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GENERALIZED THEORY OF IMPEDANCE LOADED MULTICONDUCTOR TRANSMISSION LINES IN AN INCIDENT FIELD

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ABSTRACT

A general theory is advanced for determining the currents in the load impedances of an N-conductor isolated transmission line excited by an electromagnetic field with the electric vector directed parallel to the wires. The number of impedance loads in the circuit is 2N. An impedance is connected in series with each conductor at its ends. At each end of the transmission line the impedances emanate from a common node. There is no requirement that the conductors be of the same radius, be equally spaced, or lie in a common plane; however, their axes must be parallel. Evidently the cross section of the line must be sufficiently small in terms of the wavelength that transmission line theory applies.

Numerical values for the load currents in a three-conductor model are given.

Scattering from end-loaded multiconductor transmission lines is considered. It is shown that for configurations lacking geometrical symmetry such problems become arduous if not solved by computer.

scattering, impedance, transmission lines

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GENERALIZED THEORY OF IMPEDANCE LOADED MULTICONDUCTOR TRANSMISSION LINES IN AN INCIDENT FIELD

Introduction

The problem of determining the currents in the load impedances of a multiconductor transmission line excited by a plane wave incident field was encountered in studies at Sandia Laboratories related to the electromagnetic compatibility of rockets. The purpose of this paper is to set forth the techniques and procedures required to effect solution of the problem when the structure consists of N parallel wires with a load impedance at the end of each conductor. All impedances at each end of the transmission line emanate from a common node. The theory is summarized by solving the problem for three coplanar conductors containing six load impedances illustrated by Figure 1. A portion of the paper is devoted to the theory of scattering from such obstacles.

Description of the Circuit Analyzed

Figure 1 represents a transmission line consisting of three conductors terminated at their ends in load impedances. The length of the line is s. The spacing between wires is b and their radii are a. The phase reference is taken at the lower end of the middle wire labeled 1. The outer wires are designated 2 and 3. Coincident with the point of phase reference is the origin for Cartesian and cylindrical coordinate systems. The conductors of the transmission line are parallel to the z axis and lie in the y0z plane. The incident electric field E_z^{inc} arrives at the angle θ measured counterclockwise from the positive x axis. The field is orthogonal to the x0y plane and is directed upward. The currents $I_1(z)$, $I_2(z)$ and $I_3(z)$ flow in the positive z direction in wires 1, 2, and 3, respectively. The notation used to designate a load impedance is made clear by two illustrations: The impedance Z_{s1} is connected in series with wire 1 at $z = s$; Z_{02} is connected in series with wire 2 at $z = 0$.

