reader to show that $L^2(I)$ satisfies all the postulates of a Hilbert space.

In section III we have used the Hilbert space $\underline{\mathcal{H}}$ consisting of all elements $\underline{f} = (f_1,f_2)$ such that f_1 and f_2 belongs to $L^2(S)$, where $S = \partial\Omega$, and Ω is some region of finite extent in the three-dimensional Euclidean space, and the elements g in $L^2(S)$ are all complex-valued functions $g(\underline{r})$ such that

$$\int_S |g(\underline{r})|^2 \, dS < \infty.$$

The scalar product in $\underline{\mathcal{H}}$ is defined by

$$\langle \underline{f},\underline{h} \rangle = \int_S [f_1(\underline{r})g_1^*(\underline{r}) + f_2(\underline{r})g_2^*(\underline{r})]dS.$$

C. Two Types of Convergence on $\mathcal{H}$

Different types of convergence are used in the theory of Hilbert space. We will here define two types of convergence

1. the sequence of elements $\{f_i\}$ of $\mathcal{H}$ converges strongly to an element f of $\mathcal{H}$ if $\|f - f_i\| \to 0$ as $i \to \infty$, or $f_i \to f$,

2. the sequence of elements $\{f_i\}$ of $\mathcal{H}$ converges weakly to an element f of $\mathcal{H}$ if for any g of $\mathcal{H}$ we have $\langle f_i, g\rangle \to \langle f, g\rangle$ as $i \to \infty$, or $f_i \rightharpoonup f$.

D. Fundamental Sets on $\mathcal{H}$

A set of elements $\{f_k\}$ of $\mathcal{H}$ such that $\|f_k\| = 1$ and $\langle f_k, f_j\rangle = 0$ for $k \neq j$ is said to form an orthonormal system in $\mathcal{H}$. The set $\{f_k\}$ is called a fundamental set if any element g of $\mathcal{H}$ can be expanded in terms of f_i as

$$g = \sum_k \langle g, f_k\rangle f_k$$

and the coefficients $\langle g, f_k\rangle$ are called the Fourier coefficients of g with respect to $\{f_k\}$. The set $\{f_k\}$,

$$f_k(x) = \pi^{-\frac{1}{2}} \exp(ikx),$$

forms a fundamental set on $L^2(-\pi,\pi)$. We have

$$g = \sum_{k=-\infty}^{\infty} \langle g, f_k\rangle f_k$$

where

$$\langle g, f_k\rangle = \int_{-\pi}^{\pi} g(x) f_k^*(x)dx = \pi^{-\frac{1}{2}} \int_{-\pi}^{\pi} g(x)\exp(-ikx)dx.$$

E. Isomorphism Between Two Hilbert Spaces

We will now go on to consider the concept of isomorphism between two Hilbert spaces. Two Hilbert spaces, $\mathcal{H}$ and $\mathcal{H}'$, are said to be isomorphic if there is a one-to-one correspondence between their elements

$$f \leftrightarrow f', \qquad g \leftrightarrow g'$$

implies that

1. $f + g \leftrightarrow f' + g'$
2. $\lambda f \leftrightarrow \lambda f'$
3. $\langle f,g\rangle \leftrightarrow \langle f',g'\rangle$.

Here f and g belong to $\mathcal{H}$ and f' and g' to $\mathcal{H}'$. Let g belong to $L^2(I)$ and let $\{f_i\}$ be a fundamental set on $L^2(I)$. It then follows that

$$\sum_k |a_k|^2 < \infty, \qquad a_k = \langle g,f_k\rangle$$

so that the sequence $(a_1,a_2,\ldots)$ is an element in $\mathcal{H}_o$. Moreover, from the Riesz-Fischer theorem it follows that for each sequence $(a_1,a_2,\ldots)$ of $\mathcal{H}_o$ there is an f belonging to $L^2(I)$ with Fourier coefficients $a_1,a_2,\ldots$ This correspondence between $L^2(I)$ and $\mathcal{H}_o$ is one-to-one. It also follows from the Parseval's equality that

$$\langle g,h\rangle = \sum a_k\, b_k^*$$

where h belongs to $L^2(I)$ and $b_k = \langle h,f_k\rangle$. From this it now follows immediately that $\mathcal{H}_o$ is isomorphic to $L^2(I)$.

F. Different Types of Operators in $\mathcal{H}$

We will now go on to discuss operators in the Hilbert space. A mapping T which associates with each element f belonging to a subset $\mathcal{E}$ of a Hilbert space $\mathcal{H}$ an element g belonging to a Hilbert space $\mathcal{H}'$ is said to be an operator with domain of definition $\mathcal{E}$.

I. The operator T is called linear if
   1. $\mathcal{E}$ is a linear manifold, i.e., if f_1 and f_2 belong to $\mathcal{E}$, so does $\lambda_1 f_1 + \lambda_2 f_2$ for any complex numbers λ_1 and λ_2,
   2. $T(\lambda_1 f_1 + \lambda_2 f_2) = \lambda_1 Tf_1 + \lambda_2 Tf_2$.

II. The operator T is called bounded if the domain of T is all of $\mathcal{H}$ and if there exists a finite constant C such that

$$\|Tf\| \le C\|f\| \tag{B.2}$$

for all f belonging to $\mathcal{H}$. The smallest value such that (B.2) is valid is called the norm of T, and is denoted by $\|T\|$. We have

$$\|T\| = \sup_{\|f\|=1} \|Tf\|.$$

III. An operator T is called closed if it has the property that for every sequence $\{f_n\}$ of elements in $\mathcal{E}$ such that

$$f_n \to f \quad \text{and} \quad Tf_n \to g$$

the limit element also belongs to $\mathcal{E}$ and $Tf = g$.

