INTERATION NOTE 80

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INTERIM REPORT

RESPONSE OF A MULTICONDUCTOR TRANSMISSION LINE TO EXCITATION BY AN ARBITRARY MONOCHROMATIC IMPRESSED FIELD ALONG THE LINE

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for
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1. Introduction

This report covers the derivation of the response of a multiconductor transmission line to excitation by an arbitrary monochromatic field impressed along the line. Excitation by a localized monochromatic field (space impulse) is a special case of this general result, and constitutes the second type of problem specified in the request for proposal attached to D. E. Merewether's letter to Frankel Associates of June 17, 1970.

For identification in subsequent discussions in this report the four problems listed in the proposal request are labeled as follows:

Type I: Broken Shield Problem [1]*

Type II: Exposed Line Externally Excited**

Type III: End-Excited Cable [2]

Type IV: End-Excited Exposed Line [2]

Recognition of problem types I, III, and IV is given here because it turns out that, insofar as line behavior is concerned, these problems can be shown to be special cases of a slight generalization of type II. Thus, a simple overall analysis suffices for all cases.

Of course, if one wishes to determine the responses internal to the line terminations, a mixed network/transmission line problem is involved as explained in Section 2.1.1 of Reference 2, and exemplified by the problem of Reference 3. However, the procedure involved is common to all four types and requires no further elaboration in this report.

Derivation of the general result and its reduction to the special cases of interest is the subject of Section 2 of this report. Section 3 summarizes this unity of results. A brief concluding statement is given in Section 4.

2. Analysis of General Problem

The notation used is generally that of References 1 and 2, with additional terms defined where necessary. Consult Fig. 1. As usual this represents an

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*Numbers in [ ] correspond to those of Reference List, page 48.

**Subject of present report.

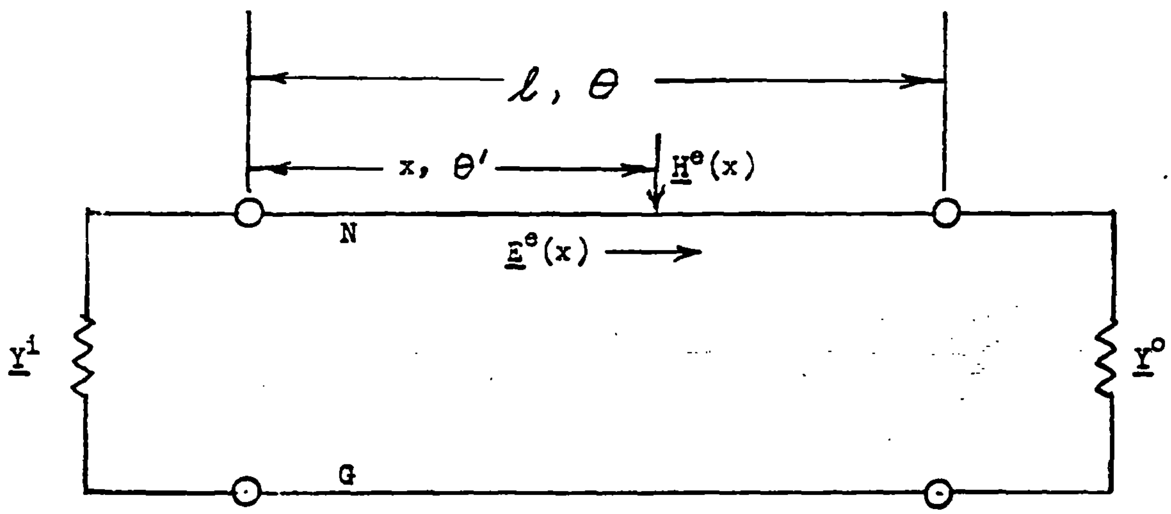


Fig. 1. N-line with external excitation, $\underline{E}^e(x)$, (volts per meter), and $\underline{H}^e(x)$, (amps. per meter).

N-line, with G representing the ground-, or reference conductor, while the N conductors above ground are indicated by the single line labelled "N". Termination admittance matrices are $\underline{Y}^i$ and $\underline{Y}^O$; x is the distance from the left end of the line to a representative point on the line, where the externally-impressed voltage gradient is $\underline{E}^e$ (x), and the impressed current gradient is $\underline{H}^e(x)$. Sources of $\underline{E}^e$ and $\underline{H}^e$ are discussed in Appendix A. According to the proposal request it would be sufficient to limit the analysis to a source of the forms $\underline{E}^e(x) = V_0\delta(x - x_0)$ where $\delta(u)$ is the impulse function:

$$\delta(u) = \infty, \; u = 0$$

$$= 0 \text{ otherwise} \qquad (1)$$

$$\int_{-\infty}^{\infty} \delta(u)du = 1 \qquad (2)$$

The more general analysis is tractable with insignificant increase in complexity and may prove useful for future problems.

The general result, obtained in Appendix B, runs as follows:

Let

$\underline{V}(x)$ = line voltage vector at distance x from the left-hand end of the line

$$= \begin{bmatrix} V_1(x) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ V_N(x) \end{bmatrix}.$$

$\underline{I}(x)$ = line current vector at x

$$= \begin{bmatrix} I_1(x) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ I_N(x) \end{bmatrix}$$

$$\theta' = \frac{\omega x}{v} = \beta x$$

$$\theta = \frac{\omega \ell}{v} = \beta \ell$$

ξ = dummy variable of integration

The remaining quantities in the following Equations (3) are defined in References 1 and 2 [see Glossary, Ref. 2] or in the following Equations (4) and (5):

$$\left.\begin{aligned} \underline{V}(x) &= \underline{R}\,\underline{S}^{-1}\,\underline{K}(\ell) + \underline{U}(x) \\ \underline{I}(x) &= -\underline{Y}\left\{\underline{T}\,\underline{S}^{-1}\,\underline{K}(\ell) - \underline{Z}\,\underline{W}(x)\right\} \end{aligned}\right\} \quad (3)$$

where

$$\left.\begin{aligned} \underline{R} &= \underline{\mathcal{I}} \cos\theta' + j\underline{P}^{i} \sin\theta' \\ \underline{S} &= (\underline{P}^{i} + \underline{P}^{o}) \cos\theta + j(\underline{\mathcal{I}} + \underline{P}^{o}\underline{P}^{i}) \sin\theta \\ \underline{T} &= \underline{P}^{i} \cos\theta' + j\,\underline{\mathcal{I}} \sin\theta' \end{aligned}\right\} \quad (4)$$

and

$$\left.\begin{aligned} \underline{K}(\ell) &= \int_0^{\ell} \left\{\underline{\mathcal{I}} \cos[\beta(\ell - \xi)] + j\,\underline{P}^{o} \sin[\beta(\ell - \xi)]\right\} \underline{Z}\,\underline{H}^{e}(\xi)\,d\xi \\ &\quad - \int_0^{\ell} \left\{\underline{P}^{o} \cos[\beta(\ell - \xi)] + j\,\underline{\mathcal{I}} \sin[\beta(\ell - \xi)]\right\} \underline{E}^{e}(\xi)\,d\xi \\ \underline{U}(x) &= \int_0^{x} \left\{\underline{E}^{e}(\xi) \cos[\beta(x - \xi)] - j\,\underline{Z}\,\underline{H}^{e}(\xi) \sin[\beta(x - \xi)]\right\} d\xi \\ \underline{Z}\,\underline{W}(x) &= \int_0^{x} \left\{\underline{Z}\,\underline{H}^{e}(\xi) \cos[\beta(x - \xi)] - j\,\underline{E}^{e}(\xi) \sin[\beta(x - \xi)]\right\} d\xi \end{aligned}\right\} \quad (5)$$

We proceed immediately to the special case represented by Equation (1) and corresponding to problem Type II (see Section 1).