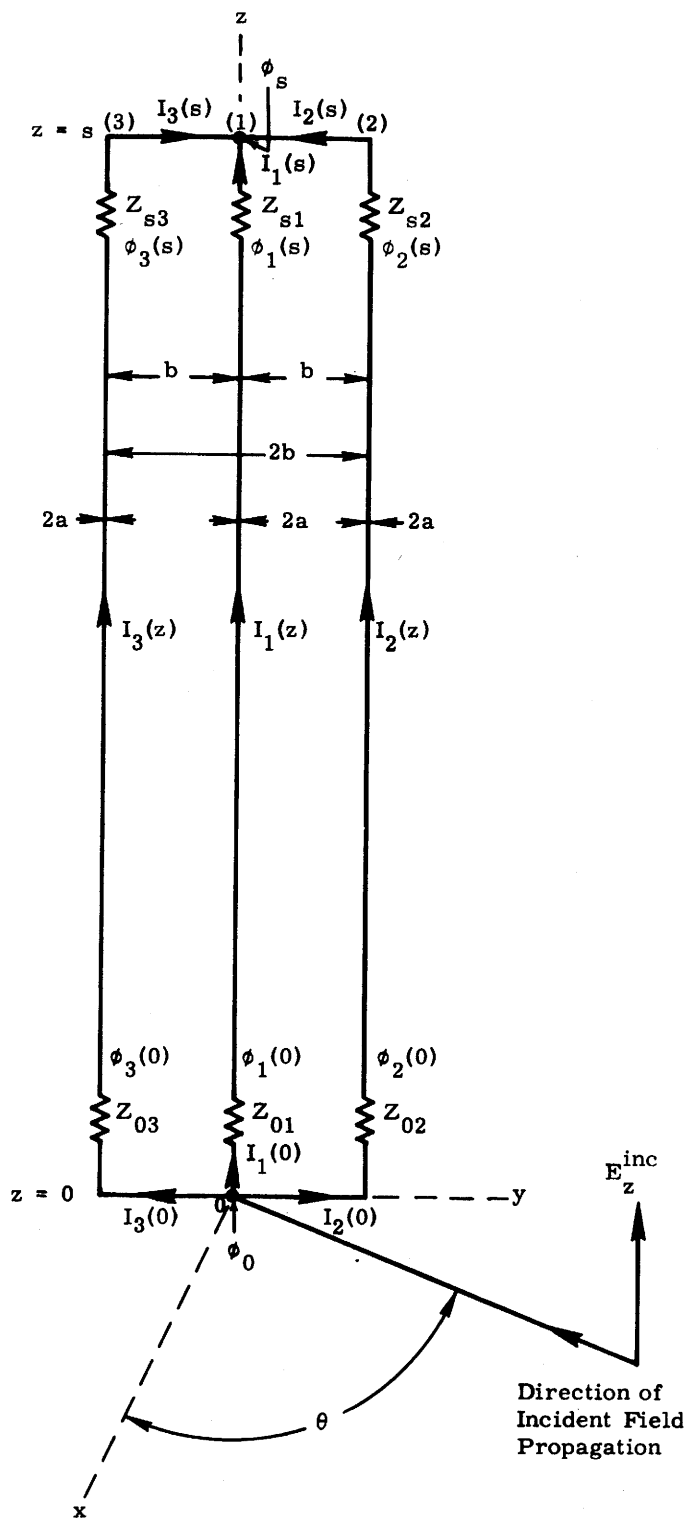

Figure 1. Three Impedance Loaded Coplanar Conductors Driven by an Incident Field

The Boundary Condition for the Total Electric Field on the Surface of Each Conductor

The wires composing the transmission line are assumed to be perfectly conducting. The boundary condition for the total electric field on the surface of each conductor is

$$E_z^{inc} + E_z^{scat} = 0 \ . \tag{1}$$

The requirement stipulated by (1) is expressed by the simple differential equation

$$E_z^{inc} - j \frac{\omega}{\beta^2} \left[\frac{\partial^2 A_z(z)}{\partial z^2} + \beta^2 A_z(z) \right] = 0 \ . \tag{2}$$

Here $\omega = 2\pi f$ is the radian wave number, $\beta = 2\pi/\lambda$ is the free space propagation constant, and $A_z(z)$ is the total vector potential on the surface of a particular wire. The solution of (2) is

$$4\pi \mu_0^{-1} A_{1z}(z) = - j \frac{4\pi}{\zeta_0} (C_1 \cos \beta z + D_1 \sin \beta z + U_1) \tag{3}$$

$$4\pi \mu_0^{-1} A_{2z}(z) = - j \frac{4\pi}{\zeta_0} (C_2 \cos \beta z + D_2 \sin \beta z + U_2) \tag{4}$$

$$4\pi \mu_0^{-1} A_{3z}(z) = - j \frac{4\pi}{\zeta_0} (C_3 \cos \beta z + D_3 \sin \beta z + U_3) \ . \tag{5}$$

In (3) through (5), $\mu_o = 4\pi \times 10^{-7}$ H/m and $\zeta_0 = 120\pi$ ohms are the permeability and characteristic impedance of free space, respectively. The constants of integration are C and D. The conductor under consideration is designated by the integer subscript. For example (4) applies to conductor 2. The excitation functions are denoted by U.

The Excitation Functions

As mentioned earlier the phrase reference is taken at the point x = y = z = 0, i.e., at the lower extremity of conductor 1, Figure 1. For E_z^{inc} in the positive z direction the solution of (2) yields

$$U_1 = E_z^{inc}(0)/\beta \tag{6}$$

Inspection of Figure 1 for the particular value of θ selected shows that conductor 2 leads conductor 1 by the phase angle $\beta b \sin\theta$, and that 3 lags 1 by the same angle. Hence

$$U_2 = \frac{E_z^{inc}(0)}{\beta} e^{j\beta b \sin\theta} \tag{7}$$

$$U_3 = \frac{E_z^{inc}(0)}{\beta} e^{-j\beta \sin\theta} \quad . \tag{8}$$

For a complicated configuration of wires care must be exercised in writing down the excitation functions, especially when the angle of wave arrival θ is arbitrary. Often interest centers in obtaining the response of an impedance loaded transmission line for a fixed value of θ. This specialization may simplify the problem. In any event only plane geometry and simple trigonometry are involved. Suppose, for example, $\theta = 3\pi/2$ radians. The incident wave is traveling in the positive y direction, so that $E_z^{inc}(y) = E_z^{inc}(0)\, e^{-j\beta y}$. When $y = -b$ and $\theta = 270°$ (8) is obtained. Similarly, when $y = b$ and $\theta = 270°$, (7) results.