IV. An operator T is called compact if it transforms any weakly convergent sequence $\{f_n\}$ in $\mathcal{H}$ into a strongly convergent sequence $\{g_n\}$ in $\mathcal{H}'$ and if the domain of definition of T is $\mathcal{H}$. Thus, if T is compact we have

$$f_n \rightharpoonup f \quad \text{and} \quad g_n \to g$$

where

$$Tf_n = g_n \quad \text{and} \quad Tf = g.$$

Let $\{f_i\}$ be a fundamental set in $\mathcal{H}$. It can then be shown that the reduced transformations T_n, defined by

$$T_n g = \sum_{i,k=1}^{n} \langle g, f_i \rangle \langle Tf_i, f_k \rangle f_k$$

tend uniformly to the transformation T when $n \to \infty$.

V. Let $\{f_i\}$ be a fundamental set in $\mathcal{H}$. A bounded linear operator T in $\mathcal{H}$ is said to be a Hilbert-Schmidt operator if

$$\|T\|_h^2 = \sum_i \|Tf_i\|^2$$

is finite. The number $\|T\|_h$ is sometimes called the Hilbert-Schmidt norm of T. It can be shown that any Hilbert-Schmidt operator also is a compact operator.

G. Examples of Operators in $\mathcal{H}$

We will now give some examples of linear operators in different Hilbert spaces. First consider an operator that maps the Hilbert space $\mathcal{H}_o$ into itself. Symbolically we write

$$Tf \doteq g$$

where $f = (a_1, a_2, \ldots)$ and $g = (b_1, b_2, \ldots)$ both belong to $\mathcal{H}_o$. The operator T can be represented by an infinite dimensional matrix with elements t_{ij} so that

$$\sum_j t_{ij} a_j = b_i.$$

Next consider an operator T in $L^2(I)$ defined by

$$(Tg)(x) = \int_a^b K(x,y)g(y)dy.$$

Assume that T is a Hilbert-Schmidt operator so that

$$Tg = \sum_{i,j} \langle g, f_i \rangle \langle Tf_i, f_j \rangle f_j$$

where $\{f_i\}$ is a fundamental set in $L^2(I)$. Since T is a Hilbert-Schmidt operator it follows from the Schwartz inequality that

$$\|Tg\| = \left\| \sum_{i,j} \langle g, f_i \rangle \langle Tf_i, f_j \rangle f_j \right\| \leq \|g\| \, \|T\|_h$$

where

$$\|T\|_h^2 = \sum_{i,j} |\langle Tf_i, f_j \rangle|^2.$$

From the isomorphism between $L^2(I)$ and $\mathcal{H}_o$ it follows that

$$\|T\|_h^2 = \int_{I \times I} |K(x,y)|^2 \, dxdy.$$

Thus, the integral operator T on $L^2(I)$ is a Hilbert-Schmidt operator if the kernel defining the operator is square integrable over I×I.

H. Spectral Theory of Linear Operators in $\mathcal{H}$

Finially in this section we will quote some theorems from the spectral theory of linear operators in a Hilbert space, $\mathcal{H}$. We first give the definitions:

1. The resolvent set $\rho(T)$ of T is the set of complex numbers λ, for which $(T - \lambda I)^{-1}$ exists as a bounded operator with domain $\mathcal{H}$.
2. The spectrum set $\sigma(T)$ is the complement of $\rho(T)$.

Without proof we quote three theorems from the spectral theory of linear operators in $\mathcal{H}$ that we have made use of in this note.

Theorem 1. If T^n is a compact operator for some positive integer n then the spectrum of T is at most denumerable and has no point of accumulation in the complex plane except possibly $\lambda = 0$. Every nonzero number in $\sigma(T)$ is a pole of some finite order of $(T - \lambda I)^{-1}$.

Theorem 2. Let T be a compact operator in $\mathcal{H}$, $\{\lambda_i\}$ a sequence of distinct scalars, and $\{f_i\}$ a sequence of nonzero elements of $\mathcal{H}$ such that $(T - \lambda_i I)f_i = 0$ for $i = 1,2,\ldots$ Then λ_i approaches zero as i approaches infinity.

Theorem 3. Let T_o be a closed operator in $\mathcal{H}$, and let λ_o be an isolated point of the spectrum of T_o such that λ_o is an eigenvalue of multiplicity one. Moreover, let $T(\varepsilon)$ be an analytic operator-valued function of ε in a neighborhood of $\varepsilon = 0$ such that $T(0) = T_o$. Then there exists a neighborhood of $\varepsilon = 0$ such that $T(\varepsilon)$ also has an eigenvalue, $\lambda(\varepsilon)$, of multiplicity one in a neighborhood of λ_o and $\lambda(0) = \lambda_o$. Furthermore, we have

$$T_o f_o = \lambda_o f_o$$

$$T(\varepsilon)f(\varepsilon) = \lambda(\varepsilon)f(\varepsilon)$$

where f_o and $f(\varepsilon)$ belong to $\mathcal{H}$ and $f(0) = f_o$. In a neighborhood of $\varepsilon = 0$ we can expand $\lambda(\varepsilon)$ and $f(\varepsilon)$ in an entire series of ε.

Appendix C

The Adjoint Operator in the Magnetic Field Formulation

In this appendix we will investigate the adjoint operator, $\underline{\underline{L}}^{\dagger}$, of the operator $\underline{\underline{L}}$ defined by equation (3.4) in section III.

