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Axial Current Induced on A Truncated Cone: Part I Theory

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ABSTRACT

A general formulation is presented for determining the total axial current induced on a body of revolution. Considering arbitrary illumination an exact integral equation is derived for the axial current distribution. As an example of an application for the formulation the truncated cone is treated and sample numerical results are obtained.

INTRODUCTION

Theoretical-numerical techniques exist for the determination of the current induced on a body of revolution, but they are very tedious becoming intractable with increase in frequency.[1-3*] However, it is shown in this paper that the total axial current may be obtained from a relatively simple, yet exact, integral equation. The integral equation is obtained by using the so called extended boundary condition, where field relationships are imposed within a perfect conductor.[4] A convenient extended boundary condition for a body of revolution is the requirement that the total electric field vanish on the axis of revolution within the conducting body.

The presented formulation applies for a completely general body with arbitrary illumination. As an example, the formulation is applied to the plane wave illumination of a truncated cone. The truncation considered is a flat end plate perpendicular to the axis of the cone. When the body of revolution possesses surface discontinuities special techniques must be employed. For the truncated cone, the end plate is considered to have a sufficiently small diameter that the leading term in a quasi-static expansion for the end plate current may be used, i.e.

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*The superscripts refer to the list of references at the end of the paper.

$ka_m \lesssim$ where a_m is the radius of the plate, and $\theta_o \ll 1$ where θ_o is the cone half angle.

Sample numerical data are presented for the truncated cone. Extensive data are to appear in a subsequent report.

ANALYSIS

General Considerations

In the presence of a metallic body the total electric field may be expressed in terms of the induced surface current on the body as

$$\vec{E}(\vec{R}) = \vec{E}^{inc}(\vec{R}') - i \frac{\eta}{4\pi k} \left\{ \nabla \int_S \nabla' \cdot \vec{K}(\vec{R}')\Psi(\vec{R},\vec{R}')ds' + k^2 \int_S \vec{K}(\vec{R}')\Psi(\vec{R},\vec{R}')ds' \right\} \qquad (1)$$

where E^{inc} is the incident electric field, K is the induced surface current on the metallic surface S, k is the propagation constant, $\eta = \sqrt{\mu/\varepsilon} \approx 120\pi$ ohms is the intrinsic wave impedance of free space, R is the radius vector to the field point,

$$\Psi(\vec{R},\vec{R}') = \frac{e^{-jk|\vec{R}-\vec{R}'|}}{|\vec{R}-\vec{R}'|} \qquad (2)$$

Consider a body of revolution with the axis coincident with the z axis of a cylindrical coordinate system (See Figure 1). Then in cylindrical coordinates

$$ds' = \frac{a(z')}{\cos\theta\ (z')} d\phi' dz' \qquad (3)$$

$$\theta(z') = \tan^{-1} \left[\frac{d}{dz'}\ a(z')\right] \qquad (4)$$

$$\Psi(\vec{R},\vec{R}') = \frac{\exp[-ik\sqrt{r^2+r'^2-2rr'\cos(\phi-\phi')+(z-z')^2}]}{\sqrt{r^2+r'^2-2rr'\cos(\phi-\phi')+(z-z')^2}} \qquad (5)$$

$$\nabla' \cdot \vec{K}(\vec{R}') = \frac{1}{a(z')} \left[\frac{\partial}{\partial t'} (a(z')K_{t'}) + \frac{\partial}{\partial \phi'} K_{\phi'}\right] \tag{6}$$

$$\hat{t}' = \hat{z}' \cos\theta(z') + \hat{r}' \sin\theta(z') \tag{7}$$

$$dt' = dz'/\cos\theta(z') \tag{8}$$

Ultimately (1) will be evaluated at r=0, i.e. on axis of the body of revolution. To that end consider the first integral of (1)

$$\int_s \nabla' \cdot \vec{K}(\vec{R}')\Psi(\vec{R},\vec{R}')ds' \Bigg]_{r=0} = \int_0^L dz' \frac{\Psi_o(z,z')}{\cos\theta(z')} \frac{\partial}{\partial t'} \left\{a(z')\int_{-\pi}^{\pi} d\phi' K_t(z',\phi')\right\}$$

$$+ \int_0^L dz' \frac{\Psi_o(z,z')}{\cos\theta(z')} \int_{-\pi}^{\pi} d\phi' \frac{\partial}{\partial \phi'} K_\phi(z',\phi') \tag{9}$$

where

$$\Psi_o(z,z') = \frac{\exp[-ik\sqrt{(z-z')^2+a^2(z')}]}{\sqrt{(z-z')^2+a^2(z')}} \tag{10}$$