2.1 Response to Localized Impressed Field (Problem Type II)

Using Equations (1) and (2) in Equations (3) and (5) yields, for $0 < x_o < \ell$,

$$\left.\begin{aligned}
U(x) &= \int_0^x \underline{V}_o\, \delta(\xi - x_o) \cos [\beta(x - \xi)]\, d\xi \\
&= 0, \; x < x_o \\
&= \underline{V}_o(x_o) \cos [\beta(x - x_o)], \; x > x_o \\
&= \underline{V}_o \cos (\theta' - \theta_o), \; x > x_o \qquad \text{(a)} \\
\underline{Z}\,\underline{W}(x) &= 0, \; x < x_o \\
&= -\underline{V}_o(x_o) \sin [\beta(x - x_o)], \; x > x_o \\
&= -\underline{V}_o \sin (\theta' - \theta_o), \; x > x_o \qquad \text{(b)}
\end{aligned}\right\} \quad (6)$$

$$\left.\begin{aligned}
\underline{K}(\ell) = \underline{K}_o &= -\int_0^{\ell} \left\{\underline{P}^o \cos [\beta(\ell - \xi)] + j\, \underline{\mathcal{J}} \sin [\beta(\ell - \xi)]\right\} \underline{V}_o\, \delta(\xi - x_o)\, d\sigma \\
&= -\left\{\underline{P}^o \cos [\beta(\ell - x_o)] + j\, \underline{\mathcal{J}} \sin [\beta(\ell - x_o)]\right\} \underline{V}_o \\
\underline{K}_o &= -\,[\underline{P}^o \cos (\theta - \theta_o) + j\, \underline{\mathcal{J}} \sin (\theta - \theta_o)]\, \underline{V}_o \qquad (7)
\end{aligned}\right\}$$

In Equations (6a), (6b), and (7), $\theta_o = \beta\, x_o$.

Using these results in Equations (3),

$$\left.\begin{aligned} \underline{V}(x) &= \underline{R}\ \underline{S}^{-1}\ \underline{K}_o + \underline{V}_o \cos(\theta' - \theta_o)\cdot u(\theta' - \theta_o) \\ \underline{I}(x) &= -\underline{Y}\left\{\underline{T}\ \underline{S}^{-1}\ \underline{K}_o + j\ \underline{V}_o \sin(\theta' - \theta_o)\cdot u(\theta' - \theta_o)\right\} \end{aligned}\right\} \qquad (8)$$

Where $u(\phi)$ is the unit step function:

$$\left.\begin{aligned} u(\phi) &= 0,\ \phi < 0 \\ &= 1,\ \phi > 0 \end{aligned}\right\} \qquad (9)$$

and $\underline{K}_o$ is given by (7).

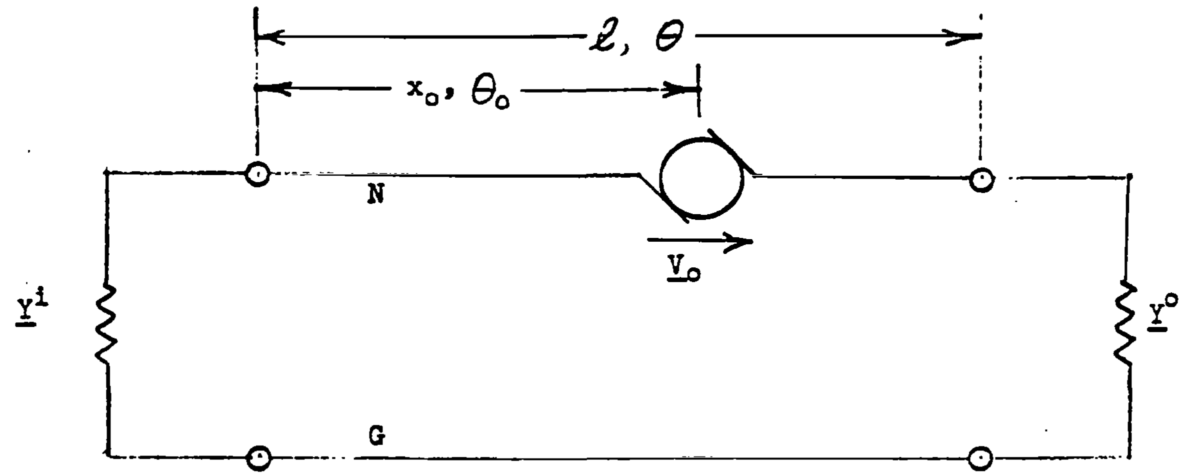
We show next that this result includes the requirements of problem Types I, III, and IV (see Section 1).

2.2 Relation to Problem Types I, III, and IV

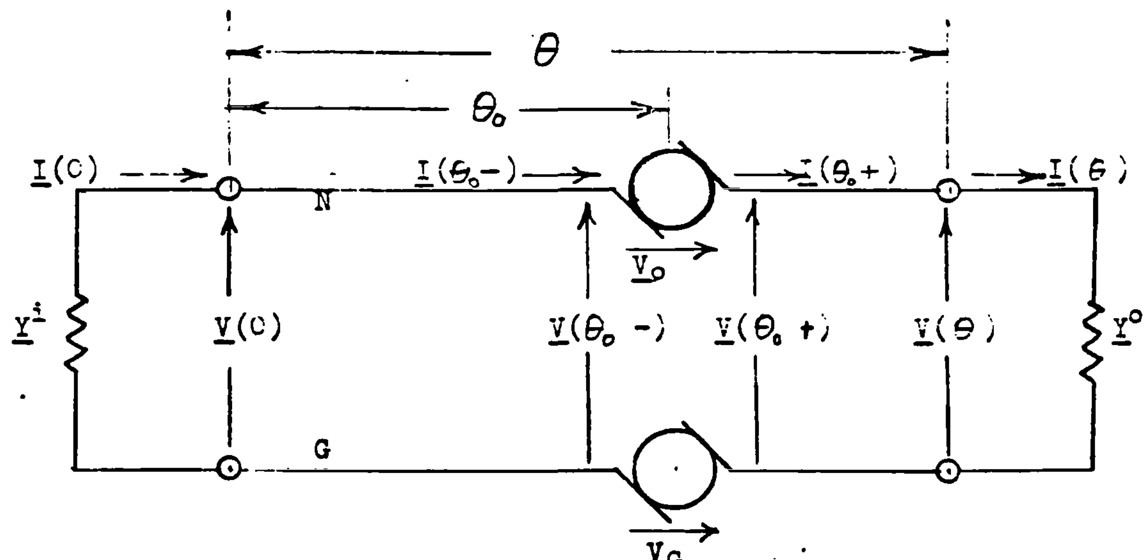
In this section we show that a slight generalization of the foregoing discussion yields a formulation which includes all four problem types.

2.2.1 Problem Type I. Broken Cable Shield

Problem Type II, the subject of the present report, is represented schematically by Figure 2a, which is a special case of Figure 1, with $\underline{E}^e(x)$ specified as an impulse, $\underline{V}_o\ \delta(x - x_o)$, or a generator vector $\underline{V}_o$ in series with the line. A slight generalization of this situation includes a generator, V_G, in the ground return as well, as shown in Figure 2b. In that case the net generator for the k^{th} conductor (with its ground return) is $(V_o^k - V_G)$. The net generator vector for the line is $(\underline{V}_o - V_G\ \underline{\mathcal{J}}_c)$. If, now, we specialize this by making $\underline{V}_o = 0$, the net generator is $-V_G\ \underline{\mathcal{J}}_c$ and the situation is that of the broken shield. Thus the formulation for problem type II should yield the results obtained previously for problem type I when $\underline{V}_o$ is replaced by $-V_G\ \underline{\mathcal{J}}_c$. That this is, in fact, true, is shown in Section 2.3 and Appendix C.



(a) without shield generator



(b) with shield generator

Fig. 3. N-line with series voltage source vector at θ_o

2.2.2 Problem Types III and IV. End-Excited Line

Problem types III and IV are essentially similar [2]. They result as a special case of type II when $\theta_o \rightarrow 0$ in Equations (8).