Let $\underline{f}$ and $\underline{g}$ both belong to the Hilbert space $\underline{\mathcal{H}}$ defined in section III. The adjoint operator $\underline{\underline{L}}^{\dagger}$ of $\underline{\underline{L}}$ is then defined by

$$\langle \underline{\underline{L}}\cdot\underline{f},\underline{g}\rangle = \langle \underline{f},\underline{\underline{L}}^{\dagger}\cdot\underline{g}\rangle . \tag{C.1}$$

We have

$$\begin{aligned}\langle \underline{\underline{L}}\cdot\underline{f},\underline{g}\rangle &= \int_{S\times S}\{\underline{n}(\underline{r})\times[\nabla G(\underline{r},\underline{r}')\times\underline{f}(\underline{r}')]\}\cdot\underline{g}^{*}(\underline{r})dSdS' \\ &= \int_{S\times S}[\underline{n}(\underline{r})\cdot\underline{f}(\underline{r}')][\nabla G(\underline{r},\underline{r}')\cdot\underline{g}^{*}(\underline{r})]dSdS' \\ &\quad - \int_{S\times S}[\underline{n}(\underline{r})\cdot\nabla G(\underline{r},\underline{r}')][\underline{f}(\underline{r}')\cdot\underline{g}^{*}(\underline{r})]dSdS'\end{aligned} \tag{C.2}$$

where the star denotes the complex conjugate value. Introduce the operator $\underline{\underline{L}}_1$

$$\underline{\underline{L}}_1\cdot\underline{g} = \int_S \nabla G\times(\underline{n}\times\underline{g})dS \tag{C.3}$$

and

$$\begin{aligned}\langle \underline{f},\underline{\underline{L}}_1\cdot\underline{g}\rangle &= \int_{S\times S}\underline{f}(\underline{r})\cdot\{\nabla G^{*}(\underline{r},\underline{r}')\times[\underline{n}(\underline{r}')\times\underline{g}^{*}(\underline{r}')]\}dSdS' \\ &= -\int_{S\times S}[\underline{n}(\underline{r}')\cdot\underline{f}(\underline{r})][\nabla' G^{*}(\underline{r},\underline{r}')\cdot\underline{g}^{*}(\underline{r}')]dSdS' \\ &\quad + \int_{S\times S}[\underline{n}(\underline{r}')\cdot\nabla' G^{*}(\underline{r},\underline{r}')][\underline{f}(\underline{r})\cdot\underline{g}^{*}(\underline{r}')]dSdS' .\end{aligned} \tag{C.4}$$

Comparing equations (C.1) through (C.4) we get

$$\underline{\underline{L}}^{\dagger}\cdot\underline{f} = -\int_S \nabla G^{*}\times(\underline{n}\times\underline{f})dS . \tag{C.5}$$

Let $\underline{h}$ be the tangential component of the magnetic field on S and let $\underline{j} = \underline{n}\times\underline{h}$, where $\underline{n}$ is the outward normal to S. The integral equations of the exterior electromagnetic scattering problem can then be written as

$$(\frac{1}{2}\,\underline{\underline{I}} - \underline{\underline{L}})\cdot\underline{j}^{ext} = \underline{j}^{inc} = \underline{n}\times\underline{h}^{inc} \tag{C.6}$$

$$(\frac{1}{2}\,\underline{\underline{I}} - \underline{\underline{L}}_1)\cdot\underline{h}^{ext} = \underline{h}^{inc} \tag{C.7}$$

and from (C.5) it follows that (C.7) can be written as

$$(\frac{1}{2}\,\underline{\underline{I}} + \underline{\underline{L}}^T)\cdot\underline{h}^{ext} = \underline{h}^{inc} \tag{C.8}$$

where

$$\underline{\underline{L}}^T = \underline{\underline{L}}^{\dagger *}. \tag{C.9}$$

The integral equations for the interior problem have the form

$$(\frac{1}{2}\,\underline{\underline{I}} + \underline{\underline{L}})\cdot\underline{j}^{int} = \underline{j}^{inc} = -\underline{n}\times\underline{h}^{inc} \tag{C.10}$$

$$(\frac{1}{2}\,\underline{\underline{I}} + \underline{\underline{L}}_1)\cdot\underline{h}^{int} = \underline{h}^{inc} \tag{C.11}$$

and (C.11) takes also the form

$$(\frac{1}{2}\,\underline{\underline{I}} - \underline{\underline{L}}^T)\cdot\underline{h}^{int} = \underline{h}^{inc}. \tag{C.12}$$

We will now go on and discuss some of the implications of the similarity between (C.6) and (C.12). In section IV we have shown that the operator $\underline{\underline{L}}^2$ is of Hilbert-Schmidt type and thus compact. It then follows that the spectrum of $\underline{\underline{L}}$, $\sigma(\underline{\underline{L}})$, is denumerable[21]. For fixed γ let $\tau(\gamma)$ be the spectral set of $\underline{\underline{L}}$ so that