The Simultaneous Integral Equations for the Currents on the Three Conductors

The three integral equations for the currents in the circuit pictured in Figure 1 are

$$\int_0^s I_1(z')K_a(z, z')dz' + \int_0^s I_2(z')K_b(z, z')dz' + \int_0^s I_3(z')K_b(z, z')dz' =$$

$$- j\,\frac{4\pi}{\zeta_0}\,(C_1 \cos\beta z + D_1 \sin\beta z + U_1) \tag{9}$$

$$\int_0^s I_1(z')K_b(z, z')dz' + \int_0^s I_2(z')K_a(z, z')dz' + \int_0^s I_3(z')K_c(z, z')dz' =$$

$$- j\,\frac{4\pi}{\zeta_0}\,(C_2 \cos\beta z + D_2 \sin\beta z + U_2) \tag{10}$$

$$\int_0^s I_1(z') K_b(z, z')dz' + \int_0^s I_2(z') K_c(z, z')dz' + \int_0^s I_3(z') K_a(z, z')dz' =$$

$$- j \frac{4\pi}{\zeta_0} (C_3 \cos \beta z + D_3 \sin \beta z + U_3) . \qquad (11)$$

Equations (9) through (11) apply to conductors 1 through 3 in sequence. Here

$$\left. \begin{aligned} K_a &= \frac{1}{R_a} \exp(-j\beta R_a) \\ K_b &= \frac{1}{R_b} \exp(-j\beta R_b) \\ K_c &= \frac{1}{R_{2b}} \exp(-j\beta R_c) \end{aligned} \right\} \qquad (12)$$

where

$$R_a = \sqrt{(z - z')^2 + a^2} ; \; R_b = \sqrt{(z - z')^2 + b^2} ; \; R_c = \sqrt{(z - z')^2 + 4b^2} . \qquad (13)$$

It is now desirable to employ the device that relates linear antenna theory to transmission line theory.[1] It consists, for example, in writing

$$\int_0^s I_1(z') K_a(z, z')dz' \simeq \int_0^s I_1(z') K_d(z, z')dz' + I_1(z)\psi_a . \qquad (14)$$

In this expression

$$K_d = \frac{1}{R_d} \exp(-j\beta R_d) \qquad (15)$$

$$R_d = \sqrt{(z - z')^2 + d^2} \qquad (16)$$

and

$$\psi_a = 2 \ln \left(\frac{d}{a}\right) . \qquad (17)$$

---

[1]R. W. P. King and C. W. Harrison, Jr., Antennas and Waves: A Modern Approach, MIT Press, 1969, Chapter 7, p. 489, Eq. 7.3.6 through 7.3.8.

The parameter d is a quantity related to the effective radius of the conductor cage. It must be known to determine the scattering cross section of the circuit, as explained later. The principle set forth in (14) may be applied to every integral appearing in (9) through (11), provided that $\beta a < \beta b \ll 1$. When this is carried out, (9) through (11) yield

$$J_d(z) + I_1(z)\psi_a + I_2(z)\psi_b + I_3(z)\psi_b = -j\frac{4\pi}{\zeta_0}(C_1 \cos\beta z + D_1 \sin\beta z + U_1) \tag{18}$$

$$J_d(z) + I_1(z)\psi_b + I_2(z)\psi_a + I_3(z)\psi_c = -j\frac{4\pi}{\zeta_0}(C_2 \cos\beta z + D_2 \sin\beta z + U_2) \tag{19}$$

$$J_d(z) + I_1(z)\psi_b + I_2(z)\psi_c + I_3(z)\psi_a = -j\frac{4\pi}{\zeta_0}(C_3 \cos\beta z + D_3 \sin\beta z + U_3) \tag{20}$$

In the above equations:

$$J_d(z) = \int_0^s \sum_{n=1}^{3} I_n(z')K_d(z, z')dz' \tag{21}$$

ψ_a is given by (17)

$$\psi_b = 2\,\ell n\left(\frac{d}{b}\right) \tag{22}$$

$$\psi_{2b} = 2\,\ell n\left(\frac{d}{2b}\right). \tag{23}$$

Also,

$$\sum_{n=1}^{3} I_n(z) = I_T(z) = I_1(z) + I_2(z) + I_3(z) \tag{24}$$

By Kirchoff's current law,

$$I_T(0) = I_T(s) = 0.$$

That is

$$I_1(0) + I_2(0) + I_3(0) = 0 \tag{25}$$

$$I_1(s) + I_2(s) + I_3(s) = 0\,. \tag{26}$$

In order to eliminate $J_d(z)$ from (18) through (20), it is convenient to subtract (19) from (18) and (20) from (18). Thus,

$$I_1(z)\psi_{ab} - I_2(z)\psi_{ab} + I_3(z)\psi_{bc} = -j\frac{4\pi}{\zeta_0}(C_{12}\cos\beta z + D_{12}\sin\beta z + U_{12}) \quad (27)$$

$$I_1(z)\psi_{ab} + I_2(z)\psi_{bc} - I_3(z)\psi_{ab} = -j\frac{4\pi}{\zeta_0}(C_{13}\cos\beta z + D_{13}\sin\beta z + U_{13}). \quad (28)$$

