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Interaction Notes

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An Approach to Certain Cable Shielding Calculations

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Abstract

At frequencies high enough that a cable shield may be considered to be a perfect conductor, and yet not so high that the cable diameter is comparable to a wavelength, leakage through the shield can occur by both inductive and capacitive coupling. An explicit statement of the two static problems that must be solved to determine these two coupling effects is given in this note, along with a method of fitting terms representing the two leakage sources into the conventional transmission-line equations. Two simple cable models are examined for illustrative purposes, more realistic models being left for study in future notes.

cables, shielding, conductors

Contents

Acknowledgement

Dr. C. E. Baum conjectured that there should be a source term in the current-change transmission-line equation describing cable shielding. A derivation of that source term is given in section III of this note.

I. Introduction

This note is about an approach to certain periodic cable shield calculations. "Periodic", in this case, means that the shield geometry is a periodic function of the distance along the cable. It will be assumed that the shield and cable are constructed of perfectly conducting material and that the maximum diameter of the cable, with its shield, is much less than the free-space wavelength corresponding to the frequency of the electromagnetic field. The minimum length that must be traveled along the cable before the geometry becomes completely indistinguishable from the geometry at some starting point will be denoted by Δ. It will be assumed that Δ is much less than an electromagnetic wavelength. As a special case Δ can be zero, and thus the shield can be uniform. Another special case is a cable shield with an infinite row of identical apertures along it. A third special case is a perfectly regular braided shield.

In this particular note, no calculations will be made of realistic models of practical cable shields. Rather, a general approach and a theoretical justification of this approach will be given. The calculations will be reduced to electrostatic and magnetostatic considerations. Transmission-line equations will be derived for the current on the inner conductor of the cable and the voltage between the inner conductor and the shield. The sources in these equations will be the total current on the shielded cable and the total linear charge density on the shielded cable. The coefficients of these source terms can be derived from static considerations, and that is the point of this note. A third type of source term in the transmission-line equations, due to direct coupling with some incident wave, will be stated and then shown to be identically zero for most practical shields. Since no calculations for practical shields will be made here, no extensive numerical results will be presented. However, two particular examples will be discussed for illustrative purposes, and the second example is a fairly realistic model of an impractical shield.

A discussion of how to fit the effect of isolated leaks in the shield into the transmission-line equations, and a discussion of the effect of finite shield conductivity on those equations, will be left to future notes.

In the next section is a derivation of the transmission-line equation for voltage change and a statement of the particular magnetostatic problem that must be solved to specify its source term completely.

In the third section is a derivation of the transmission-line equation for current change and a statement of the particular electrostatic problem that must be solved to specify its source term completely.

In the fourth and fifth sections, simple examples of the use of the previous results are given. The fourth section deals with a shielded cable where the inner cable is a circular cylinder and the shield is a unidirectionally conducting circular cylindrical shell. The fifth section deals with a shielded cable where the inner cable is a circular cylinder and the shield consists of a finite number of uniformly spaced thin wire helices on a circular cylindrical surface concentric with the inner cable.

In the last section is a discussion of a few points about the transmission-line equations derived in the second and third sections.

II. Voltage-Change Equation

To obtain the voltage-change transmission-line equation, one can integrate the normal component of Maxwell's equation,

$$\nabla \times \underline{E} = i\omega\mu_o \underline{H}, \tag{1}$$

over a surface, S_1, such as that bounded by path 1 of figure 1. In equation (1), and in the rest of this note, a harmonic time dependence of the form $\exp(-i\omega t)$ is assumed and suppressed. Figure 1 is a picture of a periodic shielded cable. Segments a and c of path 1 are two identical lines in planes perpendicular to the cable direction, and a distance Δ apart along the direction of the cable, that join the cable itself to its shield. The other two segments of path 1 lie along the cable and its shield. The segment on the shield, b, is restricted to lie along a path of perfect conductivity, thus all apertures must be avoided and, if the shield conducts in only one direction, the path must lie along that direction. A further restriction on segments b and d is that they should not make any unnecessary loops around the shielded cable. Of course, if segment b is required to make a loop around the shield by the particular shield geometry then segment d should make a loop around the inner cable.

Once path 1 has been chosen, the pair of segments, b and d, can be repeated indefinitely along the length of the cable and so can be used to define a one-dimensional space. The distance along the cable, z, seems the most natural and appropriate coordinate of this space.

Now, performing the integral of the normal component of equation (1) over the surface bounded by path 1, and applying Stokes' theorem to the left-hand side of the equation, one obtains

$$-\delta V \equiv V(z_a) - V(z_c) = -i\omega\Phi \tag{2}$$

where

$$V(z_a) \equiv \int_a \underline{E}\cdot d\underline{\ell} \tag{3}$$

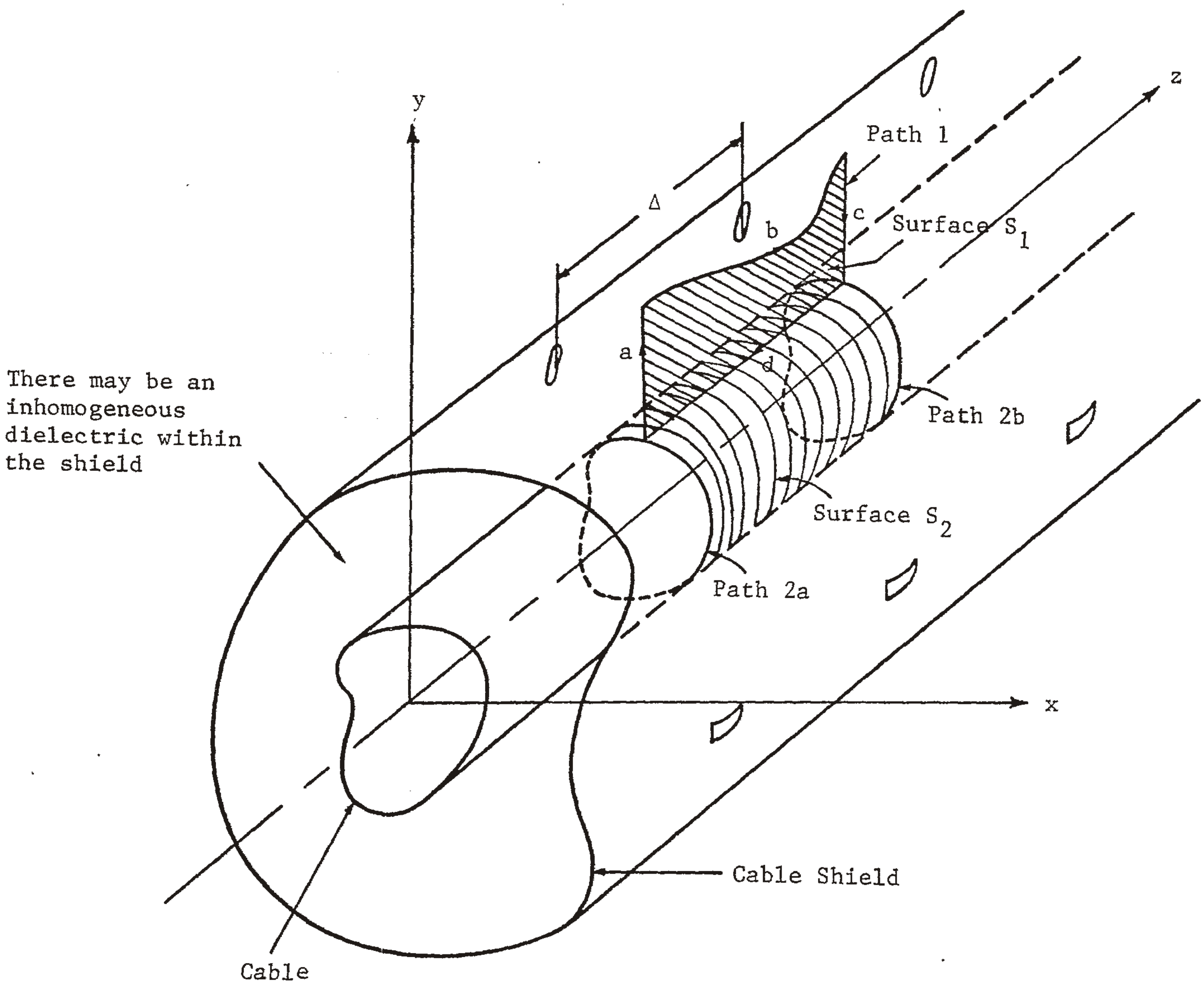

Figure 1. A Periodic Shielded Cable.

and the direction of $\underline{\ell}$ is assumed to be from the cable to the shield. In equation (2),

$$\Phi \equiv \mu_o \int_{S_1} \underline{H} \cdot d\underline{S}, \tag{4}$$

and the normal to S_1 has a positive component in the clockwise direction when one is looking along the z-axis.