Equations (8) give the voltages and currents at any point on the line when it is excited by a series voltage vector, V_o, at electrical distance, θ_o, from the reference end of the line, while the line terminations at either end are passive. However, if, for instance, we let $\theta_o \rightarrow 0$, then the resulting combination of passive termination and voltage vector at the end of the line is the equivalent of a Thevenin source at the reference end [Ref. 2, Appendix A]. Therefore, Equations (8) with $\theta_o \rightarrow 0$, represent problem types III and IV.

Reduction to the forms previously given in Reference 2 is derived in Section 2.3.1.

2.3 Generalized Formulation for Localized Voltage Source, Including a Generator in Series with Ground, or Shield. Problem Types I and II

As stated in Section 2.2.1, the generalized formulation requires only that $\underline{V}_o$ in Equations (8) be replaced by a new vector, $\underline{V}^e$, say, where

$$\underline{V}^e = \underline{V}_o - V_G \, \underline{\mathcal{J}}_c \tag{10}$$

For reference we write, from Equations (7) and (8),

$$\left.\begin{aligned} \underline{V}(x) &= \underline{R}\;\underline{S}^{-1}\,\underline{K}^e + \underline{V}^e \cos(\theta' - \theta_o)\cdot u(\theta' - \theta_o) \\ \underline{I}(x) &= -\underline{Y}\left\{\underline{T}\;\underline{S}^{-1}\,\underline{K}^e + j\,\underline{V}^e \sin(\theta' - \theta_o)\cdot u(\theta' - \theta_o)\right\} \end{aligned}\right\} \tag{11}$$

where

$$\underline{K}^e = -\left[\underline{P}^o \cos(\theta - \theta_o) + j\,\underline{\mathcal{J}} \sin(\theta - \theta_o)\right] \underline{V}^e \tag{12}$$

At the time of this writing the computer program for problem Type I has been prepared. The generalization to include all four types is so slightly different from the type I problem that it seems desirable to recast the general result in a form resembling type I as nearly as possible, with a view toward slight modification of the existing program. The resulting program should then be usable for all specified problem types.

The required transformations are discussed in detail in Appendix C. From that Appendix we reproduce the following table showing the relationships between the parameters of the locally excited, exposed N-line (Present Report) and those of the broken shield problem (Ref. 1):

Table I

Quantity	Present Report		Reference 1
Schematic Diagram	Fig. 2(b)		Fig. 2
Left-Hand Terminal Voltage	$\underline{V}(o)$	=	$\underline{V}^{o}$
Left-Hand Terminal Current	$\underline{I}(o)$	=	$-\underline{I}^{o}$
Voltage at Left of Source	$\underline{V}(\theta_o-)$	=	$\underline{V}^{i}$
Current at Left of Source	$\underline{I}(\theta_o-)$	=	$-\underline{I}^{i}$
Voltage at Right of Source	$\underline{V}(\theta_o+)$	=	$\underline{V}+^{i}$
Current at Right of Source	$\underline{I}(\theta_o+)$	=	$\underline{I}+^{i}$
Right-Hand Terminal Voltage	$\underline{V}(\theta)$	=	$\underline{V}+^{o}$
Right-Hand Terminal Current	$\underline{I}(\theta)$	=	$\underline{I}+^{o}$
Ground Generator	V_G	=	$-V_g$
N-Conductor Generator	V_o	=	$\underline{0}$
Left-Hand Length of Line	θ_o	=	$\theta-$
Right-Hand Length of Line	$\theta - \theta_o$	=	$\theta+$
Total Length of Line	θ	=	$\theta- + \theta+$
Left-Hand Termination	$\underline{Y}^{i}$	=	$Y-^{o}$
Right-Hand Termination	$\underline{Y}^{o}$	=	$Y+^{o}$
Left Termination Impedance Factor	$\underline{P}^{i}$	=	$\underline{P}-$
Right Termination Impedance Factor	$\underline{P}^{o}$	=	$\underline{P}+$

Then, starting with Equations (10) - (12), the following results are obtained in Appendix C:

$$\left.\begin{array}{lr}
\underline{V}(0) = \underline{V}^{o}_{-} = b_{-} \, (\underline{\Lambda} \, \underline{N}_{-})^{-1} \, \underline{V}^{e} & \text{(a)} \\
\underline{I}(0) = -\underline{I}^{o}_{-} = -b_{-} \, \underline{Y}_{-}^{o} \, (\underline{\Lambda} \, \underline{N}_{-})^{-1} \, \underline{V}^{e} & \text{(b)} \\
\underline{V}(\theta_{o}-) = \underline{V}^{i}_{-} = -\underline{M}_{-} \, (\underline{\Lambda} \, \underline{N}_{-})^{-1} \, \underline{V}^{e} & \text{(c)} \\
\underline{I}(\theta_{o}-) = -\underline{I}^{i}_{-} = (\underline{\Lambda} \, \underline{Z})^{-1} \, \underline{V}^{e} & \text{(d)} \\
\underline{V}(\theta_{o}+) = \underline{V}^{i}_{+} = \underline{M}_{+} \, (\underline{\Lambda} \, \underline{N}_{+})^{-1} \, \underline{V}^{e} & \text{(e)} \\
\underline{I}(\theta_{o}+) = \underline{I}^{i}_{+} = -\underline{I}^{i}_{-} = (\underline{\Lambda} \, \underline{Z})^{-1} \, \underline{V}^{e} & \text{(f)} \\
\underline{V}(\theta) = \underline{V}^{o}_{+} = -b_{+} \, (\underline{\Lambda} \, \underline{N}_{+})^{-1} \, \underline{V}^{e} & \text{(g)} \\
\underline{I}(\theta) = \underline{I}^{o}_{+} = -b_{+} \, \underline{Y}^{o}_{+} \, (\underline{\Lambda} \, \underline{N}_{+})^{-1} \, \underline{V}^{e} & \text{(h)}
\end{array}\right\} \quad (13)$$

where $\underline{V}^{e}$ is given by Equation (10), and the remaining quantities are defined in Ref. 1.

To obtain the results of Ref. 1, one sets $\underline{V}^{o} = \underline{0}$ and $V_G = -V_g$ in Equation (10) [cf. Table I].

That is, for the broken shield problem (Type I),

$$\underline{V}^{e} = V_{g} \, \underline{\mathcal{J}}_{c} \qquad (14)$$

in Equations (13). This result compares with Equation (32) of Ref. 1.

On the other hand, for the present case, locally excited exposed N-line (problem Type II), $V_G = 0$ in Equation (10) and, consequently

$$\underline{V}^{e} = \underline{V}_{o} \qquad (15)$$

in Equations (13). Thus the forms of the solutions for Types I and II are identical, and differ only in the special values of $\underline{V}^e$, at most.

2.3.1 Special Case When $\theta_o = \theta_- \to 0$. Problem Types III and IV

As $\theta_- \to 0$, we have

$$a_- = -j \cot \theta_- \to -j\,\infty$$

$$\underline{M}_- = a_- \underline{\mathcal{J}} + \underline{P}_- = a_- (\underline{\mathcal{J}} + a_-^{-1} \underline{P}_-) \to a_- \underline{\mathcal{J}} \to -j\,\infty \cdot \underline{\mathcal{J}}$$

$$\underline{N}_- = \underline{\mathcal{J}} + a_- \underline{P}_- = a_- (a_-^{-1} \underline{\mathcal{J}} + \underline{P}_-) \to a_- \underline{P}_- \to -j\,\infty \cdot \underline{\mathcal{J}}$$

Thus, in order to use the Type I computer program for this case, it is necessary to assign some small, nonvanishing value to θ_-. How small this has to be depends on the terminations.