In these expressions

$$\left.\begin{aligned}
\psi_{ab} &= \psi_a - \psi_b = 2\,\ell n\left(\frac{b}{a}\right) \\
\psi_{bc} &= \psi_b - \psi_c = 2\,\ell n 2 = 1.3863 \\
C_{12} &= C_1 - C_2 \\
C_{13} &= C_1 - C_3 \\
D_{12} &= D_1 - D_2 \\
D_{13} &= D_1 - D_3 \\
U_{12} &= U_1 - U_2 = \frac{E_z^{inc}(0)}{\beta}\left(1 - e^{j\beta b\sin\theta}\right) \\
U_{13} &= U_1 - U_3 = \frac{E_z^{inc}(0)}{\beta}\left(1 - e^{-j\beta b\sin\theta}\right)
\end{aligned}\right\}. \quad (29)$$

When z = 0, (27) and (28) may be written

$$\zeta_0\psi_{ab}I_1(0) - \zeta_0\psi_{ab}I_2(0) + \zeta_0\psi_{bc}I_3(0) + j4\pi C_{12} = -j4\pi U_{12} \quad (30)$$

$$\zeta_0\psi_{ab}I_1(0) + \zeta_0\psi_{bc}I_2(0) - \zeta_0\psi_{ab}I_3(0) + j4\pi C_{13} = -j4\pi U_{13}. \quad (31)$$

Also, when z = s, (27) and (28) become

$$\zeta_0\psi_{ab}I_1(s) - \zeta_0\psi_{ab}I_2(s) + \zeta_0\psi_{bc}I_3(s) + j4\pi C_{12}\cos\beta s + j4\pi D_{12}\sin\beta s =$$

$$-j4\pi U_{12} \quad (32)$$

$$\zeta_0\psi_{ab}I_1(s) + \zeta_0\psi_{bc}I_2(s) - \zeta_0\psi_{ab}I_3(s) + j4\pi C_{13}\cos\beta s + j4\pi D_{13}\sin\beta s =$$
$$- j4\pi U_{13} \cdot \quad (33)$$

The Voltages Across the Load Impedances

The scalar potential on each conductor is obtained from the vector potential with the help of the Lorentz condition. It is

$$\phi(z) = j\frac{\omega}{\beta^2}\frac{dA_z(z)}{dz} \quad (34)$$

Applying (34) to (3) through (5),

$$\phi_1(z) = - C_1 \sin\beta z + D_1 \cos\beta z \quad (35)$$

$$\phi_2(z) = - C_2 \sin\beta z + D_2 \cos\beta z \quad (36)$$

$$\phi_3(z) = - C_3 \sin\beta z + D_3 \cos\beta z \ . \quad (37)$$

In obtaining these formulas, use was made of relations $\mu_0/\zeta_0 = 1/v_p$ and $v_p = \omega/\beta$. The potential of the shorting bar at z = 0 is designated ϕ_0 (refer to Figure 1); the potential of the bar at $z = s$ is designated ϕ_s. These potentials are unknown but serve as an aid to the writer in avoiding errors in sign. Since all currents are assumed to flow in the positive z direction,

$$\phi_0 - \phi_1(0) = I_1(0)Z_{01} \quad (38)$$

$$\phi_0 - \phi_2(0) = I_2(0)Z_{02} \quad (39)$$

$$\phi_0 - \phi_3(0) = I_3(0)Z_{03} \quad (40)$$

$$\phi_1(s) - \phi_s = I_1(s)Z_{s1} \quad (41)$$

$$\phi_2(s) - \phi_s = I_2(s)Z_{s2} \quad (42)$$

$$\phi_3(s) - \phi_s = I_3(s)Z_{s3} \ . \quad (43)$$

By subtraction ϕ_0 is eliminated from (38) through (40) and ϕ_s from (41) through (43). The result is

$$D_{12} - I_2(0)Z_{02} + I_1(0)Z_{01} = 0 \tag{44}$$

$$D_{13} - I_3(0)Z_{03} + I_1(0)Z_{01} = 0 \tag{45}$$

$$C_{12} \sin \beta s - D_{12} \cos \beta s + I_1(s)Z_{s1} - I_2(s)Z_{s2} = 0 \tag{46}$$

$$C_{13} \sin \beta s - D_{13} \cos \beta s + I_1(s)Z_{s1} - I_3(s)Z_{s3} = 0 \tag{47}$$

where use has been made of (35) through (37).

The unknowns are $I_1(0)$, $I_2(0)$, $I_3(0)$, $I_1(s)$, $I_2(s)$, $I_3(s)$, C_{12}, C_{13}, D_{12} and D_{13}—ten in number. The following independent equations are available for determining these unknowns: (25), (26), (30), (31), (32), (33), (44), (45), (46), and (47)—again ten in number. One is now in a position to write a system of equations, shown in Table 1, for computer solution. The constants C_{12}, C_{13}, etc., need not be read out inasmuch as they are not needed. If the programmer does not trust the available routine for solving complex matrix equations, he may separate the equations into real and imaginary parts forming a 20 x 20 matrix to solve the problem presently under consideration.