The basic assumptions that the electromagnetic wavelength is long compared to both the maximum diameter of the shield and Δ imply that Φ can be calculated at the static limit. They further imply that, when the sources of any incident wave are remote, one can assume any incident field to be uniform over a period of the cable.

It follows, therefore, from magnetostatic considerations, that Φ can be assumed to depend linearly on only four quantities: the total current along the cable, the total current along the shield, and the two components of the external magnetic field perpendicular to the direction of the cable. A further examination of the magnetostatic problem reveals that the two proportionality constants connecting Φ with the components of the external field are negligible for most practical shielded cables. In particular these constants are identically zero if the shielded cable has two planes of symmetry intersecting in the inner cable (see Appendix A). Perhaps one could even define a shield as a conductor configuration where the direct coupling with the external field is negligible. Of course for some structures these terms are dominant [1].

This nevertheless leaves two quantities on which Φ depends linearly, the current on the inner cable and the current on the shield. One can therefore write

$$\frac{\Phi}{\Delta} = L_c I_c + L_s I_s \tag{5}$$

This equation defines two magnetostatic problems that must be solved to completely specify the voltage-change transmission-line equation. ΔL_c is the flux through S_1 when one ampere flows on the cable and no current flows on the shield. ΔL_s is the flux through S_1 when one ampere flows on the shield and no current flows on the cable. It should be noted that if the shield consists of more than one conductor, then there should be no net flux linking any pair of conductors of the shield. Of course, for practical shields,

$$|L_s| << |L_c|$$

The only externally measurable current on a cable is

$$I_T \equiv I_c + I_s,$$

so equation (5) should be rewritten as

$$\frac{\Phi}{\Delta} = (L_c - L_s)I_c + L_s I_T \tag{6}$$

or, defining

$$L \equiv L_c - L_s$$

and

$$I \equiv I_c$$

it follows from equations (4) and (6) that

$$\frac{\delta V}{\Delta} = i\omega LI + i\omega L_s I_T.$$

Since Δ is very small compared to a wavelength, it is quite accurate to write

$$\frac{dV}{dz} = i\omega LI + i\omega L_s I_T \tag{7}$$

Equation (7) is the usual transmission-line voltage-change equation with a source term proportional to the measurable, total current on the shielded cable.

It has been shown in principle how to compute the two proportionality constants, L and L_s, that appear in equation (7), but it may be easier in practice to use some good way of measuring these quantities [2].

An important particular case of equation (7) should be mentioned. If the inner cable can be assumed to be a circular cylinder of radius ρ_1 and the shield can be assumed to be within a very thin cylindrical shell, of radius ρ_2, that is coaxial with the inner cable, then it is easy to show that (see Appendix B)

$$L_c = \frac{\mu_o}{2\pi} \ln(\rho_2/\rho_1)$$

and thus

$$L = \frac{\mu_o}{2\pi} \ln(\rho_2/\rho_1) - L_s .$$

For this special case, therefore, there is only one nontrivial magnetostatic problem to be solved. Two simple examples of this special case will be studied in the fourth and fifth sections of this note.

It should be pointed out that equation (7) is not new. Equations equivalent to it have been used in previous cable shielding calculations [3]. However, the derivation and discussion in this section may be useful and suggestive. In the next section the other transmission-line equation will be derived, and the source term in that equation does seem to be new.

For ease of reference in future notes a name will be given to the quantity L_s. It seems appropriate to call it the inductive coupling coefficient per unit length.

III. Current-Change Equation

To obtain the current-change transmission-line equation, one can integrate the normal component of Maxwell's equation

$$\nabla\times\underline{H} = -\, i\omega\varepsilon\underline{E} + \underline{J} \tag{8}$$

over a surface, S_2, such as that bounded by path 2 of figure 1. Surface S_2 is that portion of the surface of the inner cable bounded by two planes perpendicular to the cable and a distance Δ apart. The intersections of the two planes bounding S_2 with the surface of the inner cable define the two segments, a and b, of path 2.

Now, performing the integral of the normal component of equation (8) over surface S_2, and applying Stokes' theorem to the left-hand side of the equation, one obtains

$$-\,\delta I \equiv I(z_a) - I(z_b) = -\, i\omega q_c \tag{9}$$

where

$$I(z_a) \equiv \int_{2a} \underline{H}\cdot d\underline{\ell} \tag{10}$$

and the direction of $\underline{\ell}$ is as indicated in figure 1. Of course, I is the total current on the inner cable. In equation (9),

$$q_c \equiv \varepsilon \int_{S_2} \underline{E}\cdot d\underline{S}, \tag{11}$$

or, in other words, q_c is the total charge on surface S_2.

One can now use the basic assumptions that the electromagnetic wavelength is long compared to both the maximum diameter of the shield and Δ to justify the calculation of q_c at the static limit. If the sources of any external fields are remote, the external field may be considered uniform over any period of the cable, as in the previous section.

It follows, from electrostatic considerations, that V, the potential difference between the cable and the shield can be assumed to depend linearly on only four quantities: the charge per cable period on the inner cable, the charge per cable period on the shield, and the two components of the external electric field perpendicular to the direction of the cable. For practical shielded cables the direct coupling with the external field is negligible, and is identically zero if the shielded cable has two planes of symmetry intersecting in the inner cable (see Appendix A). The two constants of proportionality between V and the external field will therefore be neglected. This allows one to write

$$\Delta V = S_c q_c + S_s q_s \tag{12}$$

This equation defines two electrostatic problems that must be solved to completely specify the current-change transmission-line equation. S_c is the potential difference between the cable and its shield when there is one coulomb per cable period on the inner cable and no net charge on the shield. S_s is the potential difference between the cable and its shield when there is one coulomb per cable period on the shield and no net charge on the inner cable. It should be noted that, if the shield consists of more than one conductor, there should be no potential difference between any two conductors of the shield. For most practical shields

$$|S_s| << |S_c|$$

The externally measurable charge per unit length on a cable is

$$Q_T \equiv \frac{q_c}{\Delta} + \frac{q_s}{\Delta} ,$$

so it is useful to rewrite equation (12) as

$$V = (S_c - S_s) \frac{q_c}{\Delta} + S_s Q_T. \tag{13}$$

or, defining

$$C \equiv \frac{1}{S_c - S_s},$$

from equations (9) and (13) it follows that

$$\frac{\delta I}{\Delta} = i\omega CV - i\omega CS_s Q_T. \tag{14}$$

Since Δ is very small compared to a wavelength it is quite accurate to write

$$\frac{dI}{dz} = i\omega CV - i\omega CS_s Q_T. \tag{15}$$

Equation (15) is the usual transmission-line equation for current change with a source term proportional to the total charge per unit length on the shielded cable. This source term does not seem to have been used much in cable shielding calculations. Perhaps it has been assumed that this source term is unimportant, but it is important sometimes. A further discussion of this point may be found in the sixth section.