$$\tau(\gamma) = \sigma(\underline{\underline{L}}). \tag{C.13}$$

We then have

$$\frac{1}{2} \in \tau(\gamma_n). \tag{C.14}$$

(c.f. (6.4)). Moreover, we have

$$\sigma(\underline{\underline{L}}^{\dagger}) = \tau^{*}(\gamma) \tag{C.15}$$

where

$$\tau^{*}(\gamma) = \{\lambda : \lambda^{*} \in \tau(\gamma)\} \tag{C.16}$$

which follows from the fact that if the operator $\underline{\underline{B}}_{\lambda} = \underline{\underline{L}} - \lambda\underline{\underline{I}}$ has a unique inverse then the operator $\underline{\underline{B}}_{\lambda}^{\dagger} = \underline{\underline{L}}^{\dagger} - \lambda^{*}\underline{\underline{I}}$ also has a unique inverse[21]. From (C.9) it then follows that

$$\sigma(\underline{\underline{L}}^{T}) = \tau(\gamma). \tag{C.17}$$

Comparing (C.11), (C.12), (C.13) and (C.17) we see that if γ is such that the operator $\frac{1}{2}\,\underline{\underline{I}} - \underline{\underline{L}}$ has an inverse then the operator $\frac{1}{2}\,\underline{\underline{I}} + \underline{\underline{L}}_1$ also has an inverse, and vice versa. Since (C.10) and (C.11) describe the same physical situation we have

$$\underline{j}_1 = \underline{n}\times\underline{h}_1$$

where $\underline{j}_1$ is the solution of (C.10) with right hand side equals to $\underline{n}\times\underline{h}_o$ and $\underline{h}_1$ is the solution of (C.11) with right hand side equals to $\underline{h}_o$. Thus, the operators $\frac{1}{2}\,\underline{\underline{I}} + \underline{\underline{L}}$ and $\frac{1}{2}\,\underline{\underline{I}} + \underline{\underline{L}}_1$ have inverse for the same values of γ. It now follows that the inverse operator $(\frac{1}{2}\,\underline{\underline{I}} - \underline{\underline{L}})^{-1}$ exists for all those values of γ for which the inverse operator $(\frac{1}{2}\,\underline{\underline{I}} + \underline{\underline{L}})^{-1}$ exists, and vice versa.

Appendix D

Solution of the Integral Equation (3.3) From Spectral Theory

In this appendix we will derive a solution of the integral equation

$$\underline{\underline{A}}(\gamma)\cdot\underline{j} = \underline{j}^{inc} \tag{D.1}$$

from the spectral decomposition of the operator $\underline{\underline{L}}$.

First, we are going to consider the eigenvalue problem

$$\underline{\underline{L}}\cdot\underline{f} = \lambda\underline{f}. \tag{D.2}$$

From the spectral theorems listed in appendix B it follows that $\underline{\underline{L}}$ has a denumerable spectrum, $\sigma(\underline{\underline{L}})$, with eigenvalues λ_k approaching zero as k is approaching infinity. Let us denote the adjoint operator of $\underline{\underline{L}}$ by $\underline{\underline{L}}^{\dagger}$ so that

$$\langle\underline{\underline{L}}\cdot\underline{f},\underline{g}\rangle = \langle\underline{f},\underline{\underline{L}}^{\dagger}\cdot\underline{g}\rangle \tag{D.3}$$

where $\underline{f}$ and $\underline{g}$ belong to the Hilbert space $\mathcal{H}$ defined in section III. Explicit expressions for $\underline{\underline{L}}^{\dagger}$ are given in appendix C. If λ_i and $\underline{\phi}_i$ are eigenvalues and eigenfunctions of $\underline{\underline{L}}$,

$$\underline{\underline{L}}\cdot\underline{\phi}_i = \lambda_i\,\underline{\phi}_i, \tag{D.4}$$

then $\underline{\underline{L}}^{\dagger}$ has eigenvalues λ_i^* and eigenfunctions $\underline{\psi}_i$,

$$\underline{\underline{L}}^{\dagger}\cdot\underline{\psi}_i = \lambda_i^*\,\underline{\psi}_i, \tag{D.5}$$

where the asterisk denotes the complex conjugate value. This follows from the fact that if $\underline{\underline{B}}_\lambda = \underline{\underline{L}} - \lambda\underline{\underline{I}}$ has a unique inverse then $\underline{\underline{B}}_\lambda^{\dagger} = \underline{\underline{L}}^{\dagger} - \lambda^*\underline{\underline{I}}$ also has a unique inverse[21]. Furthermore,

$$\langle\underline{\underline{L}}\cdot\underline{\phi}_i,\underline{\psi}_j\rangle = \lambda_i\langle\underline{\phi}_i,\underline{\psi}_j\rangle \tag{D.6}$$

$$\langle \underline{\underline{L}}\cdot\underline{\phi}_i, \underline{\psi}_j \rangle = \langle \underline{\phi}_i, \underline{\underline{L}}^\dagger\cdot\underline{\psi}_j \rangle = \langle \underline{\phi}_i, \lambda_j^* \underline{\psi}_j \rangle = \lambda_j \langle \underline{\phi}_i, \underline{\psi}_j \rangle \tag{D.7}$$

from which it follows that

$$\langle \underline{\phi}_i, \underline{\psi}_j \rangle = 0 \quad \text{if} \quad \lambda_i \neq \lambda_j . \tag{D.8}$$

We now assume that $\underline{\underline{L}}$ has simple eigenvalues. By proper normalization of the sets $\{\underline{\phi}_i\}$ and $\{\underline{\psi}_i\}$ we form a biorthonormal system

$$\langle \underline{\phi}_i, \underline{\psi}_j \rangle = \delta_{ij} . \tag{D.9}$$