Suggestions for Solving More Complicated Problems

The reader may get the impression that determination of the load currents in a multiconductor transmission line driven by an incident plane wave field is a formidable undertaking. Actually this is not the case. The following suggestions for solving these problems are proffered:

(a) Write down expressions for the excitation functions U.

(b) Write down a system of equations like (18) through (20) by inspection of the circuit. There will be one equation for each conductor involved. Eliminate $J_d(z)$ by subtraction, longhand. (Note that this can be done by computer.) Observe that the wires in the model must be of the same length but need not be of the same radius.

TABLE I

Complex Matrix for Determination of the Load Currents

$$
\left\{\begin{matrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
\zeta_0\psi_{ab} & -\zeta_0\psi_{ab} & \zeta_0\psi_{bc} & 0 & 0 & 0 & j4\pi & 0 & 0 & 0 \\
\zeta_0\psi_{ab} & \zeta_0\psi_{bc} & -\zeta_0\psi_{ab} & 0 & 0 & 0 & 0 & j4\pi & 0 & 0 \\
0 & 0 & 0 & \zeta_0\psi_{ab} & -\zeta_0\psi_{ab} & \zeta_0\psi_{bc} & j4\pi\cos\beta s & 0 & j4\pi\sin\beta s & 0 \\
0 & 0 & 0 & \zeta_0\psi_{ab} & \zeta_0\psi_{bc} & -\zeta_0\psi_{ab} & 0 & j4\pi\cos\beta s & 0 & j4\pi\sin\beta s \\
Z_{01} & -Z_{02} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
Z_{01} & 0 & -Z_{03} & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & Z_{s1} & -Z_{s2} & 0 & \sin\beta s & 0 & -\cos\beta s & 0 \\
0 & 0 & 0 & Z_{s1} & 0 & -Z_{s3} & 0 & \sin\beta s & 0 & -\cos\beta s
\end{matrix}\right\}
\left\{\begin{matrix}
I_1(0) \\ I_2(0) \\ I_3(0) \\ I_1(s) \\ I_2(s) \\ I_3(s) \\ C_{12} \\ C_{13} \\ D_{12} \\ D_{13}
\end{matrix}\right\}
=
\left\{\begin{matrix}
0 \\ 0 \\ -j4\pi U_{12} \\ -j4\pi U_{13} \\ -j4\pi U_{12} \\ -j4\pi U_{13} \\ 0 \\ 0 \\ 0 \\ 0
\end{matrix}\right\}
$$

(c) Write down a system of equations like (35) through (37). There is certainly no difficulty in doing this!

(d) Write down expressions corresponding to (38) through (43), leading to equations like (44) through (47).

(e) Write the complex matrix, and request a programmer to invert it by machine, reading out the magnitude of the load currents.

Back-Scattering Cross Section of an Impedance Loaded Multiconductor Transmission Line for Parallel Polarization of the Incident Field

Normally, interest centers in obtaining the load currents in a transmission line exposed to an incident field. Now and then it is desirable to obtain the radar cross section of the structure for parallel polarization of the incident field. This problem is very straightforward, provided that all conductors have the same radius and their axes are oriented on the vertices of a regular polygon. If either of the above conditions is not met, the problem may become horrendous. Inasmuch as the structure illustrated by Figure 1 does not possess the symmetry needed to achieve simplicity (there is a middle conductor), it is a suitable circuit to illustrate the fact that determination of the back-scattering cross section is sometimes arduous.

It is convenient to shift the origin of coordinates to the middle of conductor 1, Figure 1. The ends to the transmission line are now at $z = \pm h$ $(s = 2h)$. The definition of $J_d(z)$ in (18) through (20) now becomes

$$J_d(z) = \int_{-h}^{h} I_T(z')\, K_d(z, z')dz' \qquad (48)$$

Since

$$I_T(\pm h) = 0 \qquad (49)$$

it is not possible for any of the load impedances, which are all located at $z = \pm h$, to have any effect on $I_T(z)$. It follows that for normal incidence of the electric field the total current $I_T(z)$ satisfies the symmetry condition

$$I_T(z) = I_T(-z) . \tag{50}$$

It is possible to obtain an expression for $I_1(z)$ in terms of $I_T(z)$ by solving (27) and (28) for $I_1(z)$, adding $I_1(z)$ to both sides of the equation, and summing. The result is

$$3I_1(z) = I_T(z) - \left[I_2(z) + I_3(z)\right] \frac{\psi_{bc}}{\psi_{ab}} - j \frac{4\pi}{\zeta_0 \psi_{ab}} \left[(C_{12} + C_{13}) \cos \beta z + (D_{12} + D_{13}) \sin \beta z + (U_{12} + U_{13})\right] . \tag{51}$$