An important particular case of equation (15) occurs when the inner cable is a circular cylinder of radius ρ_1, and the shield can be assumed to lie within a very thin cylindrical shell, of radius ρ_2, that is coaxial with the inner cable. For this case it is easy to show that, if ε is a function of ρ only, then (see Appendix B)

$$S_c = \frac{1}{2\pi}\int_{\rho_1}^{\rho_2} \frac{d\rho}{\rho\varepsilon(\rho)}$$

and thus

$$C^{-1} = \frac{1}{2\pi}\int_{\rho_1}^{\rho_2} \frac{d\rho}{\rho\varepsilon(\rho)} - S_s$$

For this special case, therefore, there is only one nontrivial electrostatic problem to be solved. Simple examples of this special case will be studied in the next two sections.

Again, for ease of future reference one could call the combination CS_s the capacitive coupling coefficient per unit length, while the quantity S_s alone could be called a mutual susceptance per unit length.

IV. Unidirectionally Conducting Shield

As an example, in this section the necessary static problems will be solved for the case where the inner conductor is a circular cylinder of radius a and the shield is a circular cylindrical shell of radius b that conducts in only one direction. The conduction direction will be assumed to maintain a constant angle, ψ, with the z axis of the cable (see figure 2).

This cable falls into the category of the special cases discussed at the ends of sections II and III. Therefore, one can immediately say that

$$L_c = \frac{\mu_o}{2\pi} \ln(b/a) \tag{16}$$

and

$$S_c = \frac{1}{2\pi\varepsilon_o} \ln(b/a). \tag{17}$$

Furthermore, it is irrelevant to the electrostatic problem that the shield conducts in only one direction; since the potential of the shield is constant at any given cross section, it might as well be perfectly conducting. From this it is clear that

$$S_s = 0. \tag{18}$$

Now the only static problem that requires further investigation is the calculation of L_s. This quantity may be calculated by considering the period of the cable to be the distance along the cable corresponding to exactly one circulation of a conductance path of the shield about the axis of the cable, i.e.,

$$\Delta = 2\pi b \cot \psi$$

Now the longitudinal component of current density in the shield will give rise to no field inside the shield while the circulating component, given by

$$K_\phi = \frac{I_s}{2\pi b} \tan \psi,$$

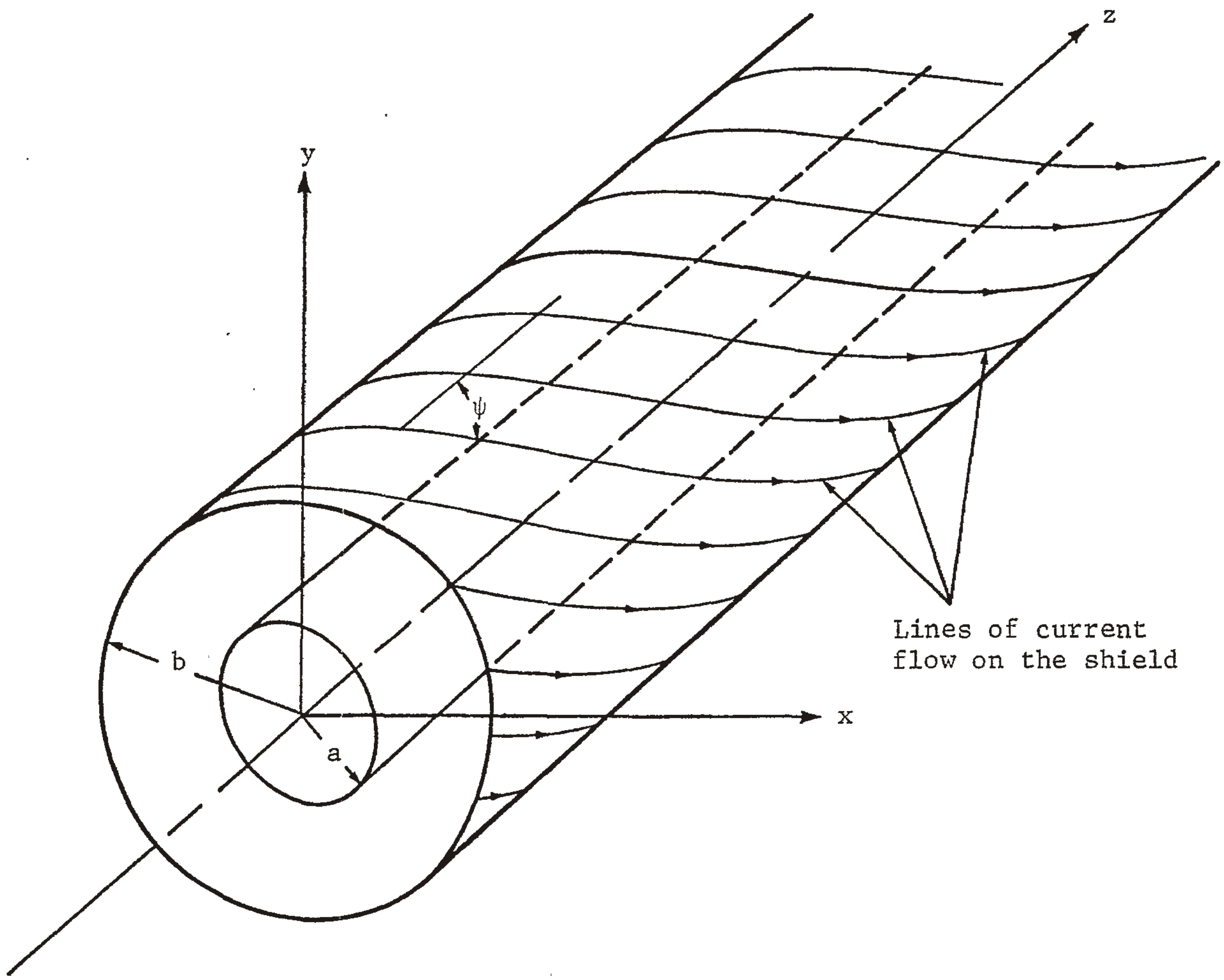

Figure 2. A Unidirectionally Conducting Shield.

will give rise to a z component of the H field inside the shield of magnitude

$$H_z = K_\phi = \frac{I_s}{2\pi b} \tan \psi.$$

Now, taking account of the direction of the normal to the surface involved in the definition of Φ, one can immediately state that

$$\Phi = - \mu_o H_z \pi (b^2 - a^2)$$

$$= - \mu_o \frac{I_s}{2\pi b} (\tan \psi) \pi (b^2 - a^2).$$

From this equation it is clear that

$$L_s = \frac{\Phi}{\Delta I_s}$$

$$= - \frac{\mu_o}{4\pi} \tan^2 \psi (1 - a^2/b^2). \tag{19}$$

Suppose now that the cable under examination is infinitely long and immersed in an incident plane wave whose propagation vector is perpendicular to the cable axis. In such a case there can be no variation of any quantity with z; thus, it follows from equation (7) that

$$I = - \frac{L_s}{L} I_T.$$

Taking account of equations (16) and (19) the above equation may be rewritten as

$$\frac{I}{I_T} \equiv F = \frac{1}{1+[2b^2/(b^2-a^2)]\cot^2 \psi \ln(b/a)} . \tag{20}$$

The scattering of a plane wave by the structure studied in this section may be treated exactly by the method of the separation of variables. This treatment may be found in Appendix C. In Appendix C it is also shown that the exact equation for F reduces precisely to equation (20) in the low-frequency limit. Curves of F as a function of a/b for various ψ's are given in figure 3.