We also assume that the sets $\{\underline{\phi}_i\}$ and $\{\underline{\psi}_i\}$ are complete so that the following expansions are valid for any arbitrary $\underline{f}$ belonging to $\mathcal{H}$,

$$\underline{f} = \sum_{i=1}^{\infty} \langle \underline{f}, \underline{\psi}_i \rangle \underline{\phi}_i = \sum_{i=1}^{\infty} \langle \underline{f}, \underline{\phi}_i \rangle \underline{\psi}_i . \tag{D.10}$$

Let us order these sets so that

$$|\lambda_1| \geq |\lambda_2| \geq \cdots \geq \lambda_k \geq \lambda_{k+1} \geq \cdots$$

and $\lambda_k \to 0$ as $k \to \infty$. We have

$$\underline{f}_k = \underline{f} - \sum_{i=1}^{k} \langle \underline{f}, \underline{\psi}_i \rangle \underline{\phi}_i \tag{D.11}$$

and it follows that

$$||\underline{\underline{L}}\cdot\underline{f}_k|| \leq |\lambda_{k+1}| \; ||\underline{f}_k|| \to 0 \quad \text{as } k \to \infty \tag{D.12}$$

so that

$$\underline{\underline{L}}\cdot\underline{f}_k = \underline{\underline{L}}\cdot\underline{f} - \sum_{i=1}^{k} \lambda_i \langle \underline{f}, \underline{\psi}_i \rangle \underline{\phi}_i \to 0 \quad \text{as} \quad k \to \infty . \tag{D.13}$$

Thus,

$$\underline{\underline{L}}\cdot\underline{f} = \sum_{i=1}^{\infty} \lambda_i \langle \underline{f}, \underline{\psi}_i \rangle \underline{\phi}_i \tag{D.14}$$

and, of course, this series contains only a finite number of terms when λ_i is zero after some index.

Having the representation (D.14) of $\underline{\underline{L}}$ we go on to consider the equation

$$(\frac{1}{2}\underline{\underline{I}} - \underline{\underline{L}})\cdot\underline{j} = \underline{j}^{inc} \tag{D.15}$$

where $\underline{j}^{inc}$ belongs to $\underline{\mathcal{H}}$ and we are looking for solutions, $\underline{j}$, such that $\underline{j}$ belongs to $\underline{\mathcal{H}}$. Formal manipulations give

$$\underline{j} = 2\underline{j}^{inc} + 4\sum_{i=1}^{\infty} \lambda_i (1 - 2\lambda_i)^{-1} \langle \underline{j}^{inc}, \underline{\psi}_i \rangle \underline{\phi}_i, \qquad 1 - 2\lambda_i \neq 0. \tag{D.16}$$

Conversely, when the series (D.16) converges it is a solution of (D.15). Since $\underline{\mathcal{H}}$ is complete the convergence of (D.16) follows from the fact that the partial sums, $\underline{v}_n$ of (D.16), satisfy the Cauchy condition

$$\|\underline{v}_n - \underline{v}_m\| = 4\|\sum_{i=m+1}^{n} \lambda_i (1 - 2\lambda_i)^{-1} \langle \underline{j}^{inc}, \underline{\psi}_i \rangle \underline{\phi}_i\| \leq \beta \|\sum_{i=m+1}^{n} \langle \underline{j}^{inc}, \underline{\psi}_i \rangle \underline{\phi}_i\| \to 0 \tag{D.17}$$

as $n, m \to \infty$, since $\{\underline{\phi}_i\}$ and $\{\underline{\psi}_i\}$ are complete in $\underline{\mathcal{H}}$ and β,

$$\beta = 4 \sup_i |\lambda_i (1 - 2\lambda_i)^{-1}|, \tag{D.18}$$

exists since $1 - 2\lambda_i \neq 0$ and $\lambda_i \to 0$ as $i \to \infty$.

We now go on to consider the analytical properties of the solution (D.16) of (D.15) in the complex frequency plane. The series (D.16) is a solution of (D.15) for all finite γ under the assumption that the eigenvalues are simple for all finite γ. Suppose that we have this solution for one special value of γ, say γ'. Since $\underline{\underline{L}}$ is an analytic operator-valued function of γ and since the eigenvalues of $\underline{\underline{L}}$ at γ' are of multiplicity one it follows from the perturbation theory of closed operators that λ_i and $\underline{\phi}_i$ are analytic functions of γ at γ'.[24] From (C.5) of appendix C it also follows that $\underline{\underline{L}}^T = \underline{\underline{L}}^{\dagger *}$ is an analytic operator-valued function of γ. From the relationship

$$\underline{\underline{L}}^T \cdot \underline{\psi}_i^* = \lambda_i \underline{\psi}_i^* \tag{D.19}$$

it follows that $\underline{\psi}_i^*$ is an analytic function at γ'. Since γ' can be any point in the finite complex γ-plane it follows that λ_i, $\underline{\phi}_i$ and $\underline{\psi}_i^*$ are analytic functions of γ for γ finite. Thus, the solution $\underline{j}$ of (D.15) is an analytic function of γ except at the singularities of $\underline{j}^{inc}$ and at those points, γ_{ik}, for which $2\lambda_i(\gamma_{ik}) = 1$. Because λ_i is an analytic function the solution $\underline{j}$ has a pole of some finite order at γ_{ik}. Moreover, since $\lambda_i \to 0$ as $i \to \infty$ there exists for any finite γ, $|\gamma| < R$, a finite number, $N = N(R)$, such that

$$2|\lambda_i| < 1 \quad \text{for} \quad i \geq N. \tag{D.20}$$

The locations of the poles in $|\gamma| < R$ are then given by the solutions of the finite number of equations

$$2\lambda_i + 1 = 0, \qquad 1 \leq i < N. \tag{D.21}$$

Each equation in (D.21) has a finite number of roots in $|\gamma| < R$ since λ_i is analytic in $|\gamma| < R$. Thus, we can only have a finite number of poles in $|\gamma| < R$.