Since the current distribution must be single valued the second integral on the righthand side of (9) vanishes. Now consider the second integral of (1) evaluated at r = 0.

$$\int_s K_z(\vec{R}')\Psi(\vec{R},\vec{R}')ds' \Bigg]_{r=0} = \int_0^L dz' \frac{\Psi_o(z,z')}{\cos\theta(z')} a(z') \int_{-\pi}^{\pi} d\phi' K_z(z',\phi') \tag{11}$$

Further it is noted

$$K_z(z',\phi') = \cos\theta(z')K_t(z',\phi') \tag{12}$$

and

$$I_t(z') = a(z')\int_{-\pi}^{\pi} d\phi'\; K_t(z',\phi') \tag{13}$$

is the total current through the cross section at z' along the axis of the body.

Using (9), (11), (12) and (13) in (1) evaluated at r=0 yields

$$E_z(0,\phi,z) = E_z^{inc}(0,\phi,z) - i\,\frac{\eta}{4\pi k}\left\{\int_0^L dz'\,\frac{\partial}{\partial z'}\,I_t(z')\,\frac{\partial}{\partial z}\,\Psi_o(z,z') + k^2\int dz'\; I_t(z')\Psi_o(z,z')\right\} \tag{14}$$

At the center of the body of revolution the total electric field must be zero then (14) becomes

$$\int_0^L dz'\left[\frac{\partial}{\partial z'}\,I_t(z')\,\frac{\partial}{\partial z}\,\Psi_o(z,z') + k^2\,I_t(z')\Psi_o(z,z')\right] = i\frac{4\pi k}{\eta}E_z^{inc}(o,\phi,z) \tag{15}$$

which is an exact integral equation for the total axial current on the body of revolution. The first integral may be integrated by parts to cast (15) into a more common form,

$$\int_0^L dz'\; I_t(z')\;\left[k^2 - \frac{\partial^2}{\partial z\partial z'}\,\Psi_o(z,z')\right] + I_t(z')\,\frac{\partial}{\partial z}\,\Psi_o(z,z')\Big|_o^L = -\,i\,\frac{4\pi k}{\eta}\,E_z^{inc}\;(0,\phi,z) \tag{16}$$

If the current vanishes at both ends of the body of revolution then (15) takes the form derived by Hallén[5] and later by Albert and Synge.[6]

Truncated Cone

If the body of revolution has surface discontinuities, such as occur for a truncated cone, the integral equation must be modified. This modification consists of two parts: (1) continuity of current must be preserved on the surface discontinuity and (2) the incident field on the continuous portion of the body is forced to include the scattered field from the discontinuity. And to solve the problem rigorously requires the solution for the current on the discontinuity including the interaction with the current on the continuous portion of the body. However, for surfaces with characteristic dimensions much less than a wavelength a quasi-static approximation for the functional form of the current and the use of the aforementioned modifications should yield a highly accurate current distribution for the body. This is to be the procedure followed in the analysis of the current induced on a truncated cone.

According to the quasi-static approximation the leading term in the expansion for the radial current distribution on the end plate is[7]

$$K_t(r) \approx \text{Const.} \times r \qquad (17)$$

Preserving continuity of current at the edge yields

$$K_r(r) \simeq \frac{r}{2\pi a_m^2} I_t(0) \tag{18}$$

and the contribution to the incident electric field component along the cone axis is

$$\Delta E_z(z) \simeq \frac{i\eta\, I_t(0)}{2\pi k a_m^2} e^{-ikz} \left[1 - \frac{z e^{ik(z-\sqrt{z^2+a_m^2}}}{\sqrt{z^2+a_m^2}}\right] \tag{19}$$