Alternatively one can introduce a supplementary computation into the program subject to conditional control. Note that for problem Types III and IV all dynamic quantities in Equation (13) with "-" subscripts are internal to the Thevenin source and therefore of secondary interest. On the other hand, in the expressions for the remaining quantities [(e) - (h)], only Λ requires modification. In fact, since

$$\underline{M}_- \underline{N}_-^{-1} = (a_- \underline{\mathcal{J}} + \underline{P}_-)(\underline{\mathcal{J}} + a_- \underline{P}_-)^{-1} \to \underline{P}^-$$

we have

$$\underline{\Lambda} = \underline{M}_+ \underline{N}_+^{-1} + \underline{P}_-^{-1}$$

Therefore it is only necessary to include in the program the instruction

$$\left.\begin{array}{l} \text{If } \theta_- \neq 0, \text{ compute } \underline{\Lambda} = \underline{M}_+ \underline{N}_+^{-1} + \underline{M}_- \underline{N}_-^{-1} \\ \text{If } \theta_- = 0, \text{ compute } \Lambda = \underline{M}_+ \underline{N}_+^{-1} + \underline{P}_-^{-1} \end{array}\right\} \qquad (16)$$

(In fact, this difficulty and its resolution occurs for <u>any</u> of the problem types whenever θ_+ or θ_- is an integral multiple of π.)

It is of interest to complete the identification of this case with the results of Ref. 2. As noted above, we are interested only in Equations (13e - h). Furthermore, since the line is one-sided, the subscripts may be dropped from $\underline{M}_+$ and $\underline{N}_+$. Finally, in the notation of Ref. 2 (and Table I)

$$\underline{P}- = \underline{P}^i = (Q^i)^{-1}$$

Using these changes in notation, the second of Equations (16) is

$$\left.\begin{aligned} \underline{\Lambda} &= \underline{M}\,\underline{N}^{-1} + \underline{Q}^i = (\underline{M} + \underline{Q}^i\,\underline{N})\,\underline{N}^{-1} && \text{(a)} \\ &= \underline{N}^{-1}\,\underline{M} + \underline{Q}^i = \underline{N}^{-1}\,(\underline{M} + \underline{N}\,\underline{Q}^i) && \text{(b)} \end{aligned}\right\} \quad (17)$$

i.e., two slightly different alternative forms are possible. We will use (17a). Substituting in Equations (13e - h) we have

$$\left.\begin{aligned} \underline{V}\,(\theta_o+) &= \underline{V}^i_+ = \underline{V}^i = \underline{M}(\underline{M} + \underline{Q}^i\,\underline{N})^{-1}\,\underline{V}^e && \text{(a)} \\ \underline{I}\,(\theta_o+) &= \underline{I}^i_+ = \underline{I}^i = \left[(\underline{M} + \underline{Q}^i\,\underline{N})\,\underline{N}^{-1}\,\underline{Z}\right]^{-1}\,\underline{V}^e && \\ &= \underline{Y}\,\underline{N}\,(\underline{M} + \underline{Q}^i\,\underline{N})^{-1}\,\underline{V}^e && \text{(b)} \\ \underline{V}\,(\theta) &= \underline{V}^o_+ = \underline{V}^o = -b\,(\underline{M} + \underline{Q}^i\,\underline{N})^{-1}\,\underline{V}^e && \text{(c)} \\ \underline{I}\,(\theta) &= \underline{I}^o_+ = \underline{I}^o = -b\,\underline{Y}^o\,(\underline{M} + \underline{Q}^i\,\underline{N})^{-1}\,\underline{V}^e && \text{(d)} \end{aligned}\right\} \quad (18)$$

since $\underline{Y}^o_+ = \underline{Y}^o$ in the notation of Ref. 2.

Equations (18a - 18d) identify with Equation (16) of Ref. 2 if $\underline{V}^e$ is identified with the Thevenin open-circuit voltage vector, $\underline{V}^g$.

3. Summary

In Ref. 1 we derived formulas for the broken-shield case (problem Type I) and a computer program to implement the model was prepared at Sandia. The more general case shown in Figure 2b of the present report yields results identical to those of Ref. 1 when the impressed voltage, $V_g \underline{\mathcal{J}}_c$ of the latter is replaced by

$$\underline{V}^e = \underline{V}_o - V_G \underline{\mathcal{J}}_c$$

in the present report.* To obtain the results of Ref. 1 set

$$\underline{V}_o = \underline{0}, \; V_G = -V_g,$$

so that

$$\underline{V}^e = (\underline{V}^e)_I = V_g \underline{\mathcal{J}}_c$$

To obtain the formulation for the exposed line with localized external excitation, set $V_G = 0$, so that

$$\underline{V}^e = (\underline{V}^e)_{II} = \underline{V}_o$$

The formulas for the two cases preceding include a parameter

$$\underline{\Lambda} = \underline{M}_+ \underline{N}_+^{-1} + \underline{M}_- \underline{N}_-^{-1} \quad .$$

For the end-excited line [2], $\underline{V}^e$ is identified with the open-circuited Thevenin voltage vector of the source

*See Equations (13).

$$\underline{V}^e = (\underline{V}^e)_{III} = (\underline{V}^e)_{IV} = \underline{V}^g$$

and $\underline{\Lambda}$ degenerates to

$$\underline{\Lambda} = \underline{M}_+ \underline{N}_+^{-1} + \underline{P}^{-1}$$

4. Conclusions

In the process of deriving the dynamic behavior of an exposed N-line locally excited at some arbitrary point along the line (one of the four problem types proposed by Sandia) we have shown that all four problems can be formulated as special cases of a single problem. A computer program, differing only slightly from that of the broken shield case (problem Type I), can be written to cover all four cases.

Appendix A

Line Differential Equations With Impressed Fields

The important components of impressed field become apparent when one undertakes to write the differential equations of a line. Figure 1-A shows Faraday's Law applied to the i^{th} conductor of an N-line. The law is

$$\oint_c \bar{E} \cdot d\bar{s} = - \frac{\partial \phi_n}{\partial t} = - j\omega \phi_n \tag{1-A}$$

where the integral is taken in a clockwise direction around C and ϕ_n is the normal flux through the plane of C in the negative-z direction. $-\phi_n$ is then the flux in the positive-z direction.

ϕ_n is composed of flux, ϕ_j (j = 1, ..., N) due to the currents, I_j, in the line conductors, and of flux due to an impressed magnetic intensity, $H_z^i(x, y)$, independent of y and z near the ground plane and conductor system.

We have, for the i^{th} conductor,

$$\phi_n = \sum_{j=1}^{N} \phi_{ji} - L_i^e \, i_z^e \tag{2-A}$$

where L_i^e is a constant (in henrys) for the i^{th} conductor. When the conductor dimensions (except ground) are small compared to spacings and to distances to the ground plane, the impressed intensity seen by the i^{th} conductor is approximately H_z^e, and the resulting flux component is

$$\phi_n^e = - u \int_o^{h_i} H_z^e (x) \, dy \cdot \Delta x \approx - \mu h_i \, H_z^e (x) \cdot \Delta x$$