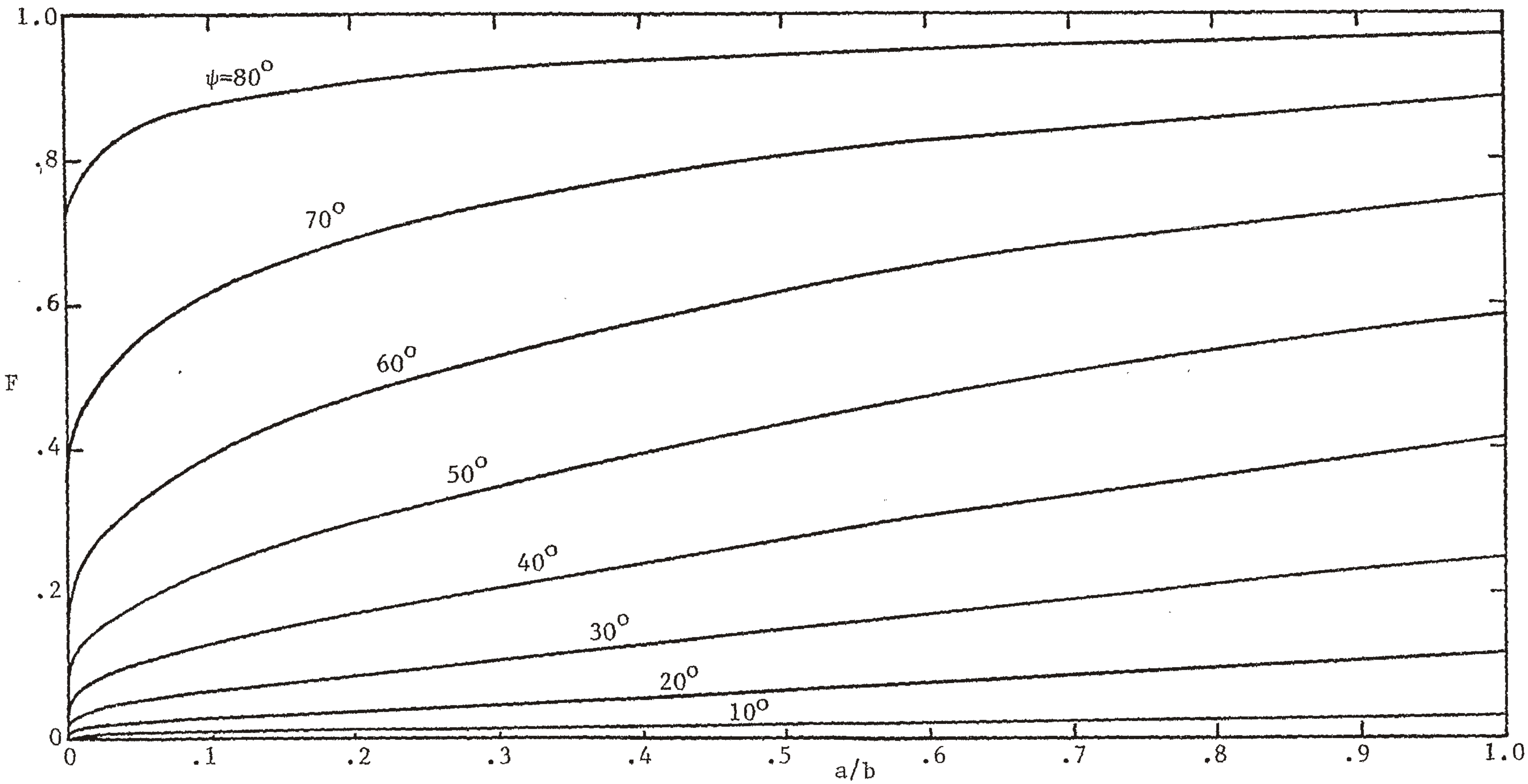

Figure 3. F for a Unidirectionally Conducting Shield.

V. Multi-Helix Shield

As a second example, in this section the necessary static problems will be solved for the case where the inner conductor is a circular cylinder of radius a and the shield is made up of N uniformly spaced thin-wire helices confined to a circular cylindrical shell of radius b as shown in figure 4. The helices are coaxial with the inner conductor and the angle, on the surface to which they are confined, between any helical wire and a line parallel to the z-axis will be denoted by ψ. The space in the cylindrical shell between the inner conductor and the helices is assumed to be filled with a dielectric material of permittivity ε.

This cable falls into the category of the special cases discussed at the ends of sections II and III. Therefore, one can immediately say that

$$L_c = \frac{\mu_o}{2\pi} \ln(b/a)$$

$$S_c = \frac{1}{2\pi\varepsilon} \ln(b/a).$$

There are two quantities that still have to be calculated, S_s and L_s. The calculation of S_s will be given first.

To compute S_s, it will be assumed that the wires of the helices are thin compared with a, b, and the shortest distance between helices. With this assumption, the potential anywhere outside the wires can be computed by assuming a uniform line charge along the center of each helical wire. The potential due to one helix will be computed first; the total potential may then be computed by simple superposition. Using the usual cylindrical coordinates (ρ,ϕ,z), it will be assumed that the line charge passes through the z plane at the point $\phi = \phi_o$. The surface charge density of the line charge on the surface $\rho = b$ is given by

$$\sigma(b,\phi,z) = \frac{q_s}{b} \delta(\phi - \phi_o - z \frac{\tan \psi}{b}) \equiv \frac{q_s}{b} \delta(\theta) \tag{21}$$

where q_s is the total charge per unit length along the z axis. It is possible to write the potential in the region outside the helical line charge (region I) in the form (see Appendix D)

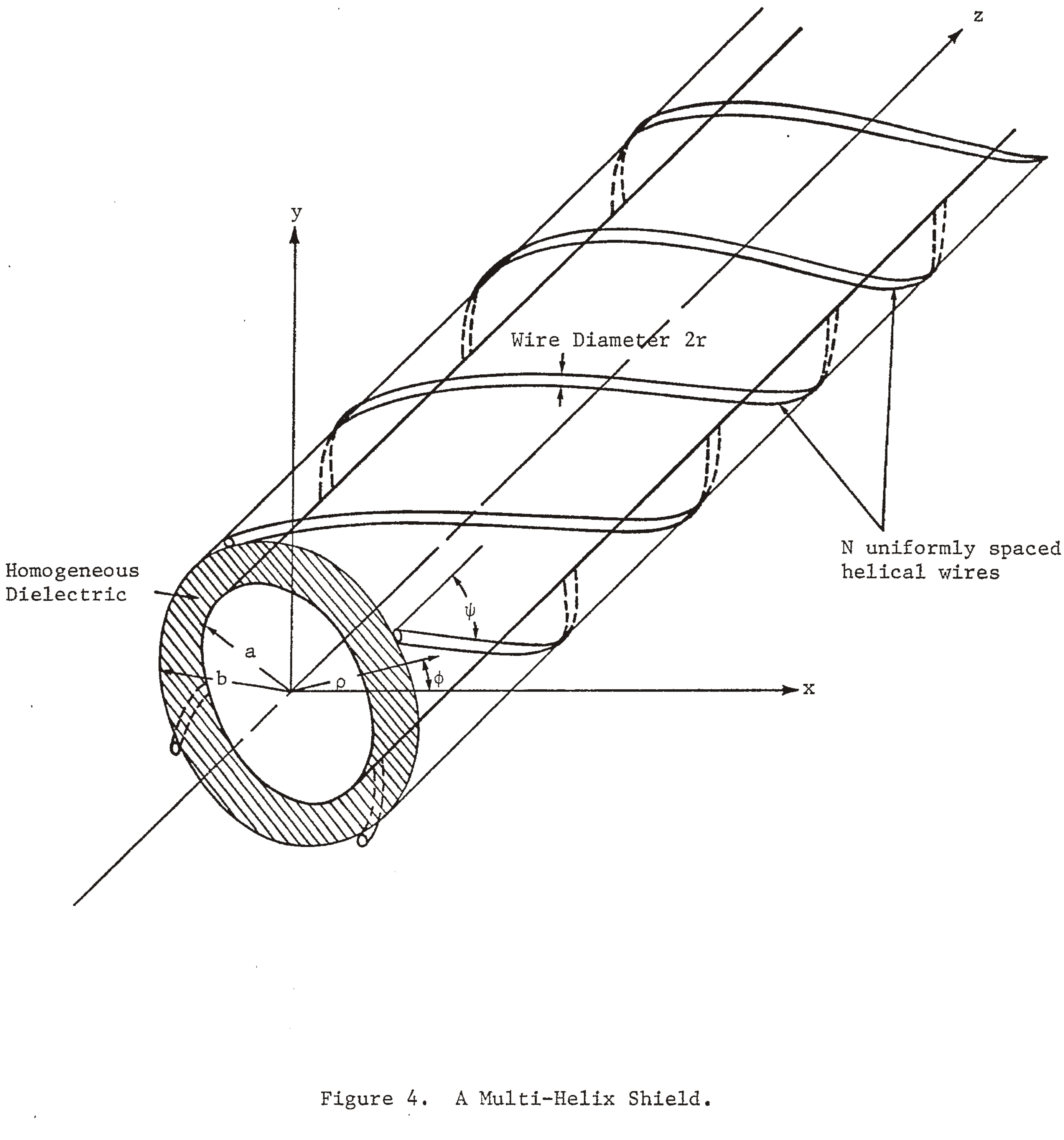

Figure 4. A Multi-Helix Shield.