Therefore using the foregoing contribution to the incident field yields the following integral equation for the current distribution induced on a cone truncated with a flat end plate

$$\int_0^L dz' I_t(z') K_1(z,z') = -i \frac{4\pi k}{\eta} E_z^{inc}(z) \tag{20}$$

where

$$K_1(z,z') = (k^2 - \frac{\partial^2}{\partial z \partial z'}) K(z,z') - 2\delta(z') \frac{\partial}{\partial z} K(z,z')$$

$$- \frac{4\delta(z')}{a_m^2} e^{-kz} \left\{1 - \frac{z e^{ik(z-\sqrt{z^2+a_m^2})}}{\sqrt{z^2+a_m^2}}\right\} \tag{21}$$

$$K(z,z') = \frac{\exp[ik\sqrt{(z-z')^2 + (L-z')^2\tan^2\theta_o}]}{\sqrt{(z-z')^2 + (L-z')^2\tan^2\theta_o}} \tag{22}$$

$$a_m = \tan\theta_o$$

The solution for the current density may now be effected by solving (20) subject to the boundary condition

$$I_t(L) = 0 \tag{23}$$

If the cone is illuminated by a plane wave then

$$E_z^{inc}(z) = E_o \sin\theta \, e^{-i(k\cos\theta)z} \tag{24}$$

where the direction of propagation forms an angle θ with the positive z axis and E_o is the complex amplitude of the electric field.

The only restrictions in the foregoing analysis results from the use of the leading term in the quasi-static expansion current distribution and they are

$$ka_m < 1. \tag{25}$$

$$\theta_o \ll 1.$$

NUMERICAL SOLUTION TECHNIQUE

The integral equation for the current distribution is readily solved by use of the method of moments. It has been shown that the piecewise sinusoidal expansion for the current distribution provides a rapidly convergent solution. According to this technique the current is

$$I_t(z') = \sum_{m=1}^{N} I_m(z')\, U(z';z_{m+1},z_m) \tag{26}$$

where

$$I_m(z') = \alpha_{m+1}\, \text{sink}(z'-z_m) + \alpha_m\, \text{sink}(z_{m+1}-z') \tag{27}$$

$$U(z';z_{m+1},z_m) = 1 \qquad z_m < z' < z_{m+1} \tag{28}$$

$$= 0 \qquad \text{otherwise}$$

with $\{z_m\}$ as the set of N+1 equally spaced points on the domain of the current distribution including the end points.

Substituting the current expansion into (20) and forcing the resulting equation to be satisfied at a suitable set of points yields a system of linear equations for the expansion coefficients. Also (23) must be satisfied. Accordingly the following system of equations is obtained:

$$\sum_{m=1}^{N} \Pi_{nm}\, \alpha_m = \Gamma_n \tag{29}$$

where

$$\Pi_{nm} = (1-\delta_{m1}) \int_{z_{m-1}}^{z_m} dz' \; K_2(z_n,z')\sin k(z'-z_{m-1})$$

$$+ \int_{z_m}^{z_{m+1}} dz' \; K_2(z_n,z')\sin k(z_{m+1}-z')$$

$$-\delta_{m1} \frac{2}{a_m^2} \left[\left\{ e^{-ikz_n} - z_n \; K(z_n,0) \right\} + \frac{\partial}{\partial z} K(z,0) \Big|_{z=z_n} \right] \sin k\Delta \qquad (30)$$

$$\Gamma_n = -i \frac{4\pi k}{\eta} E_o \sin\theta e^{-ik\cos\theta z_n}$$

$$z_n = (n-1)\Delta$$

$$\Delta = L/N$$

$$K_2(z_n,z') = \left[k^2 - \frac{\partial^2}{\partial z \partial z'}\right] K(z,z') \Big|_{z=z_n} \qquad (32)$$

In the foregoing $K(z,\bar{z}')$ is defined by (22). Note that applying (23) to (26) yields

$$\alpha_{n+1} = 0$$

This result is used in obtaining (29).

NUMERICAL RESULTS

The system of linear equations (29) are in a form amenable to solution by using a high speed digital computer. A fortran program was written for the IBM 360 model 40 digital computer that was available. The execution time for the program to calculate the current distributions for three angles of incidence with N = 10 is about 4 minutes.