Furthermore we have, by definition

$$\phi_{ji} = L_{ji} \, I_j \cdot \Delta x \tag{3-A}$$

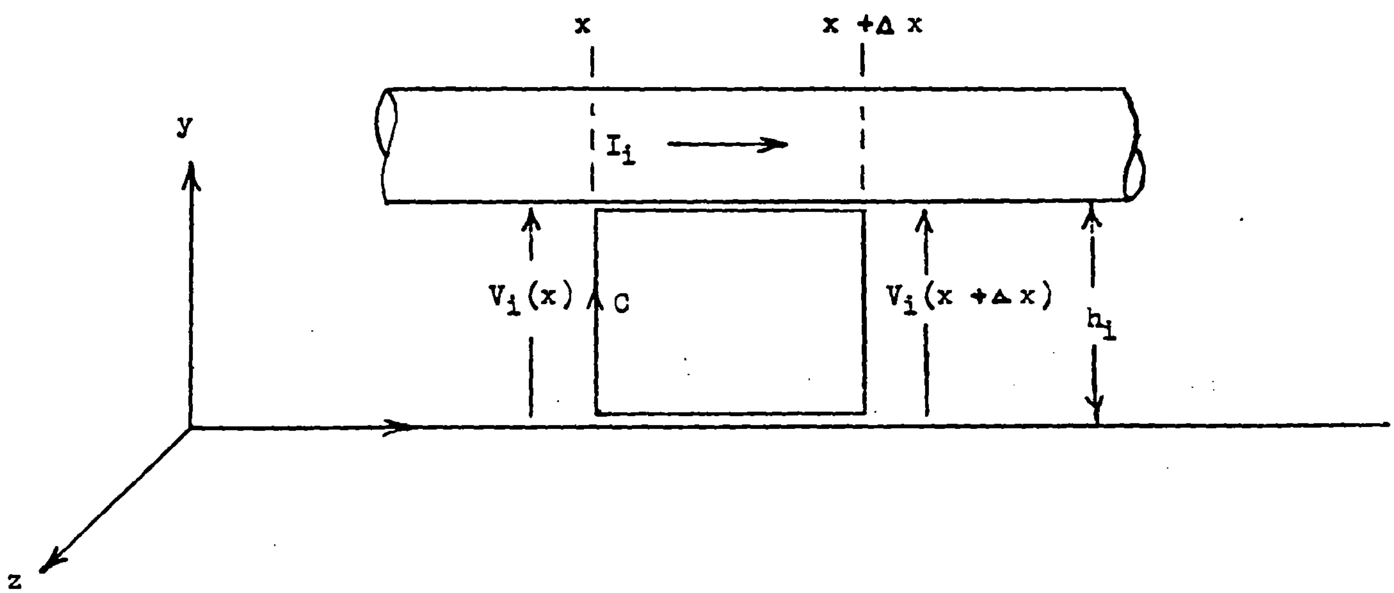


Fig. 1-A. Faraday's law applied to a short length of the i^{th} conductor.

where L_{ji} are the line inductance coefficients per unit length.

For the left member of (1-A) we have*

$$\oint_c \bar{E} \cdot \bar{ds} = -V_i(x) + (0 \cdot \Delta x) + V_i(x + \Delta x) + (0 \cdot \Delta x)$$

$$= V_i(x + \Delta x) - V_i(x) \qquad (4\text{-A})$$

By virtue of the fact that tangential $\bar{E}$ is zero along perfect conductors, there is no contribution to the integral along the i^{th} conductor and along ground.

Using these results in Equation (1-A) we get

$$V_i(x + \Delta x) - V_i(x) = -j\omega \left\{ \sum_{j=1}^{N} L_{ji} I_j - L_i^e H_z^e \right\} \Delta x$$

Dividing by $\Delta x \rightarrow 0$, letting $\Delta x \rightarrow 0$, and transposing,

$$\frac{dV_i}{dx} + \sum_{j=1}^{N} (j\omega L_{ji}) I_j = j\omega L_i^e H_z^e \qquad (5\text{-A})$$

Write

$$\zeta_{ij} = \zeta_{ji} = j\omega L_{ji}$$

*We assume that $H_x^e = 0$, so that the electric field in a transverse plane is conservative.

= series mutual impedance, i^{th} and

j^{th} conductors, per meter of line* $(j \neq i)$

ζ_{ii} = series self impedance, i^{th} conductor.

Note also that the various terms of (5-A) are in units of volts/meter, so that the right member may be interpreted as an impressed voltage per meter of line, or an impressed electric field. Writing

$$E_i^e(x) = j\omega L_i^e H_z^e \tag{6-A}$$

we have, for the first line equation

$$\frac{dV_i}{dx} + \sum_{j=1}^{N} \zeta_{ij} I_j = E_i^e(x), \; i = 1, \ldots, N \tag{7-A}$$

or, in matrix form

$$\frac{d\underline{V}}{dx} + \underline{\zeta}\, \underline{I} = \underline{E}^e(x) \tag{8-A}$$

where $\underline{E}^e$ is the column vector

$$\underline{E}^e = \left[j\omega\mu \int_0^{h_i} H_z^e\, (x, y)\, dy \right]$$

*Not to be confused with Z_{ij}, a line mutual impedance coefficient. A relation between these quantities is $Z_{ij} = v\, L_{ij}$, where v is velocity of propagation. Thus, $\zeta_{ij} = j\beta Z_{ij}$, where $\beta = \omega/v$.

Next we apply the equation of current continuity to the i^{th} conductor (Fig. 2A):

$$I_i(x + \Delta x) + \frac{dq_i}{dt} \Delta x = I_i(x)$$

where q_i is the charge per meter of length of the i^{th} conductor. Again, writing, for steady state

$$\frac{dq_i}{dt} = j\omega q_i$$

dividing by Δx, and letting $\Delta x \rightarrow 0$,

$$\frac{dI_i}{dx} + j\omega q_i = 0 \qquad (9\text{-A})$$

q_i is induced on the i^{th} conductor by the potentials of all the conductors, in accordance with

$$q_i = \sum_{i=1}^{N} C_{ij} V_j, \; i = 1, \ldots, N \qquad (10\text{-A})$$

But if a transverse impressed electric field is present, charges are induced on the conductors even if all conductors are at zero potential. Furthermore, such charge must be proportional to the impressed field. Near the ground plane it is reasonable to assume that the impressed field (in the absence of effects from the conductors) is constant and normal to the ground plane, -- say $E^e = E^e_y$. Then we can write

$$q_i = \sum_{j=1}^{N} C_{ij} V_j + C^e_i E^e_y \qquad (11\text{-A})$$

where C^e_i is in farads.

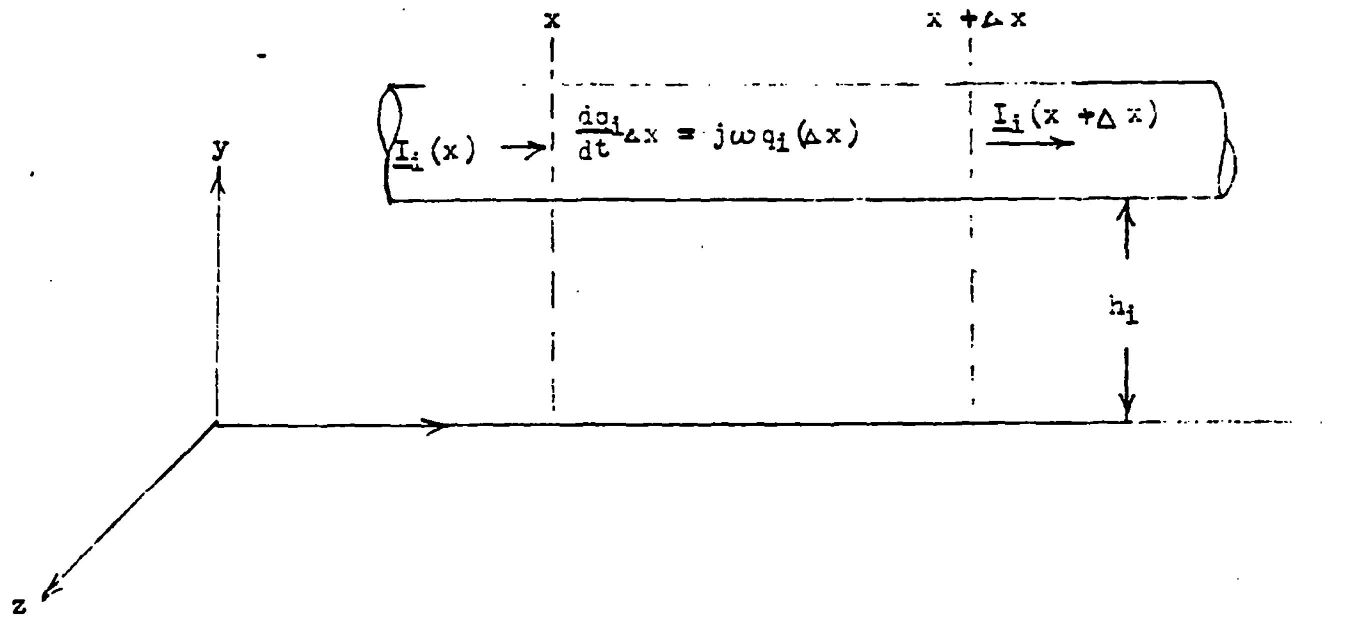


Fig. 2-A. Law of current continuity applied to i^{th} conductor.