$$\Phi^{I}(\rho,\phi,z) = V_\infty - \frac{Q_T}{2\pi\varepsilon_o} \ln(\rho/b) + \sum_{n=1}^{\infty} C_n \cos(n\theta) \frac{K_n(nt\rho/b)}{K_n(nt)} \tag{22}$$

where

$$t \equiv \tan\psi$$

In the region between the inner conductor and the helical line charge (region II) it is possible to write

$$\Phi^{II}(\rho,\phi,z) = V_c - \frac{q_c}{2\pi\varepsilon} \ln(\rho/a) + \sum_{n=1}^{\infty} C_n \cos(n\theta) \frac{F_n(nt\rho/b)}{F_n(nt)} \tag{23}$$

where

$$F_n(nt\rho/b) = K_n(nt\rho/b) I_n(nta/b) - I_n(nt\rho/b) K_n(nta/b).$$

The above two representations for Φ^{I} and Φ^{II} satisfy Laplace's equation in their respective regions and the requirement, imposed by the symmetry of the helical line charge, that

$$\Phi(\rho,\phi + \alpha, z + b\alpha\cot\psi) = \Phi(\rho,\phi,z)$$

In order that there be no net charge on the inner cylinder (a requirement of the electrostatic problem through which S_s is defined) one must set

$$q_c = 0, \qquad Q_T = q_s.$$

In order that the potential be continuous through the surface $\rho = b$, one must set

$$V_\infty = V_c.$$

The remaining constants, C_n, must be computed from the condition that the change in the normal component of the electric displacement through the surface $\rho = b$ must be equal to the surface charge density. From (21), (22), and (23), then

$$\frac{q_s}{b}\,\delta(\theta) = \frac{q_s}{2\pi b} + \sum_{n=1}^{\infty} C_n \cos(n\theta)\,\frac{nt}{b}\left[\varepsilon\frac{F_n'(nt)}{F_n(nt)} - \varepsilon_o\,\frac{K_n'(nt)}{K_n(nt)}\right]$$

where primes denote differentiation with respect to the total argument of a function. Thus,

$$\frac{q_s}{b} = \pi C_n\,\frac{nt}{b}\left[\varepsilon\,\frac{F_n'(nt)}{F_n(nt)} - \varepsilon_o\,\frac{K_n'(nt)}{K_n(nt)}\right]. \tag{24}$$

From (23) and (24) one can then write the potential at the surface $\rho = b$ in the form

$$\Phi(b,\phi,z) = V_c + \sum_{n=1}^{\infty}\frac{q_s\cos(n\theta)}{\pi nt}\left[\varepsilon\,\frac{F_n'(nt)}{F_n(nt)} - \varepsilon_o\,\frac{K_n'(nt)}{K_n(nt)}\right]^{-1}$$

$$\equiv V_c + q_s\sum_{n=1}^{\infty}\frac{D_n}{nt}\cos(n\theta)$$

Now, if there are N wires, each with charge q_s/N, one may write the total potential on $\rho = b$ in the form

$$\Phi_N(b,\phi,z) = V_c + \frac{q_s}{N}\sum_{n=1}^{\infty}\frac{D_n}{nt}\sum_{m=1}^{N}\cos(n[\theta + 2\pi m/N])$$

$$= V_c + q_s\sum_{p=1}^{\infty}\frac{D_{pN}}{pNt}\cos(pN\theta)$$

Now, if each wire has radius r, the intersection of the surface of one wire with the surface $\rho = b$ is given by the equation

$$\theta = r/(b\cos\psi).$$

Thus, the potential at the surface of a wire is given by

$$V_s = V_c + q_s\sum_{p=1}^{\infty}\frac{D_{pN}}{pNt}\cos\left(\frac{pNt}{b\cos\psi}\right),$$

Now by defining the optical coverage, c, as $c \equiv Nr/(\pi b \cos\psi)$ one can write

$$- S_s = \frac{V_s - V_c}{q_s} = \sum_{p=1}^{\infty} \frac{D_{pN} \cos(p\pi c)}{pNt} \tag{25}$$

In the important special case where $\varepsilon = \varepsilon_o$, it follows after a little algebra that

$$\pi\varepsilon_o D_n = nt \frac{K_n(nt)}{K_n(nta/b)} \left[I_n(nt)K_n(nta/b) - K_n(nt)I_n(nta/b) \right]$$

and thus

$$- S_s = \frac{1}{\pi\varepsilon_o} \sum_{p=1}^{\infty} \frac{K_{pN}(pNt)}{K_{pN}(pNta/b)} \left[I_{pN}(pNt)K_{pN}(pNta/b) - K_{pN}(pNt)I_{pN}(pNta/b) \right] \cos(p\pi c). \tag{26}$$

If one further specializes this equation to the case where ψ approaches zero the result is

$$- S_s = \frac{1}{2\pi\varepsilon_o} \sum_{p=1}^{\infty} \left[1 - (a/b)^{2Np} \right] \frac{\cos(p\pi c)}{pN} ,$$

which can also be written in the closed form

$$- S_s = \frac{1}{4\pi\varepsilon_o N} \ln \left\{ (a/b)^{2N} + \left[\frac{1-(a/b)^{2N}}{2 \sin(c\pi/2)} \right]^2 \right\} \tag{27}$$

There are several other special cases of equation (25) that one could examine. For example it can be shown that if a/b approaches zero then

$$\pi\varepsilon_o D_n = \frac{I_n(nt)K_n(nt)}{(\varepsilon/\varepsilon_o)K_n(nt)I_n'(nt) - K_n'(nt)I_n(nt)}$$

If, in addition, $\varepsilon = \varepsilon_o$ it follows that

$$\pi\varepsilon_o D_n = ntI_n(nt)K_n(nt)$$

and

$$- S_s = \frac{1}{\pi\varepsilon_o} \sum_{p=1}^{\infty} I_{pN}(pNt)K_{pN}(pNt)\cos(p\pi c)$$

If ψ approaches zero in this equation the result is a special case of equation (27), while if N is large the result is

$$S_s = \frac{\cos\psi}{2\pi\varepsilon_o N} \ln[2 \sin(c\pi/2)].$$

This completes the treatment of S_s. The next item is the computation of L_s.

In order to compute L_s one may make use of Ω, a magnetic scalar potential, in regions I and II. In region I one can set

$$\underline{B}^{I} = \frac{\mu_o I_s}{2\pi\rho} \underline{e}_{\phi} - \nabla\Omega_I \tag{28}$$

and in region II

$$\underline{B}^{II} = \frac{\mu_o I_s \tan\psi}{2\pi b} \underline{e}_z - \nabla\Omega_{II}. \tag{29}$$

Again the wire will be assumed to be very thin, thus allowing one to write, for the surface current on the surface $\rho = b$,

$$K_z = \frac{I_s}{b} \delta(\phi - \phi_o - z \tan\psi/b) \equiv \frac{I_s}{b} \delta(\theta) \tag{30}$$

$$K_{\phi} = \frac{I_s t}{b} \delta(\theta) \tag{31}$$

A solution for Ω that can be made to match these surface current conditions is (see Appendix D)

$$\Omega_I = \sum_{n=1}^{\infty} C_n \sin n\theta \frac{K_n(nt\rho/b)}{K_n'(nt)} \tag{32}$$

$$\Omega_{II} = \sum_{n=1}^{\infty} C_n \sin n\theta \frac{G_n(nt\rho/b)}{G_n'(nt)} \tag{33}$$

where

$$G_n(nt\rho/b) = K_n(nt\rho/b)I_n'(nta/b) - I_n(nt\rho/b)K_n'(nta/b)$$

Equations (32) and (33) assure that B_ρ is zero at the surface of the inner conductor and B_ρ is continuous through the surface $\rho = b$.