The locations of these poles are completely determined by the shape of the scattering body, and we call them the body poles. The singularities of $\underline{j}$ due to the singularities in $\underline{j}^{inc}$ we call the waveform singularities.

Suppose the incident field is a delta-function (in time) plane wave so that

$$\underline{j}^{inc}(\underline{r}) = \underline{u}(\underline{r})e^{-x\gamma} \tag{D.22}$$

(c.f. (5.1) and (6.15)). The induced current density, $\underline{j}(\underline{r},\gamma)$, on S is then

$$\underline{j}(\underline{r},\gamma) = 2\underline{u}(\underline{r})e^{-x\gamma} + 4\sum_{i=1}^{\infty} \lambda_i(\gamma)[1 - 2\lambda_i(\gamma)]^{-1}\langle \underline{u}(\underline{r}')e^{-x'\gamma}, \underline{\psi}_i(\underline{r}',\gamma)\rangle \underline{\phi}_i(\underline{r},\gamma). \tag{D.23}$$

Performing an inverse Laplace transform on (D.23) and assuming that λ_i only has simple poles and that $\underline{\phi}_i$ and $\underline{\psi}_i$ are bounded for all γ we get the following expression for the current in the time-domain $\underline{J}(\underline{r},t)$,

$$\underline{J}(\underline{r},t) = 2\underline{n}(\underline{r})\times \underline{I}_o\, \delta(x - ct) - \sum_{i,k} a_{ik} C_{ij}(t)\underline{\phi}_i(\underline{r},\gamma_{ij})e^{ct\gamma_{ik}} \tag{D.24}$$

where

$$a_{ik} = 2\lambda_i(\gamma_{ik})/\lambda_i'(\gamma_{ik})$$

$$C_{ij}(t) = \langle U(ct - x')\underline{u}(\underline{r}')e^{-x'\gamma},\underline{\psi}_i\rangle$$

$$\lambda_i'(\gamma) = \frac{d\lambda_i}{d\gamma}$$

and $U(x)$ is Heavisides unit step function. Notice that $C_{ij}(t) = 0$ for $t < t_o$ and $C_{ij}(t)$ = constant for $t > t_1$. The quantities t_o and t_1 are defined in (5.11) and (5.12), respectively.

Appendix E
The Case Where $\underline{\underline{A}}^{-1}(\gamma)$ Has a Double Pole at γ_n

In this appendix we are going to treat the case where $\underline{\underline{A}}^{-1}(\gamma)$ has a double pole at $\gamma = \gamma_n$. Assuming that $\underline{j}^{inc}$ is an analytic function of γ in a neighborhood of γ_n it follows from (6.19) of section VI that the solution, $\underline{j}$, of the integral equation

$$\underline{\underline{A}}(\gamma)\cdot\underline{j} = \underline{j}^{inc} \tag{E.1}$$

has the following representation in a neighborhood of γ_n

$$\underline{j} = \sum_{m=1}^{M_2} [a_{nm}(\gamma - \gamma_n)^{-2} + b_{nm}(\gamma - \gamma_n)^{-1}]\underline{j}_{n2m} + \sum_{m=1}^{M_1} c_{nm}(\gamma - \gamma_n)^{-1}\,\underline{j}_{n1m} + \underline{j}_n. \tag{E.2}$$

Here a_{nm}, b_{nm} and c_{nm} are unknown constants, $\underline{j}_n$ is an analytic function of γ, M_2 is the number of linearly independent solutions, $\underline{j}_{n2m}$, such that

$$\underline{\underline{A}}(\gamma_n)\cdot\underline{j}_{n2m} = \underline{\underline{B}}_{n1}\cdot\underline{j}_{n2m} = 0 \tag{E.3}$$

but

$$\underline{\underline{B}}_{n2}\cdot\underline{j}_{n2m} \neq 0 \tag{E.4}$$

and M_1 is the number of independent solutions, $\underline{j}_{n1m}$, such that

$$\underline{\underline{A}}(\gamma_n)\cdot\underline{j}_{n1m} = 0 \tag{E.5}$$

but

$$\underline{\underline{B}}_{n1}\cdot\underline{j}_{n1m} \neq 0. \tag{E.6}$$

The operators $\underline{\underline{B}}_{n1}$ and $\underline{\underline{B}}_{n2}$ are defined in (6.22) of section VI.

It is also easy to show that there exist M_2 linearly independent functions, $\underline{h}_{n2m}$, such that

$$\underline{\underline{A}}^{\dagger}(\gamma_n)\cdot\underline{h}_{n2m} = \underline{\underline{B}}_{n1}^{\dagger}\cdot\underline{h}_{n2m} = 0 \tag{E.7}$$

but

$$\underline{\underline{B}}_{n2}^{\dagger}\cdot\underline{h}_{n2m} \neq 0 \tag{E.8}$$

and that there exist M_1 linearly independent solutions, $\underline{h}_{n1m}$, such that

$$\underline{\underline{A}}^{\dagger}(\gamma_n)\cdot\underline{h}_{n1m} = 0 \tag{E.9}$$

and

$$\underline{\underline{B}}_{n1}^{\dagger}\cdot\underline{h}_{n1m} \neq 0. \tag{E.10}$$