If one remembers that

$$I_2(z) + I_3(z) = I_T(z) - I_1(z) , \tag{52}$$

the following formula for $I_1(z)$ may be obtained:

$$I_1(z) = I_T(z) \left[\frac{\psi_{ab} - \psi_{bc}}{3\psi_{ab} - \psi_{bc}}\right] - j \frac{4\pi}{\zeta_0(3\psi_{ab} - \psi_{bc})} \left[(C_{12} + C_{13}) \cos \beta z + (D_{12} + D_{13}) \sin \beta z + (U_{12} + U_{13})\right]. \tag{53}$$

Adding (18), (19), and (20) yields

$$3J_d(z) + I_1(z) (\psi_a + 2\psi_b) + \left[I_2(z) + I_3(z)\right]\left[\psi_a + \psi_b + \psi_c\right] = - j \frac{4\pi}{\zeta_0} \left[(C_1 + C_2 + C_3) \cos \beta z + (D_1 + D_2 + D_3) \sin \beta z + (U_1 + U_2 + U_3)\right] . \tag{54}$$

In this equation the current $[I_2(z) + I_3(z)]$ drops out if one sets

$$\psi_a + \psi_b + \psi_c = 0 \tag{55}$$

so that

$$d = \sqrt[3]{2ab^2} \ . \tag{56}$$

Substituting (53) into (54) subject to (55) yields

$$J_d(z) + \frac{2}{3} \cdot \frac{\ell n\,(4)\ \ell n\left(\frac{b}{2a}\right)}{\ell n\left(\frac{b^6}{4a^6}\right)} I_T(z) =$$

$$- j\,\frac{4\pi}{\zeta_0}\left\{\left[\left(\frac{C_1 + C_2 + C_3}{3}\right) - \frac{1}{3}\,\frac{\ell n\,(4)}{\ell n\left(\frac{b^6}{4a^6}\right)}\left(C_{12} + C_{13}\right)\right] \cos \beta z\right.$$

$$+ \left[\left(\frac{D_1 + D_2 + D_3}{3}\right) - \frac{1}{3}\,\frac{\ell n\,(4)}{\ell n\left(\frac{b^6}{4a^6}\right)}\left(D_{12} + D_{13}\right)\right] \sin \beta z$$

$$\left. + \left[\left(\frac{U_1 + U_2 + U_3}{3}\right) - \frac{1}{3}\,\frac{\ell n\,(4)}{\ell n\left(\frac{b^6}{4a^6}\right)}\left(U_{12} + U_{13}\right)\right]\right\} . \tag{57}$$

The coefficient of $\cos \beta z$ is just a constant denoted hereinafter by C. The coefficient of $\sin \beta z$ must be zero because of (50). The remaining term is a complicated excitation function designated U_{eff}. Its value is determined later. Equation (57) now takes the form

$$J_d(z) + \frac{2}{3}\,\frac{\ell n\,(4)\ \ell n\left(\frac{b}{2a}\right)}{\ell n\left(\frac{b^6}{4a^6}\right)} I_T(z) = - j\,\frac{4\pi}{\zeta_0}\left(C \cos \beta z + U_{eff}\right) . \tag{58}$$

This is a Fredholm integral equation of the third kind. It may be reduced to a Fredholm integral equation of the first kind as follows: The integral $J_d(z)$ is defined by (48). Observe that the antenna current $I_T(z)$ occurs under the integral sign. Thus, the left side of (58) is in the same form as (14). Accordingly, (58) may be written

$$\int_{-h}^{h} I_T(z') \frac{e^{-j\beta\sqrt{(z-z')^2+g^2}}}{\sqrt{(z-z')^2+g^2}}\, dz' + I_T(z)\left[\psi_g + \frac{2}{3}\, \frac{\ell n\,(4)\ \ell n\left(\frac{b}{2a}\right)}{\left(\ell n\ \frac{b^6}{4a^6}\right)}\right] = -j\,\frac{4\pi}{\zeta_0}\left(C\cos\beta z + U_{eff}\right) \tag{59}$$

where

$$\psi_g = 2\ \ell n\left(\frac{g}{d}\right). \tag{60}$$

One is at liberty to set

$$\psi_g + \frac{2}{3}\ \frac{\ell n\,(4)\ \ell n\left(\frac{b}{2a}\right)}{\ell n\left(\frac{b^6}{4a^6}\right)} = 0 \tag{61}$$

in order to determine g. It is

$$g = \sqrt[3]{2ab^2}\ \exp\left[-\frac{1}{3}\ \frac{\ell n\,(4)\ \ell n\left(\frac{b}{2a}\right)}{\ell n\left(\frac{b^6}{4a^6}\right)}\right] \tag{62}$$

where use has been made of (56).