The condition that must be used to determine the C_n is that

$$\underline{e} \times (\underline{B}^{I}(b,\phi,z) - \underline{B}^{II}(b,\phi,z)) = \mu_o \underline{K}(\phi,z)$$

i.e.

$$\underline{e}_z\left[\frac{\mu_o I_s}{2\pi b} - \frac{\partial\Omega_I}{b\partial\phi} + \frac{\partial\Omega_{II}}{b\partial\phi}\right] + \underline{e}_\phi\left[\frac{\mu_o I_s t}{2\pi b} + \frac{\partial\Omega_I}{\partial z} - \frac{\partial\Omega_{II}}{\partial z}\right] = \frac{\mu_o I_s}{b}\,\delta(\theta)(e_z + e_\phi t)$$

Both components of this equation may be satisfied by setting, in equations (32) and (33),

$$\pi n C_n\left[\frac{K_n(nt)}{K_n'(nt)} - \frac{G_n(nt)}{G_n'(nt)}\right] = \mu_o I_s$$

which reduces to

$$C_n = \frac{\mu_o I_s t}{\pi} \cdot \frac{G_n'(nt)K_n'(nt)}{K_n'(nta/b)}$$

Thus, from (33),

$$\Omega_{II} = \frac{\mu_o I_s t}{\pi} \sum_{n=1}^{\infty} \sin n\theta \, \frac{K_n'(nt)}{K_n'(nta/b)}\, G_n(nt\rho/b) \qquad (34)$$

Now, if there are N uniformly spaced line currents whose sum is I_s, it is easy to show by superposition that

$$\Omega_{II} = \frac{\mu_o I_s t}{\pi} \sum_{p=1}^{\infty} \sin(Np\theta)\, \frac{K_{Np}'(Npt)}{K_{Np}'(Npta/b)}\, G_{Np}(Npt\rho/b) \qquad (35)$$

One can choose as the surface of integration used for the computation of Φ the surface

$$\theta = r/b \cos \psi \equiv \pi c/N$$

where r is again the radius of the wire. Also, one may write

$$dS_z = - \rho d\rho d\phi = - (t/b)\rho d\rho dz$$

$$dS_\phi = d\rho dz.$$

It then follows easily from (29) and (35) that

$$L_s = \frac{1}{I_s}\frac{d\Phi}{dz}$$

$$= - \frac{\mu_o t^2}{4\pi}\left(1 - \frac{a^2}{b^2}\right) - \frac{\mu_o t}{\pi}\int_a^b d\rho \sum_{p=1}^{\infty} Np \cos(p\pi c)\left[\frac{t^2}{b^2} + \frac{1}{\rho^2}\right]\rho G_{Np}(Npt\rho/b)\ \frac{K'_{Np}(Npt)}{K'_{Np}(Npta/b)}$$

$$= - \frac{\mu_o t^2}{4\pi}\left(1 - \frac{a^2}{b^2}\right) - \frac{\mu_o t}{\pi}\int_a^b d\rho \sum_{p=1}^{\infty} \frac{\cos(p\pi c)}{Np}\ \frac{K'_{Np}(Npt)}{K'_{Np}(Npta/b)}\ \frac{d}{d\rho}\ (\rho\ \frac{dG}{d\rho})$$

$$= - \frac{\mu_o t^2}{4\pi}\left(1 - \frac{a^2}{b^2}\right) - \frac{\mu_o t^2}{\pi}\sum_{p=1}^{\infty} \cos(p\pi c)\frac{K'_{Np}(Npt)}{K'_{Np}(Npta/b)}\ G'_{Np}(Npt) \tag{36}$$

where

$$G'_{Np}(Npt) = K'_{Np}(Npt)I'_{Np}(Npta/b) - K'_{Np}(Npta/b)I'_{Np}(Npt)$$

In equation (36) one can recognize the first term as being the contribution to L_s due to the restriction of the shield current to one direction. This is all there is to L_s in the shield studied in the previous section. The second term in equation (36) gives the contribution to L_s due to the fact that the shield is really made up of several thin wires rather than an infinitesimally thin uniform sheet.

There are several special cases of equation (36) that one can study. Some of them will be mentioned here.

If ψ approaches zero, the small argument asymptotic forms of the Bessel functions can be substituted and this leads directly to

$$L_s \to -\frac{\mu_o}{2\pi}\sum_{p=1}^{\infty}\frac{\cos(p\pi c)}{Np}\left[1-\left(\frac{a}{b}\right)^{2Np}\right]$$

$$= -\frac{\mu_o}{4\pi N}\ln\left\{\left(\frac{a}{b}\right)^{2N}+\left[\frac{1-(a/b)^{2N}}{2\sin(c\pi/2)}\right]^2\right\} \qquad (37)$$

Thus, from (27) and (35), as one would expect in the case where the cable is independent of z and there are no dielectrics present,

$$L_s = \mu_o\varepsilon_o S_s.$$

Another special case of equation (36) that might be of interest is the one where a/b approaches zero; in that case

$$L_s \to -\frac{\mu_o t^2}{4\pi}+\frac{\mu_o t^2}{\pi}\sum_{p=1}^{\infty}\cos(p\pi c)K'_{Np}(Npt)I'_{Np}(Npt).$$

For large N one can use the uniform asymptotic expansions for large orders of the Bessel functions [4] and sum the resulting series in closed form to give

$$L_s \to -\frac{\mu_o t^2}{4\pi}+\frac{\mu_o}{2\pi N\cos\psi}\ln(2\sin(c\pi/2))$$

In the above equation one should note, from the definition of c, that

$$c\pi/2 \le \pi/2$$

and the equality can only occur when the wires touch each other, a case that doesn't fit the basic assumption of this section that the wires are thin compared to their spacing and the cable diameter. Of course similar remarks apply to the equivalent equation for S_s.

VI. Discussion

There are a few points about differential equations (7) and (15) that should be brought up now.

(a) If the cable shield material cannot be assumed to be perfectly conducting, there should be an extra term in equation (7) to account for this fact. This term will be proportional to the current in the shield, so if the variables I and I_T are to be retained, the coefficient of I in equation (7) would be changed slightly.

(b) No method has been given, in this note, for fitting the effect of an isolated leak or irregularity in the external shield into equations (7) and (15). This topic will be discussed in a future note. It may be that terms describing the direct coupling with some external field, the type of term we have thrown away in this note, become very important for some particular isolated shield defects.

(c) The solutions of equations (7) and (15) for any finite cable involve the specification of boundary conditions on the current and voltage variables. These boundary conditions can be specified in the usual transmission-line theory way by invoking admittance and impedance concepts. In particular it can be shown that, if an admittance Y is connected between the inner cable and the shield at some point z_1, then

$$I(z_1 - 0) - I(z_1 + 0) = YV(z_1).$$

Similarly, if an impedance Z is inserted in the inner cable at some point z_2, then

$$V(z_2 - 0) - V(z_2 + 0) = ZI(z_2).$$

(d) It follows from equations (7) and (15) that the equivalent circuit for one period of the periodic line may be drawn as shown in figure 5 where the