In order to evaluate the C_i^e, proceed as follows: First, with E_y^e present, let all conductors "float" with respect to ground, so that they carry zero net charges, but take on potentials $V_j^e (j = 1, \ldots, N)$.

Next, suppose all conductors grounded. The effect is the same as applying to them a set of potentials, $- V_j^e$. But such a set must result in a set of charges

$$q_i^e = \sum_{j=1}^{N} C_{ij}(-V_j^e) = - \sum_{j=1}^{N} C_{ij} V_J^e \tag{12-A}$$

Adding these to the charges induced by the line potentials [Equation (10-A)], we have, altogether

$$q_i = \sum_{j=1}^{N} C_{ij}(V_j - V_j^e) \tag{13-A}$$

Substituting (13-A) in (9-A) and rearranging

$$\frac{dI_i}{dx} + \sum_{j=1}^{N} (j\omega C_{ij}) V_j = \sum_{j=1}^{N} (j\omega C_{ij}) V_j^e \tag{14-A}$$

In Equation (14-A) write

$$\eta_{ji} = \eta_{ij} = j\omega C_{ij}$$

= shunt mutual admittance, i^{th} and j^{th} conductors, per meter of line.* $(i \neq j)$.

η_{ii} = shunt self admittance, i^{th} conductor.

*Not to be confused with Y_{ij}, a line mutual admittance coefficient. In fact, $Y_{ij} = v\, C_{ij}$. See footnote, page 18.

Generally the V_j^e must be evaluated as a separate electrostatic problem. If the conductor cross-sections (except ground) are small compared to the distances between them and their distances to ground, then approximately,

$$V_j^e = -h_j \, E_y^e \qquad (15\text{-A})$$

where h_j is the height of the j^{th} conductor above ground. In general, we can only say that V_j^e is proportional to E_y^e:

$$V_j^e = K_j \, E_y^e$$

and

$$\sum_{j=1}^{N} C_{ij} \, V_j^e = E_y^e \sum_{j=1}^{N} C_{ij} \, K_j \qquad (16\text{-A})$$

so that C_i^e of Equation (11-A) is identified as

$$C_i^e = \sum_{j=1}^{N} C_{ij} \, K_j \qquad (17\text{-A})$$

or, in the special case then (15-A) holds,

$$C_i^e = - \sum_{j=1}^{N} C_{ij} \, h_j \qquad (18\text{-A})$$

In matrix form,

$$\underline{C}^e = -\underline{C} \, \underline{h} \qquad (19\text{-A})$$

Since the terms of Equation (14-A) are in units of amperes per meter, the right member may be interpreted as an impressed magnetic intensity or as a distributed current source. Writing

$$H_i^e = \sum_{j=1}^{N} (j\omega\, C_{ij}) V_j^e = E_y^e \sum_{j=1}^{N} \eta_{ij}\, K_j$$

$$= E_y^e\, Y_i^e \qquad (20\text{-A})$$

we have

$$\frac{dI_i}{dx} + \sum_{j=1}^{N} \eta_{ij}\, V_j = H_i^e,\ i = 1, \ldots, N \qquad (21\text{-A})$$

as the second line equation. Writing in matrix form and pairing with Equation (8-A),

$$\left.\begin{aligned} \frac{d\underline{V}}{dx} + \underline{\zeta}\ \underline{I} &= \underline{E}^e(x) \\ \frac{d\underline{I}}{dx} + \underline{\eta}\ \underline{V} &= \underline{H}^e(x) \end{aligned}\right\} \qquad (22\text{-A})$$

where

$$\underline{H}^e = \begin{bmatrix} H_1^e \\ \vdots \\ H_N^e \end{bmatrix}$$

Appendix B

Line Response to Impressed Fields

Solution of Differential Equations

From Appendix A, Equations (22-A) we have

$$\left.\begin{aligned} \frac{d\underline{V}}{dx} + \underline{\zeta}\,\underline{I} &= \underline{E}^{e}(x) \\ \frac{d\underline{I}}{dx} + \underline{\eta}\,\underline{V} &= \underline{H}^{e}(x) \end{aligned}\right\} \qquad (1\text{-B})$$

These equations will be solved using Laplace transforms in the x domain, since we may choose the N-line to lie in the range $0 < x < \ell$, where ℓ is the length of the line. Notation will be as follows:

Let $\underline{F}(x)$ be the column vector

$$\underline{F}(x) = \begin{bmatrix} F_1(x) \\ \vdots \\ F_N(x) \end{bmatrix}$$

Then for the Laplace transform of $\underline{F}$ we write

$$\mathcal{L}\,\underline{F}(x) = \underline{\tilde{F}}(p) = \int_0^\infty \underline{F}(\xi) e^{-p\xi}\, d\xi$$

$$= \begin{bmatrix} \int_0^\infty F_1(\xi) e^{-p\xi}\, d\xi \\ \vdots \\ \int_0^\infty F_N(\xi) e^{-p\xi}\, d\xi \end{bmatrix} = \begin{bmatrix} \tilde{F}_1(p) \\ \vdots \\ \tilde{F}_N(p) \end{bmatrix}$$

For the Laplace transform of $\frac{dF_i}{dx}$ it is shown in standard texts, using integration by parts once, that

$$\mathcal{L}\left\{\frac{dF_i}{dx}\right\} = -F_i(0) + p\,\tilde{F}_i(p) = \left(\frac{\widetilde{dF}_i}{dx}\right)$$

Thus

$$\mathcal{L}\left\{\frac{d\underline{F}}{dx}\right\} = \begin{bmatrix} \frac{\widetilde{dF}_1}{dx} \\ \vdots \\ \frac{\widetilde{dF}_N}{dx} \end{bmatrix} = \begin{bmatrix} -F_1(0) + p\,\tilde{F}_1(p) \\ \vdots \\ -F_N(0) + p\,\tilde{F}_N(p) \end{bmatrix}$$

$$= -\begin{bmatrix} F_1(0) \\ \vdots \\ F_N(0) \end{bmatrix} + p \begin{bmatrix} \tilde{F}_1(p) \\ \vdots \\ \tilde{F}_N(p) \end{bmatrix}$$

$$= -\underline{F}(0) + p\,\tilde{\underline{F}}(p)$$

Taking transforms in Equations (1-3)

$$\left.\begin{aligned} -\underline{V}(0) + p\tilde{\underline{V}} + \underline{\zeta}\,\tilde{\underline{I}} &= \tilde{\underline{E}}^e \\ -\underline{I}(0) + p\tilde{\underline{I}} + \underline{\eta}\,\tilde{\underline{V}} &= \tilde{\underline{H}}^e \end{aligned}\right\}$$

that is

$$\left.\begin{aligned} p\tilde{\underline{V}} + \underline{\zeta}\,\tilde{\underline{I}} &= \underline{V}(0) + \tilde{\underline{E}}^e \\ \underline{\eta}\,\tilde{\underline{V}} + p\tilde{\underline{I}} &= \underline{I}(0) + \tilde{\underline{H}}^e \end{aligned}\right\}$$

Solve for $\underline{\tilde{V}}$ by Gaussian elimination of $\underline{\tilde{I}}$:

$$\left.\begin{aligned} p^2\,\underline{\tilde{V}} + p\,\underline{\zeta}\,\underline{\tilde{I}} &= p\,\underline{V}(0) + p\,\underline{\tilde{E}}^e \\ \underline{\zeta}\,\underline{\eta}\,\underline{\tilde{V}} + p\,\underline{\zeta}\,\underline{\tilde{I}} &= \underline{\zeta}\,\underline{I}(0) + \underline{\zeta}\,\underline{\tilde{H}}^e \end{aligned}\right\}$$