The final form of the integral equation is

$$\int_{-h}^{h} I_T(z')\ \frac{e^{-j\beta R_g}}{R_g}\ dz' = -j\,\frac{4\pi}{\zeta_0}\left(C\cos\beta z + U_{eff}\right). \tag{63}$$

An approximate solution of this equation is[2]

$$I_T(z) \simeq j \frac{4\pi}{\zeta_0} U_{eff} K (\cos \beta z - \cos \beta h) . \tag{64}$$

Here

$$K = \left[\psi_{dU} \cos \beta h - \psi_U(h)\right]^{-1} \tag{65}$$

$$\psi_{dU} = (1 - \cos \beta h)^{-1} \int_{-h}^{h} (\cos \beta z' - \cos \beta h) \times$$

$$\left[K_g(0, z') - K_g(h, z')\right] dz' \tag{66}$$

$$\psi_U(h) = \int_{-h}^{h} (\cos \beta z' - \cos \beta h) K_g(h, z') dz' \tag{67}$$

$$K_g = \frac{1}{R_g} \exp(-j\beta R_g) \tag{68}$$

$$R_g = \sqrt{(z - z')^2 + g^2} . \tag{69}$$

The back-scattering cross section for parallel polarization is

$$\sigma_{11} = 4\pi r^2 \left| \frac{E_z^{rad}}{E_z^{inc}} \right|^2 \tag{70}$$

where

$$\left| E_z^{rad} \right| = \frac{\zeta_0 \beta}{4\pi r} \int_{-h}^{h} I_T(z) dz . \tag{71}$$

[2]Reference 1, Chapter 8, p. 520, Eq. 8.3.19b.

It is easily shown that

$$U_{eff} = \frac{E_z^{inc}}{\beta}\left[\frac{1 + 2\cos(\beta b \sin\theta)}{3} - \frac{4\,\ell n\,(4)}{3\,\ell n\left(\frac{b^6}{a^6}\right)}\right]\sin^2\left(\frac{\beta b \sin\theta}{2}\right)$$

$$\equiv \frac{E_z^{inc}}{\beta}\,Q(\theta) \qquad (72)$$

using (6) through (8) in the last bracketed term in (57). Substituting (72) into (64); (64) into (71) and (71) into (70), sequentially, and carrying out the indicated mathematical operations yields

$$\frac{\sigma_{11}}{\lambda^2} = \frac{4}{\pi}\left[Q(\theta)\,K\,(\sin\beta h - \beta h\cos\beta h)\right]^2 . \qquad (73)$$

This is the final formula for the normalized radar cross section of the impedance loaded multiple wire line for parallel polarization of the incident field. Since $Q(\theta)$ is defined by (72) as a function of θ, the value of σ_{11}/λ^2 varies as the angle θ of wave arrival changes.

Numerical Results

In the numerical results reported in this paper $E_z^{inc}(0) = 1$ volt/m, $\theta = 270°$, $a = 10^{-3}$ m, and $b = 10^{-2}$ m. In the tables E-n means 10^{-n}; for example, E-5 = 10^{-5}. All currents are given in amperes and impedances are expressed in ohms.

Example I: $Z_{01} = Z_{02} = Z_{03} = Z_{s1} = Z_{s2} = Z_{s3} = 500 + j0\ \Omega$

$s = 1.0$ m.

It is easily shown that

$$U_{eff} = \frac{E_z^{inc}}{\beta}\left[\frac{1 + 2\cos(\beta b \sin\theta)}{3} - \frac{4\,\ell n\,(4)}{3\,\ell n\left(\frac{b^6}{4a^6}\right)}\sin^2\left(\frac{\beta b \sin\theta}{2}\right)\right]$$

$$= \frac{E_z^{inc}}{\beta}\,Q(\theta) \tag{72}$$

using (6) through (8) in the last bracketed term in (57). Substituting (72) into (64); (64) into (71) and (71) into (70), sequentially, and carrying out the indicated mathematical operations yields

$$\frac{\sigma_{11}}{\lambda^2} = \frac{4}{\pi}\left[Q(\theta)\,K\,(\sin\beta h - \beta h\cos\beta h)\right]^2 . \tag{73}$$

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<u>Example I:</u> $Z_{01} = Z_{02} = Z_{03} = Z_{s1} = Z_{s2} = Z_{s3} = 500 + j0.0.$

$$s = 1.0\,m.$$

	$\beta s = 1.5$	$\beta s = 3.0$
$/I_1(0)/$ =	9.076 E-8	7.736 E-7
$/I_2(0)/$ =	1.767 E-5	5.461 E-5
$/I_3(0)/$ =	1.766 E-5	5.454 E-5
$/I_1(s)/$ =	9.076 E-8	7.736 E-7
$/I_2(s)/$ =	1.767 E-5	5.461 E-5
$/I_3(s)/$ =	1.766 E-5	5.454 E-5

Example II: $Z_{01} = 50 - j25$; $Z_{02} = 100 + j100$; $Z_{03} = 25 + j25$;

$Z_{s1} = 50 + j25$; $Z_{s2} = 100 - j50$; $Z_{s3} = 150 - j50$.