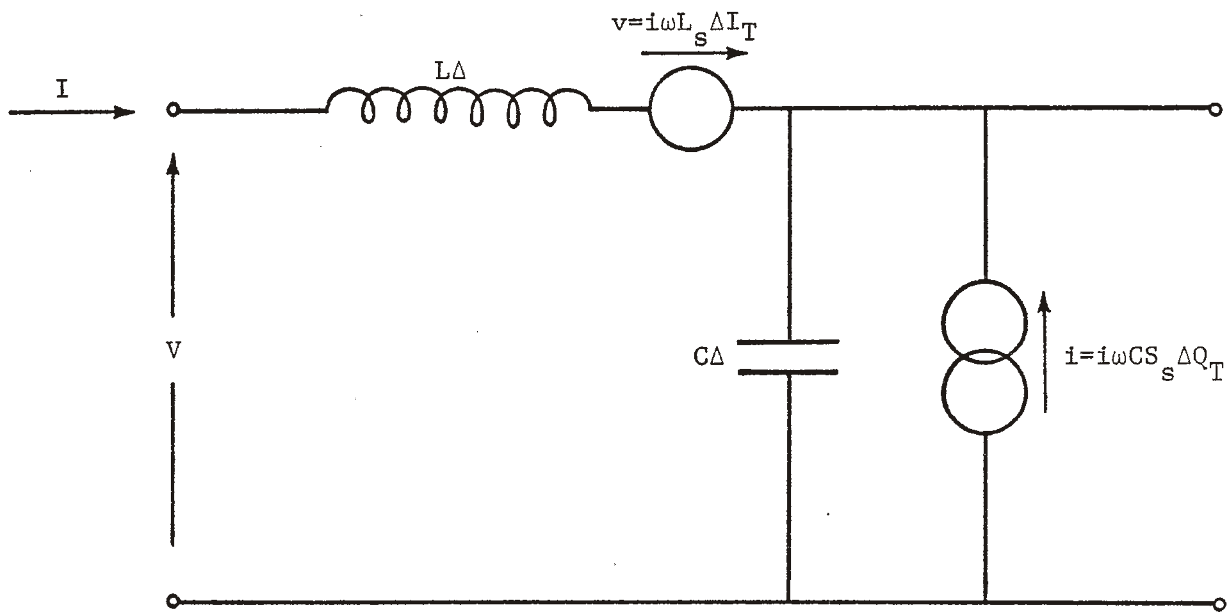

Figure 5. An Equivalent Circuit for a Periodic Section.

Appendix A
Argument for Neglecting Direct Coupling

In this appendix a brief argument will be given for the truth of the statement in the text that, if the shielded cable has two planes of symmetry intersecting in the inner cable, the coupling coefficients between V and the two transverse components of a uniform external electric are zero. An analogous argument may be made to show that the coupling coefficients between Φ and the two transverse components of a uniform external magnetic field are zero.

The uniform, transverse, external, electric field may be decomposed into its two components perpendicular to the two planes of symmetry of the shielded cable. Looking at one such component, one can define the potential of the external field to be zero on the plane of symmetry and proportional to the coordinate perpendicular to the plane of symmetry. Since the structure is symmetric about the symmetry plane and the external potential is antisymmetric about the symmetry plane, the induced potential can be assumed to be antisymmetric about the symmetry plane. This leads to the conclusions that the potential on the inner cable is zero and the potentials at symmetrical points on the shield are equal and opposite. But all points on a cross-section of the shield must have the same potential, therefore this potential is zero and there is no difference in potential between the cable and its shield.

Appendix B

Partial Solutions for a Special Case

A special case was discussed at the ends of the second and third sections. In this appendix a justification of the expressions for L_c and S_c for that special case will be given.

The special shielded cable to be studied is one whose inner cable is a circular cylinder and whose shield is confined to an infinitesimally thin circular cylindrical shell, co-axial with the inner cylinder. The radius of the inner cylinder will be called ρ_1 and the radius of the cylindrical shell will be called ρ_2.

To determine L_c, one must find the magnetic field between the cable and its shield when there is one ampere flowing on the cable and no net axial current on the shell. But, if I is the current on the inner cylinder, one can write

$$\underline{H} = \underline{e}_\phi \frac{I}{2\pi\rho} + \underline{H}^{sc}.$$

Now the boundary condition that $\underline{H}$ be tangential to all conducting surfaces, and the condition that the tangential component of $\underline{H}$ doesn't change through the outer shell (which is sufficient to assure that no net current flows in the shell), are both satisfied by setting $\underline{H}^{sc}$ to be zero. It then follows that

$$L_c = \frac{\Phi}{\Delta I} = \frac{\mu_o}{2\pi} \int_{\rho_1}^{\rho_2} \frac{d\rho}{\rho} = \frac{\mu_o}{2\pi} \ln(\rho_2/\rho_1).$$

To determine S_c, one must find the electric field between the cable and its shield. One can use an argument analogous to the one used above for the magnetic field to say that, even if there is a cylindrically stratified dielectric material present, the electric displacement, $\underline{D}$, is given by

$$\underline{D} = \varepsilon(\rho)\underline{E} = \underline{e}_\rho \frac{q_c}{2\pi\Delta\rho}$$

From this equation it follows that

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$$\underline{D} = \varepsilon(\rho)\underline{E} = \underline{e}_\rho \frac{q_c}{2\pi\Delta\rho}$$

From this equation it follows that

$$S_c = \frac{\Delta V}{q_c} = \frac{1}{2\pi} \int_{\rho_1}^{\rho_2} \frac{d\rho}{\rho \varepsilon(\rho)} ,$$

and if ε is homogeneous

$$S_c = \frac{1}{2\pi\varepsilon} \ln(\rho_2/\rho_1).$$

Appendix C

Plane Wave Scattering by a Unidirectionally Conducting Shield

In this appendix an exact calculation will be given of the scattering of a plane wave from a circular cylindrical conductor of radius a. Outside the conducting cylinder is a unidirectionally conducting shield in the form of a circular cylindrical shell of radius b. The conduction direction of the shell maintains a constant angle, ψ, with the axis of symmetry. The incident plane wave will be assumed to travel perpendicular to the axis of the cylinder and to have its electric vector parallel to the axis of the cylinder.

If ϕ is the angle from the direction of propagation of the incident wave in the plane perpendicular to the axis of the cylinder, and if ρ is the radial distance from the axis of the cylinder, then the electric field in the region outside the shield (region I) and in the region between the inner cylinder and the shield (region II) may be written in the following form [5]

$$E_z^I = \sum_{n=-\infty}^{\infty} i^n J_n(k\rho)e^{in\phi} + \sum_{n=-\infty}^{\infty} r_n H_n(k\rho)e^{in\phi}$$

$$E_z^{II} = \sum_{n=-\infty}^{\infty} t_n X_n(k\rho)e^{in\phi}$$

where

$$X_n(k\rho) = \frac{J_n(k\rho)}{J_n(ka)} - \frac{Y_n(k\rho)}{Y_n(ka)}$$

also

$$E_\phi^I = \sum_{n=-\infty}^{\infty} r_n' H_n'(k\rho)e^{in\phi}$$

$$E_\phi^{II} = \sum_{n=-\infty}^{\infty} t_n' Z_n'(k\rho)e^{in\phi}$$

where

$$Z'_n(k\rho) = \frac{J'_n(k\rho)}{J'_n(ka)} - \frac{Y'_n(k\rho)}{Y'_n(ka)}$$

In addition, one can write the following representations for the magnetic field

$$Z_o H_z^I = i \sum_{n=-\infty}^{\infty} r'_n H_n(k\rho) e^{in\phi}$$

$$Z_o H_z^{II} = i \sum_{n=-\infty}^{\infty} t'_n Z_n(k\rho) e^{in\phi}$$

and

$$Z_o H_\phi^I = i \sum_{n=-\infty}^{\infty} i^n J'_n(k\rho) + i \sum_{n=-\infty}^{\infty} r_n H'_n(k\rho) e^{in\phi}$$

$$Z_o H_\phi^{II} = i \sum_{n=-\infty}^{\infty} t_n X'_n(k\rho) e^{in\phi}$$

The above representations of the fields already satisfy the boundary conditions at infinity and at the surface of the inner cylinder. For each ϕ mode, there are four remaining undetermined coefficients, r_n, t_n, r'_n, and t'_n. These constants can be determined by imposing the following four conditions at the surface of the shield. The two components of electric field are continuous through the shield. The component of electric field along the conduction direction of the shield is zero. The component of magnetic field along the conduction direction of the shield is continuous through the shield. These conditions give the following four equations.