Subtracting, and recalling that for TEM lines, $\underline{\zeta}\,\underline{\eta} = \underline{\eta}\,\underline{\zeta} = -\beta^2\underline{\mathcal{I}}$ [4], where

$$\beta = \frac{\omega}{v} ,$$

we get

$$(p^2 + \beta^2)\underline{\tilde{V}} = p\,\underline{V}(0) - \underline{\zeta}\,\underline{I}(0) + p\underline{\tilde{E}}^e - \underline{\zeta}\,\underline{\tilde{H}}^e$$

or

$$\underline{\tilde{V}} = \frac{p}{p^2 + \beta^2}\,\underline{V}(0) - \frac{1}{p^2 + \beta^2}\,\underline{\zeta}\,\underline{I}(0) + \frac{p}{p^2 + \beta^2}\,\underline{\tilde{E}}^e(p) - \frac{\underline{\zeta}\,\underline{\tilde{H}}^e(p)}{p^2 + \beta^2} \qquad \text{(2-B)}$$

Similarly, eliminating $\underline{\tilde{V}}$,

$$\underline{\tilde{I}} = \frac{p}{p^2 + \beta^2}\,\underline{I}(0) - \frac{1}{p^2 + \beta^2}\,\underline{\eta}\,\underline{V}(0) + \frac{p}{p^2 + \beta^2}\,\underline{\tilde{H}}^e(p) - \frac{\underline{\eta}\,\underline{E}^e(p)}{p^2 + \beta^2} \qquad \text{(3-B)}$$

Tables of Laplace transforms and their inverses are generally available (e.g., Ref. 5, Section 29; Ref. 6, Section 44). Writing, for the inverse transform,

$$\mathcal{L}^{-1}\,[\tilde{F}(p)] = F(x)$$

We have from standard tables,

$$\left.\begin{aligned} \mathcal{L}^{-1}\left[\frac{p}{p^2+\beta^2}\right] &= \cos\beta x \\ \mathcal{L}^{-1}\left[\frac{1}{p^2+\beta^2}\right] &= \beta^{-1}\sin\beta x \end{aligned}\right\} \qquad (4\text{-B})$$

To obtain the transforms for

$$\underline{G}(x) = \mathcal{L}^{-1}\left\{\frac{\underline{\tilde{E}}^e(p)}{p^2+\beta^2}\right\} \qquad (5\text{-B})$$

and

$$\underline{J}(x) = \mathcal{L}^{-1}\left\{\frac{\underline{\tilde{H}}^e(p)}{p^2+\beta^2}\right\} \qquad (6\text{-B})$$

use the convolution theorem [5, 6; ibid.]. The theorem states: Given the product of two Laplace transforms

$$\tilde{G}(p) = \tilde{F}_1(p)\tilde{F}_2(p)$$

and the inverse transforms of the factors

$$F_1(x) = \mathcal{L}^{-1}\,\tilde{F}_1(p)$$

$$F_2(x) = \mathcal{L}^{-1}\,\tilde{F}_2(p)$$

then

$$G(x) = \int_0^x F_1(\xi)\,F_2(x-\xi)\,d\xi = \int_0^x F_1(x-\xi)\,F_2(\xi)\,d\xi \qquad (7\text{-B})$$

In Equation (5-B), if we let

$$\left.\begin{aligned} \tilde{F}_1(p) &= (p^2 + \beta^2)^{-1} \\ \underline{\tilde{F}}_2(p) &= \underline{\tilde{E}}^e(p) \end{aligned}\right\} \qquad (8\text{-B})$$

then the product

$$\underline{\tilde{G}}(p) = \tilde{F}_1(p)\, \underline{\tilde{F}}_2(p)$$

is a column vector with the scalar elements

$$\tilde{G}_i(p) = \tilde{F}_1(p)\, \tilde{F}_{2,i}(p) \qquad i = 1, \ldots, N$$

so that, by Equation (7-B), the i^{th} element of the inverse transform is

$$G_i(x) = \int_o^x F_1(x - \xi)\, E_i^e(\xi)\, d\xi \qquad i = 1, \ldots, N$$

or

$$\underline{G}(x) = \int_o^x F_1(x - \xi)\, \underline{E}^e(\xi)\, d\xi$$

Since

$$F_1(x) = \mathcal{L}^{-1}\left[(p^2 + \beta^2)^{-1}\right] = \beta^{-1} \sin \beta x$$

we get, for (5-B),

$$\underline{G}(x) = \beta^{-1} \int_o^x \underline{E}^e(\xi) \sin\left[\beta(x - \xi)\right] d\xi \qquad (9\text{-B})$$

In exactly the same way, Equation (6-B) becomes

$$\underline{J}(x) = \beta^{-1} \int_0^x \underline{H}^e(\xi) \sin\left[\beta(x - \xi)\right] d\xi \tag{10-B}$$

Next, from the general inversion formula [29. 2.2, Ref. 5]

$$\underline{G}(x) = \frac{1}{j2\pi} \int_{c-j\infty}^{c+j\infty} \underline{\tilde{G}}(p)e^{px}\, dp, \quad c > 0$$

we get, by differentiation with respect to x,

$$\underline{G}'(x) = \frac{1}{j2\pi} \int_{c-j\infty}^{c+j\infty} p\underline{\tilde{G}}(p)e^{px}\, dp$$

Thus, since

$$\underline{G}(x) = \mathcal{L}^{-1}\left\{\frac{\underline{\tilde{E}}^e(p)}{p^2 + \beta^2}\right\} = \beta^{-1} \int_0^x \underline{E}^e(\xi) \sin\left[\beta(x - \xi)\right] d\xi$$

it follows that

$$\underline{G}'(x) = \mathcal{L}^{-1}\left\{\frac{p\underline{\tilde{E}}^e(p)}{p^2 + \beta^2}\right\} = \int_0^x \underline{E}^e(\xi) \cos\left[\beta(x - \xi)\right] d\xi \tag{11-B}$$

Similarly,

$$\underline{J}'(x) = \mathcal{L}^{-1}\left\{\frac{p\underline{\tilde{H}}^e(p)}{p^2 + \beta^2}\right\} = \int_0^x \underline{H}^e(\xi) \cos\left[\beta(x - \xi)\right] d\xi \tag{12-B}$$

Substitution of Equations (4-B), (9-B)-(12-B) in the inverse of Equations (2-B) and (3-B) yields, respectively,

$$\underline{V} = \underline{V}(0)\cdot\cos\beta x - \beta^{-1}\underline{\zeta}\,\underline{I}(0)\cdot\sin\beta x + \int_0^x \underline{E}^e(\xi)\cos\left[\beta(x-\xi)\right]d\xi$$

$$- \beta^{-1}\underline{\zeta}\int_0^x \underline{H}^e(\xi)\sin\left[\beta(x-\xi)\right]d\xi \tag{13-B}$$

$$\underline{I} = \underline{I}(0)\cdot\cos\beta x - \beta^{-1}\underline{\eta}\,\underline{V}(0)\cdot\sin\beta x + \int_0^x \underline{H}^e(\xi)\cos\left[\beta(x-\xi)\right]d\xi$$

$$- \beta^{-1}\underline{\eta}\int_0^x \underline{E}^e(\xi)\sin\left[\beta(x-\xi)\right]d\xi \tag{14-B}$$

or, more compactly,

$$\left.\begin{aligned}\underline{V} &= \underline{V}(0)\cdot\cos\beta x - \beta^{-1}\underline{\xi}\,\underline{I}(0)\cdot\sin\beta x + \underline{G}'(x) - \underline{\zeta}\,\underline{J}(x)\\ \underline{I} &= \underline{I}(0)\cdot\cos\beta x - \beta^{-1}\underline{\eta}\,\underline{V}(0)\cdot\sin\beta x + \underline{J}'(x) - \underline{\eta}\,\underline{G}(x)\end{aligned}\right\} \tag{15-B}$$

Now (Appendix A and Ref. 4)

$$\beta^{-1}\zeta_{ij} = \frac{v}{\omega}(j\omega L_{ij}) = j(v L_{ij}) = jZ_{ij}$$

and

$$\beta^{-1}\eta_{ij} = \frac{v}{\omega}(j\omega C_{ij}) = j(v C_{ij}) = j Y_{ij} \tag{16-B}$$

where Z_{ij} and Y_{ij} are the N-line impedance - and admittance coefficients respectively. Therefore,

$$\left.\begin{aligned}\beta^{-1}\underline{\zeta} &= j\,\underline{Z}\\ \beta^{-1}\underline{\eta} &= j\,\underline{Y}\end{aligned}\right\} \tag{17-B}$$