To illustrate to the convergence of the solution a table of currents is presented for increasing values of N. It is noted that N = 10 should yield sufficiently accurate results for most practical purposes. Figure 3 illustrates the variation in current for different angles of incidence. It is noted that broadside incidence produces the largest currents and that the shape of the current distribution is relatively unchanged with the change in the angle of incidence.

TABLE: CONVERGENCE OF SOLUTION FOR THE AXIAL CURRENT DISTRIBUTION

$\theta_o = 6^o$, $kL = \frac{\pi}{2}$, $L = 1m$, $\theta = 0$ and E_o and $1 \frac{V}{m}$

$$\text{CURRENT} = I_{tR} + i\, I_{tI}$$

N	$Max(I_{tR})$	$Max(I_{tI})$	I_{tR} z=0	I_{tI} z=0
10	1.021 ma	1.706 ma	0.1497 ma	0.4145 ma
20	1.067	1.730	0.1427	0.3878
30	1.078	1.759	0.1331	0.3658

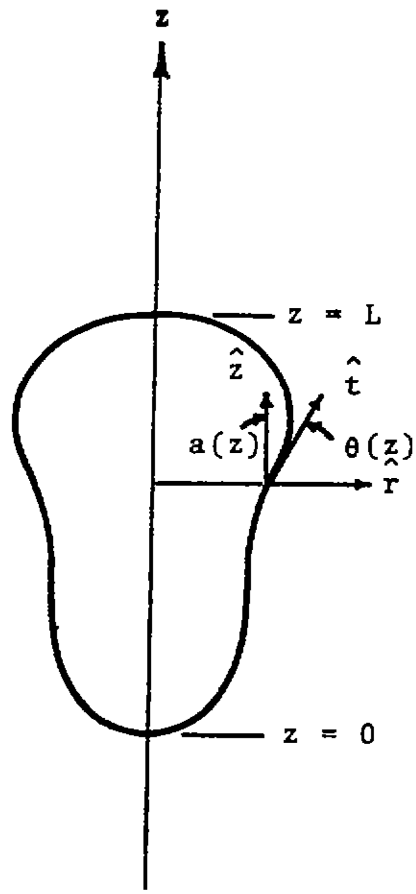

FIGURE 1: BODY OF REVOLUTION

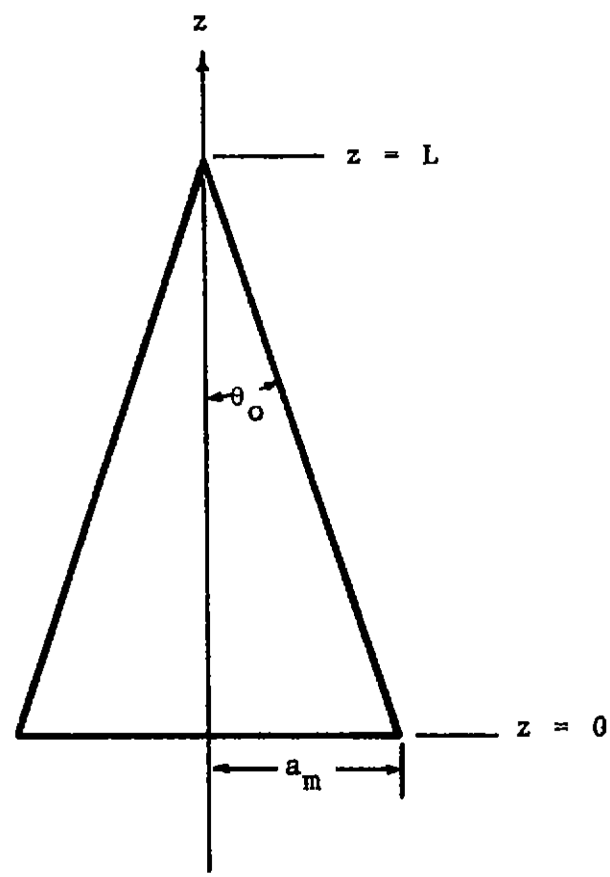

FIGURE 2: CONE TRUNCATED WITH A FLAT END PLATE

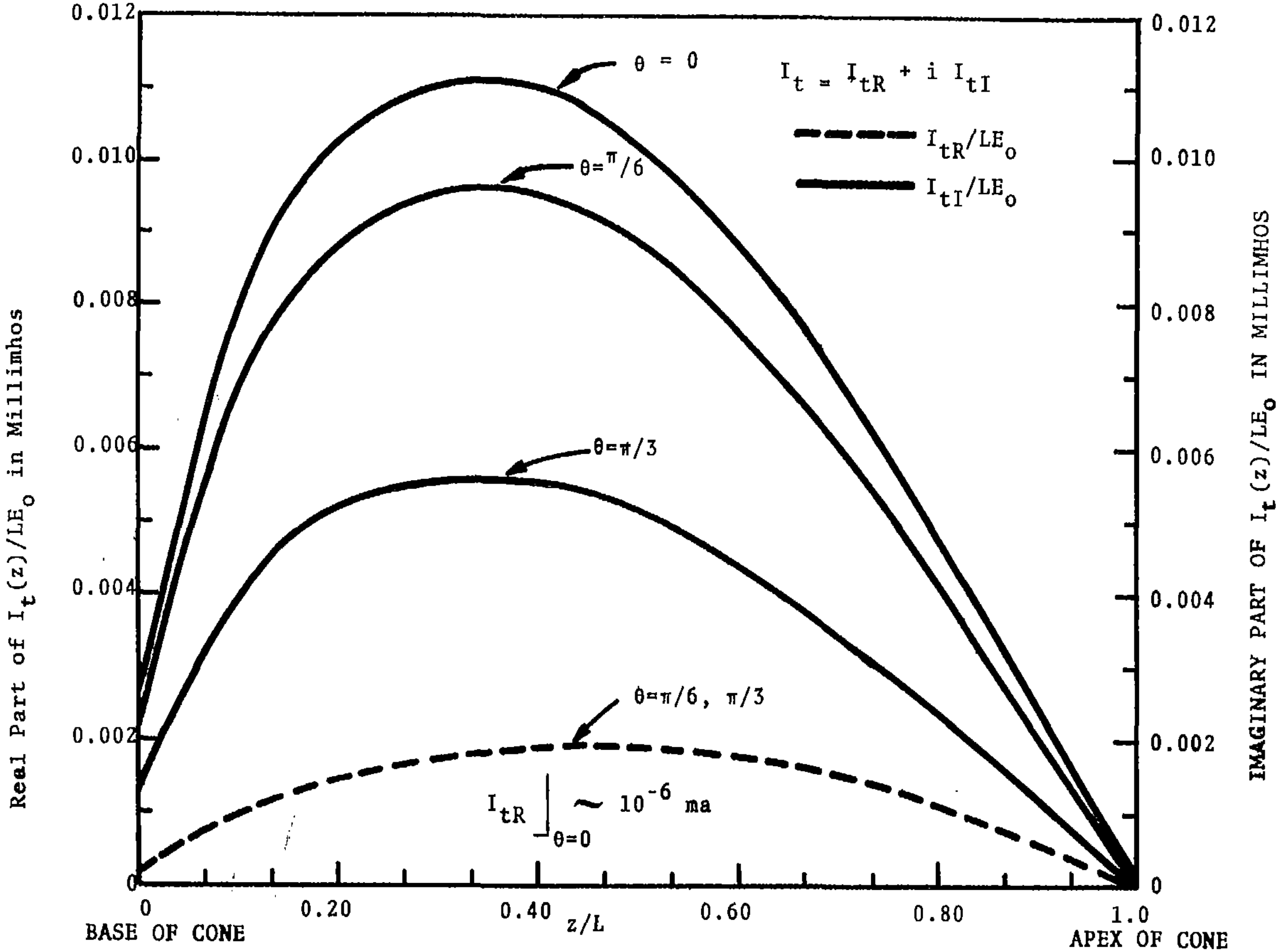

FIGURE 3: AXIAL CURRENT DISTRIBUTION ON CONE FOR VARIOUS ANGLES OF INCIDENCE, HERE $\theta_o = 5^\circ$, $N = 10$, $kL = 0.1$

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