$$r'_n H'_n(kb) = t'_n Z'_n(kb)$$

$$t_n X_n(kb) \cos\psi + t'_n Z'_n(kb) \sin\psi = 0$$

$$i^n J_n(kb) + r_n H_n(kb) = t_n X_n(kb)$$

$$t'_n Z_n(kb) \cos\psi + t_n X'_n(kb) \sin\psi = r'_n H_n(kb) \cos\psi + [i^n J'_n(kb) + r_n H'_n(kb)] \sin\psi$$

The above four equations may be reduced to give the following equation for t_n

$$t_n\left\{\cot^2\psi X_n(kb)\left[\frac{H_n(kb)}{H_n'(kb)} - \frac{Z_n(kb)}{Z_n'(kb)}\right] + \left[X_n'(kb) - \frac{X_n(kb)H_n'(kb)}{H_n(kb)}\right]\right\} = \frac{-2i^{n+1}}{\pi kbH_n(kb)}$$

but it is easy to show that

$$Z_n'(kb)H_n(kb) - Z_n(kb)H_n'(kb) = -\frac{2}{\pi kb}\frac{H_n'(ka)}{J_n'(ka)Y_n'(ka)}$$

and

$$H_n(kb)X_n'(kb) - X_n(kb)H_n'(kb) = -\frac{2}{\pi kb}\frac{H_n(ka)}{J_n(ka)Y_n(ka)} \cdot$$

Thus,

$$t_n\left[\cot^2\psi\,\frac{X_n(kb)H_n(kb)H_n'(ka)}{H_n'(kb)Z_n'(kb)J_n'(ka)Y_n'(ka)} + \frac{H_n(ka)}{J_n(ka)Y_n(ka)}\right] = i^{n+1}$$

Now the total current in the z direction on the inner cylinder, I, is given by

$$Z_oI = \int_o^{2\pi} H_\phi^{II}(a)ad\phi = 2\pi iat_oX_o'(ka)$$

while the total current in the z direction on the unidirectionally conducting shield is

$$Z_oI_s = \int_o^{2\pi}[H_\phi^{I}(b) - H_\phi^{II}(b)]bd\phi$$

$$= 2\pi ib[J_o'(kb) + r_oH_o'(kb) - t_oX_o'(kb)]$$

The ratio of the total axial current on the inner conductor to the total axial conduction current on the inner conductor plus the shield is thus

$$F \equiv \frac{I}{I+I_s} = \frac{t_oX_o'(ka)}{t_oX_o'(ka)+(b/a)[J_o'(kb)+r_oH_o'(kb)-t_oX_o'(kb)]} \cdot$$

But since

$$r_o = \frac{t_o X_o(kb) - J_o(kb)}{H_o(kb)} ,$$

one can rewrite F as

$$F = \frac{t_o X_o'(ka) H_o(kb)}{t_o \left\{ X_o'(ka) H_o(kb) + (b/a) \left[\frac{2}{\pi kb} \frac{H_o(ka)}{J_o(ka) Y_o(ka)} \right] \right\} + (b/a) \left[- \frac{2i}{\pi kb} \right]}$$

where the Wronskin relation

$$[H_o(kb) J_o'(kb) - J_o(kb) H_o'(kb)] = - \frac{2i}{\pi kb}$$

has been used. Now from the previous work

$$t_o^{-1} = - i \cot^2\psi \frac{X_o(kb) H_o(kb) H_o'(ka)}{H_o'(kb) Z_o'(kb) J_o'(ka) Y_o'(ka)} + \frac{H_o(ka)}{J_o(ka) Y_o(ka)} ,$$

thus

$$F = \frac{X_o'(ka)}{X_o'(ka) + \frac{2}{\pi ka} \frac{H_o(ka)}{J_o(ka) Y_o(ka) H_o(kb)} - \frac{2i}{\pi ka} \frac{t_o^{-1}}{H_o(kb)}}$$

$$= \frac{X_o'(ka)}{X_o'(ka) - \frac{2}{\pi ka} \frac{\cot^2\psi X_o(kb) H_o'(ka)}{H_o'(kb) Z_o'(kb) J_o'(ka) Y_o'(ka)}} .$$

But

$$X_o'(ka) = - \frac{2}{\pi ka} \frac{1}{J_o(ka) Y_o(ka)}$$

Therefore, one may write

$$F = \frac{1}{1+\cot^2\psi \dfrac{X_o(kb)H_o'(ka)J_o(ka)Y_o(ka)}{H_o'(kb)Z_o'(kb)J_o'(ka)Y_o'(ka)}}$$

$$= \frac{1}{1+\cot^2\psi G(ka,b/a)}$$

where

$$G(x,b/a) = \frac{X_o(xb/a)H_1(x)J_o(x)Y_o(x)}{Z_o'(xb/a)H_1(xb/a)J_1(x)Y_1(x)}$$

$$= \frac{H_1(x)}{H_1(xb/a)} \frac{Y_o(x)J_o(xb/a)-J_o(x)Y_o(xb/a)}{Y_1(x)J_1(xb/a)-J_1(x)Y_1(xb/a)}$$

Using the small argument expansions for the Bessel functions in the above equations, it is easy to arrive at the limiting value of G

$$G(0,b/a) = \frac{2b^2 \ln(b/a)}{b^2-a^2} .$$

In the low frequency limit, then,

$$F = \frac{1}{1+\cot^2\psi \dfrac{2b^2 \ln(b/a)}{b^2-a^2}} ,$$

in complete agreement with equation (20).

Appendix D

Justification of the Form of the Helix Potentials

In this appendix, a brief discussion will be given of the forms chosen for Φ and Ω in the fifth section of the note. In particular, a justification will be given of equations (22), (23), (32), and (33).

Since it is shown in the text that the potential representations in question can be made to satisfy all required boundary conditions, all that it is really necessary to show is that they satisfy Laplace's equation. From linearity, it is sufficient to show that each term in the sums satisfy Laplace's equation. That is to say, if

$$\psi_n(\rho,\phi,z) = \cos n(\phi - \phi_o - tz/b) f_n(nt\rho/b),$$

then

$$\nabla^2\psi_n \equiv \frac{\partial^2\psi_n}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\psi_n}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2\psi_n}{\partial\phi^2} + \frac{\partial^2\psi_n}{\partial z^2}$$

must be zero. But, by direct substitution,

$$\nabla^2\psi_n = \left[\frac{\partial^2 f_n}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial f_n}{\partial\rho} - \frac{n^2 f_n}{\rho^2} - \frac{n^2t^2}{\rho^2} f_n\right]\cos n(\phi - \phi_o - tz/b)$$

or, if

$$\xi \equiv nt\rho/b$$

then

$$\nabla^2\psi_n = \left(\frac{nt}{b}\right)^2\left[\frac{\partial^2 f_n}{\partial\xi^2} + \frac{1}{\xi}\frac{\partial f_n}{\partial\xi} - \left(1 + \frac{n^2}{\xi^2}\right)f_n\right]\cos n(\phi - \phi_o - tz/b)$$

Thus ψ_n satisfies Laplace's equation as long as f_n, as a function of ξ, is a linear combination of modified Bessel functions. The f_n's of equations (22), (23), (32), and (33) fulfill this requirement.

References

[1] C. W. Harrison, Jr., "Receiving Characteristics of Two-Wire Lines Excited by Uniform and Non-Uniform Electric Fields," Interaction Note 15, May 1964.

[2] Edward F. Vance, "Cable Transfer Impedance Measurements," forthcoming in the series of Interaction Notes.

[3] E. F. Vance and H. Chang, "Shielding Effectiveness of Braided Wire Shields," forthcoming in the series of Interaction Notes.

[4] Milton Abromowitz and Irene A. Stegun, Editors, Handbook of Mathematical Functions, National Bureau of Standards, AMS-55, June 1964, p. 378.

[5] Julius Adams Stratton, Electromagnetic Theory, McGraw-Hill, New York, 1941, p. 361.

[6] H. Kaden, Wirbelströme und Schirmung in der Nachrichtentechnik, Springer-Verlag, Berlin, 1959.

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