$s = 10m$.

	$\beta s = 1.5$	$\beta s = 3.0$
$/I_1(0)/$ =	1.065 E-5	2.728 E-5
$/I_2(0)/$ =	5.644 E-5	7.240 E-5
$/I_3(0)/$ =	6.255 E-5	8.884 E-5
$/I_1(s)/$ =	1.220 E-5	2.751 E-5
$/I_2(s)/$ =	2.784 E-5	5.918 E-5
$/I_3(s)/$ =	3.650 E-5	3.179 E-5

Example III: $Z_{s1} = Z_{s2} = Z_{s3} = 0$; $s = 10m$.

$Z_{02} = Z_{03} = 0$

	$\beta s = 1.5$	$\beta s = 3.0$
Z_{01}	$/I_1(0)/$	$/I_1(0)/$
10^0	4.023 E-8	8.040 E-8
10^1	4.023 E-8	7.530 E-8
10^2	4.020 E-8	2.066 E-8
10^3	3.760 E-8	2.137 E-9
10^4	1.023 E-8	2.138 E-10
10^5	1.057 E-9	2.138 E-11

10^6	1.058 E-10	2.138 E-12
10^7	1.058 E-11	2.138 E-13

$Z_{01} = Z_{03} = 0$

	$\beta s = 1.5$	$\beta s = 3.0$
Z_{02}	$/I_2(0)/$	$/I_2(0)/$
10^0	5.563 E-5	5.561 E-5
10^1	5.563 E-5	5.344 E-5
10^2	5.561 E-5	1.818 E-5
10^3	5.340 E-5	1.922 E-6
10^4	1.801 E-5	1.924 E-7
10^5	1.902 E-6	1.924 E-8
10^6	1.903 E-7	1.924 E-9
10^7	1.903 E-8	1.924 E-10

$Z_{01} = Z_{02} = 0$

	$\beta s = 1.5$	$\beta s = 3.0$
Z_{03}	$/I_3(0)/$	$/I_3(0)/$
10^0	5.563 E-5	5.561 E-5
10^1	5.563 E-5	5.344 E-5
10^2	5.561 E-5	1.818 E-5
10^3	5.340 E-5	1.922 E-6
10^4	1.801 E-5	1.924 E-7
10^5	1.902 E-6	1.924 E-8
10^6	1.903 E-7	1.924 E-9
10^7	1.903 E-8	1.924 E-10

<u>Example IV</u>: $Z_{02} = Z_{03} = Z_{s1} = Z_{s3} = 0$

$Z_{01} = 10 + j0.0 \quad \beta s = 1.5, \quad s = 10m.$

Z_{s2}	$/I_1(0)/$	Z_{s2}	$/I_1(0)/$
0.0-j 1.0E3	2.102 E-4	0.0-j 6.0E3	1.033 E-3
0.0-j 1.5E3	3.921 E-4	0.0-j 6.5E3	9.589 E-4
0.0-j 2.0E3	6.795 E-4	0.0-j 7.0E3	9.021 E-4
0.0-j 2.5E3	1.152 E-3	0.0-j 7.5E3	8.575 E-4
0.0-j 3.0E3	1.797 E-3	0.0-j 8.0E3	8.216 E-4
0.0-j 3.5E3	2.061 E-3	0.0-j 8.5E3	7.922 E-4
0.0-j 4.0E3	1.788 E-3	0.0-j 9.0E3	7.676 E-4
0.0-j 4.5E3	1.488 E-3	0.0-j 9.5E3	7.467 E-4
0.0-j 5.0E3	1.278 E-3	0.0-j 1.0E4	7.288 E-4
0.0-j 5.5E3	1.134 E-3		

In this work the N x N system of equations was separated into a 2N x 2N real system of equations. Also, the N x N complex system was solved directly. Identical results were obtained. Inasmuch as two different programs were employed, the writer has confidence in the accuracy of the numerical results presented.

Conclusions

A unified treatment has been developed for determination of the load currents in an impedance loaded multiconductor transmission line in an incident field. The possibility of reducing a complicated loading network to simple series impedances at the ends of the conductors forming the transmission line should not be overlooked. Although the theory presented herein employs a three conductor model, familiarity with the procedures and techniques involved in the solution of this specific problem permits the reader to write down by inspection the needed equations to determine the currents in the load impedances of a more complicated circuit.

A formula for the back-scattering cross section of impedance loaded N conductor transmission lines may be derived easily, provided that the structure possesses the required symmetry.