From the Gram-Schmidt orthogonalization procedure it follows that we can choose the sets of functions

$$\left\{\underline{j}_{n2m}\right\}_{m=1}^{M_2} \quad \text{and} \quad \left\{\underline{h}_{n2m}\right\}_{m=1}^{M_2}$$

such that

$$\langle \underline{\underline{B}}_{n2}\cdot\underline{j}_{n2m},\underline{h}_{n2k}\rangle \begin{cases} = 0 & \text{for} \quad m \neq k \\ \neq 0 & \text{for} \quad m = k. \end{cases} \tag{E.11}$$

It also follows from the Gram-Schmidt procedure that we can choose the sets of functions

$$\left\{\underline{j}_{n1m}\right\}_{m=1}^{M_1} \quad \text{and} \quad \left\{\underline{h}_{n1m}\right\}_{m=1}^{M_1}$$

such that

$$\langle \underline{\underline{B}}_{n1}\cdot \underline{j}_{n1m}, \underline{h}_{n1k}\rangle \begin{cases} = 0 & \text{for} \quad m \neq k \\ \neq 0 & \text{for} \quad m = k. \end{cases} \tag{E.12}$$

Next, we expand the equations

$$\langle [\underline{\underline{A}}(\gamma) - \underline{\underline{A}}(\gamma_n)]\cdot \underline{j}, \underline{h}_{npm}\rangle = \langle \underline{j}^{inc}, \underline{h}_{npm}\rangle \tag{E.13}$$

in a Laurent series around $\gamma = \gamma_n$. In (E.13) we have $p = 1,2$ and $1 \leq m \leq M_1$ for $p = 1$ and $1 \leq m \leq M_2$ for $p = 2$. Equations (E.11) and (E.12), together with the fact that

$$\langle \underline{\underline{B}}_{n1}\cdot \underline{j}_{n1m}, \underline{h}_{n2m}\rangle = \langle \underline{j}_{n1m}, \underline{\underline{B}}_{n1}^{\dagger}\cdot \underline{h}_{n2m}\rangle = 0 \tag{E.14}$$

and the Laurent series expansion of (E.13), enable us to get the following expressions for a_{nm} and c_{nm}

$$a_{nm} = \langle \underline{j}_n^{inc}, \underline{h}_{n2m}\rangle [\langle \underline{\underline{B}}_{n2}\cdot \underline{j}_{n1m}, \underline{h}_{n2m}\rangle]^{-1}, \qquad 1 \leq m \leq M_2 \tag{E.15}$$

$$\begin{aligned} c_{nm} = {} & \langle \underline{j}_n^{inc}, \underline{h}_{n1m}\rangle [\langle \underline{\underline{B}}_{n1}\cdot \underline{j}_{n1m}, \underline{h}_{n1m}\rangle]^{-1} \\ & - \sum_{k=1}^{M_1} \langle \underline{j}_n^{inc}, \underline{h}_{n2k}\rangle \langle \underline{\underline{B}}_{n2}\cdot \underline{j}_{n2k}, \underline{h}_{n1k}\rangle [\langle \underline{\underline{B}}_{n2}\cdot \underline{j}_{n2k}, \underline{h}_{n2k}\rangle \langle \underline{\underline{B}}_{n1}\cdot \underline{j}_{n1m}, \underline{h}_{n1m}\rangle]^{-1}, \\ & \qquad 1 \leq m \leq M_1 \end{aligned} \tag{E.16}$$

where $\underline{j}_n^{inc}$ is $\underline{j}^{inc}$ evaluated at γ_n. Moreover, b_{nm} satisfies the set of algebraic equations

$$\begin{aligned} \sum_{k=1}^{M_2} \langle \underline{\underline{B}}_{n2}\cdot \underline{j}_{n2k}, \underline{h}_{n2m}\rangle b_{nk} = {} & \langle \underline{j}_{n1}^{inc}, \underline{h}_{n2m}\rangle - \sum_{k=1}^{M_2} \langle \underline{\underline{B}}_{n3}\cdot \underline{j}_{n2k}, \underline{h}_{n2k}\rangle a_{nk} \\ & - \sum_{k=1}^{M_1} \langle \underline{\underline{B}}_{n2}\cdot \underline{j}_{n1k}, \underline{h}_{n2m}\rangle c_{nk}, \qquad 1 \leq m \leq M_2 \end{aligned} \tag{E.17}$$

where $\underline{j}_{n1}^{inc} = \frac{\partial}{\partial\gamma}\, \underline{j}^{inc}$ evaluated at γ_n.

For the special case where $M_1 = M_2 = 1$ (no degeneracy) we can drop the index m and have the following expression which is valid in a neighborhood of γ_n,

$$a_n = \langle \underline{j}_n^{inc}, \underline{h}_{n2} \rangle [\langle \underline{\underline{B}}_{n2} \cdot \underline{j}_{n2}, \underline{h}_{n2} \rangle]^{-1} \tag{E.18}$$

$$\begin{aligned} b_n = {} & \langle \underline{j}_{n1}^{inc}, \underline{h}_{n2} \rangle [\langle \underline{\underline{B}}_{n2} \cdot \underline{j}_{n2}, \underline{h}_{n2} \rangle]^{-1} \\ & - \langle \underline{j}_n^{inc}, \underline{h}_{n1} \rangle \langle \underline{\underline{B}}_{n2} \cdot \underline{j}_{n1}, \underline{h}_{n2} \rangle [\langle \underline{\underline{B}}_{n2} \cdot \underline{j}_{n2}, \underline{h}_{n2} \rangle \langle \underline{\underline{B}}_{n1} \cdot \underline{j}_{n1}, \underline{h}_{n1} \rangle]^{-1} \\ & - \langle \underline{j}_n^{inc}, \underline{h}_{n2} \rangle \{ \langle \underline{\underline{B}}_{n3} \cdot \underline{j}_{n2}, \underline{h}_{n2} \rangle - \langle \underline{\underline{B}}_{n2} \cdot \underline{j}_{n1}, \underline{h}_{n2} \rangle \langle \underline{\underline{B}}_{n2} \cdot \underline{j}_{n2}, \underline{h}_{n1} \rangle \\ & [\langle \underline{\underline{B}}_{n1} \cdot \underline{j}_{n1}, \underline{h}_{n1} \rangle]^{-1} \} [\langle \underline{\underline{B}}_{n2} \cdot \underline{j}_{n2}, \underline{h}_{n2} \rangle]^{-2} \end{aligned} \tag{E.19}$$

$$\begin{aligned} c_n = {} & \{ \langle \underline{j}_n^{inc}, \underline{h}_{n1} \rangle - \langle \underline{j}_n^{inc}, \underline{h}_{n2} \rangle \langle \underline{\underline{B}}_{n2} \cdot \underline{j}_{n2}, \underline{h}_{n1} \rangle [\langle \underline{\underline{B}}_{n1} \cdot \underline{j}_{n1}, \underline{h}_{n1} \rangle]^{-1} \} \\ & [\langle \underline{\underline{B}}_{n2} \cdot \underline{j}_{n2}, \underline{h}_{n2} \rangle]^{-1} \end{aligned} \tag{E.20}$$

Appendix F

A Method of Separating Exterior Modes From Interior Ones

In this appendix we will show one feature of the scalar product $\langle \underline{j}_n^{inc}, \underline{h}_{nm} \rangle$ which enables us to differ exterior natural modes from interior ones. Let $\underline{j}_n^{inc}$ be the incident field evluated at $\gamma = \gamma_n$ and let $\underline{h}_{nm}$ be a nontrivial solution of the integral equation (6.10). We then have

$$\langle \underline{j}_n^{inc}, \underline{h}_{nm} \rangle = \int_S \underline{j}_n^{inc} \cdot \underline{h}_{nm}^* \, dS$$

$$= \int_S (\underline{n} \times \underline{h}_n^{inc}) \cdot \underline{h}_{nm}^* \, dS = \int_S \underline{n} \cdot (\underline{h}_n^{inc} \times \underline{h}_{nm}^*) dS. \qquad \text{(F.1)}$$

From (6.10), (6.11) of section VI and (C.12) of appendix C it follows that we can choose $\underline{h}_{nm}^*$ so that it represents the tangential magnetic field on S of an interior mode. From $\underline{h}_{nm}^*$ we can then construct the magnetic field $\underline{H}_{nm}$ and electric field $\underline{E}_{nm}$ at any point in Ω. Let $\underline{H}_n^{inc}$ and $\underline{E}_n^{inc}$ be the magnetic field and electric field of the incident wave, evaluated at $\gamma = \gamma_n$. We have

$$\langle \underline{j}_n^{inc}, \underline{h}_{nm} \rangle = \int_S \underline{n} \cdot (\underline{H}_n^{inc} \times \underline{H}_{nm}) dS$$

$$= \int_\Omega \nabla \cdot (\underline{H}_n^{inc} \times \underline{H}_{nm}) dV$$

$$= \int_\Omega (\underline{H}_{nm} \cdot \nabla \times \underline{H}^{inc} - \underline{H}_n^{inc} \cdot \nabla \times \underline{H}_{nm}) dV$$

$$= \int_\Omega \underline{H}_{nm} \cdot \underline{i}_n^{inc} \, dV + Z_o^{-1} \gamma_n \int_\Omega (\underline{H}_{nm} \cdot \underline{E}_n^{inc} - \underline{H}_n^{inc} \cdot \underline{E}_{nm}) dV$$

$$= \int_\Omega \underline{H}_{nm} \underline{i}_n^{inc} \, dV + Z_o^{-2} \int_\Omega (\underline{E}_{nm} \cdot \nabla \times \underline{E}_n^{inc} - \underline{E}_n^{inc} \cdot \nabla \times \underline{E}_{nm}) dV$$

$$= \int_\Omega \underline{H}_{nm} \underline{i}_n^{inc} \, dV + Z_o^{-2} \int_\Omega \nabla \cdot (\underline{E}_n^{inc} \times \underline{E}_{nm}) dS$$

$$= \int_\Omega \underline{H}_{nm} \underline{i}_n^{inc} \, dV + Z_o^{-2} \int_S \underline{n} \cdot (\underline{E}_n^{inc} \times \underline{E}_{nm}) dS. \qquad \text{(F.2)}$$

Since S is a perfectly conducting surface we have

$$\int_S \underline{n} \cdot (\underline{E}_n^{inc} \times \underline{E}_{nm}) dS = 0. \tag{F.3}$$

Thus, we have

$$\langle \underline{j}_n^{inc}, \underline{h}_{nm} \rangle = \int_\Omega \underline{H}_{nm} \cdot \underline{i}_n^{inc} \, dV \tag{F.4}$$

where $\underline{i}^{inc}$ represents the sources of the incident wave. If the incident wave has its sources outside Ω, as is the case for a plane wave, we have

$$\langle \underline{j}_n^{inc}, \underline{h}_{nm} \rangle = 0. \tag{F.5}$$

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