Substituting in (15-B),

$$\left.\begin{aligned} \underline{V} &= \underline{V}(0)\cdot\cos\beta x - j\,\underline{Z}\,\underline{I}(0)\cdot\sin\beta x + \underline{G}'(x) - \underline{\zeta}\,\underline{J}(x) \\ \underline{I} &= \underline{I}(0)\cdot\cos\beta x - j\,\underline{Y}\,\underline{V}(0)\cdot\sin\beta x + \underline{J}'(x) - \underline{\eta}\,\underline{G}(x) \end{aligned}\right\} \quad (18\text{-B})$$

Introduction of Terminal Conditions

In Equations (17-B), in order to conform with previous notation, write

$$\underline{V}(0) = \underline{V}^{i}$$

$$\underline{I}(0) = \underline{I}^{i}$$

and note that terminal conditions require that

$$\left.\begin{aligned} \underline{I}^{i} + \underline{Y}^{i}\,\underline{V}^{i} &= 0 \\ \underline{I}^{o} - \underline{Y}^{o}\,\underline{V}^{o} &= 0 \end{aligned}\right\} \quad (19\text{-B})$$

(see Glossary, Ref. 2)

Then Equations (18-B) yield, respectively,

$$\left.\begin{aligned} \underline{V}^{o} &= \underline{V}^{i}\cos\beta\ell - j\,\underline{Z}\,\underline{I}^{i}\cdot\sin\beta\ell + \underline{G}'(\ell) - \underline{\zeta}\,\underline{J}(\ell) \\ \underline{I}^{o} &= \underline{I}^{i}\cos\beta\ell - j\,\underline{Y}\,\underline{V}^{i}\cdot\sin\beta\ell + \underline{J}'(\ell) - \underline{\eta}\,\underline{G}(\ell) \end{aligned}\right\} \quad (20\text{-B})$$

Writing $\theta = \beta\ell$ and

$$\left.\begin{aligned} \underline{U}(x) &= \underline{G}'(x) - \underline{\zeta}\,\underline{J}(x) \\ \underline{W}(x) &= \underline{J}'(x) - \underline{\eta}\,\underline{G}(x) \end{aligned}\right\} \quad (21\text{-B})$$

$$\left.\begin{aligned} \underline{V}^{o} &= \underline{V}^{i} \cos\theta - j\,\underline{Z}\,\underline{I}^{i} \sin\theta + \underline{U}(\ell) \\ \underline{I}^{o} &= \underline{I}^{i} \cos\theta - j\,\underline{Y}\,\underline{V}^{i} \sin\theta + \underline{W}(\ell) \end{aligned}\right\} \qquad (22\text{-B})$$

Using Equations (22-B) in the second of Equations (19-B) and replacing $\underline{I}^{i}$ by means of the first of Equations (19-B),

$$\underline{I}^{o} - \underline{Y}^{o}\,\underline{V}^{o} = \left[-\underline{Y}^{i}\,\underline{V}^{i} \cos\theta - j\,\underline{Y}\,\underline{V}^{i} \sin\theta + \underline{W}(\ell)\right]$$

$$-\underline{Y}^{o}\left[\underline{V}^{i} \cos\theta + j\,\underline{Z}\,\underline{Y}^{i}\,V^{i} \sin\theta + \underline{U}(\ell)\right] = 0$$

$$\left[(\underline{Y}^{i} + \underline{Y}^{o}) \cos\theta + j(\underline{Y} + \underline{Y}^{o}\,\underline{Z}\,\underline{Y}^{i}) \sin\theta\right] \underline{V}^{i} = \underline{W}(\ell) - \underline{Y}^{o}\,\underline{U}(\ell)$$

Factor $\underline{Y}$ in the left member:

$$\underline{Y}\left[(\underline{Z}\,\underline{Y}^{i} + \underline{Z}\,\underline{Y}^{o}) \cos\theta + j(\underline{\mathcal{I}} + \underline{Z}\,\underline{Y}^{o}\,\underline{Z}\,\underline{Y}^{i}) \sin\theta\right] \underline{V}^{i} = W(\ell) - \underline{Y}^{o}\,\underline{U}(\ell)$$

Using previous notation [2]

$$\underline{Y}\left[(\underline{P}^{i} + \underline{P}^{o}) \cos\theta + j(\underline{\mathcal{I}} + \underline{P}^{o}\,\underline{P}^{i}) \sin\theta\right] \underline{V}^{i} = \underline{W}(\ell) - \underline{Y}^{o}\,\underline{U}(\ell) \qquad (23\text{-B})$$

Write

$$\underline{S} = (\underline{P}^{i} + \underline{P}^{o}) \cos\theta + j(\underline{\mathcal{I}} + \underline{P}^{o}\,\underline{P}^{i}) \sin\theta \qquad (24\text{-B})$$

and

$$K(\ell) = \underline{Z}\left[\underline{W}(\ell) - \underline{Y}^{o}\,\underline{U}(\ell)\right] \qquad (25\text{-B})$$

whence (23-B) becomes

$$\underline{V}^{i} = \underline{S}^{-1}\,\underline{K}(\ell) \qquad (26\text{-B})$$

Equation (25-B) is transformed as follows:

By (21-B),

$$K(\ell) = Z\ \underline{J}'(\ell) - \underline{Z}\ \underline{\eta}\ \underline{G}(\ell) - \underline{P}^{o}\ \underline{G}'(\ell) + \underline{P}^{o}\ \underline{\zeta}\ \underline{J}(\ell)$$

$$= \underline{Z} \int_{o}^{\ell} \underline{H}^{e}(\xi) \cos\left[\beta(\ell - \xi)\right] d\xi - j \int_{o}^{\ell} \underline{E}^{e}(\xi) \sin\left[\beta(\ell - \xi)\right] d\xi$$

$$-\underline{P}^{o} \int_{o}^{\ell} \underline{E}^{e}(\xi) \cos\left[\beta(\ell - \xi)\right] d\xi + j\ \underline{Z}\ \underline{Y}^{o}\ \underline{Z} \int_{o}^{\ell} \underline{H}^{e}(\xi) \sin\left[\beta(\ell - \xi)\right] d\xi$$

$$= \int_{o}^{\ell} \left\{\underline{Z} \cos\left[\beta(\ell - \xi)\right] + j\ \underline{Z}\ \underline{Y}^{o}\ \underline{Z} \sin\left[\beta(\ell - \xi)\right]\right\} \underline{H}^{e}(\xi)\ d\xi$$

$$- \int_{o}^{\ell} \left\{\underline{P}^{o} \cos\left[\beta(\ell - \xi)\right] + j\ \underline{\mathcal{J}} \sin\left[\beta(\ell - \xi)\right]\right\} \underline{E}^{e}(\xi)\ d\xi$$

$$\underline{K}(\ell) = \int_{o}^{\ell} \left\{\underline{\mathcal{J}} \cos\left[\beta(\ell - \xi)\right] + j\ \underline{P}^{o} \sin\left[\beta(\ell - \xi)\right]\right\} \underline{Z}\ \underline{H}^{e}(\xi)\ d\xi$$

$$- \int_{o}^{\ell} \left\{\underline{P}^{o} \cos\left[\beta(\ell - \xi)\right] + j\ \underline{\mathcal{J}} \sin\left[\beta(\ell - \xi)\right]\right\} \underline{E}^{e}(\xi)\ d\xi \qquad (27\text{-B})$$

Use (26-B), (21-B) and the first of (19-B) in Equations (18-B), and note that

$$\left.\begin{aligned} \underline{V}(0) &\equiv \underline{V}^{i} \\ \underline{I}(0) &\equiv \underline{I}^{i} \\ \beta x &= \theta' \end{aligned}\right\}$$

$$V = \underline{V}^{i} \cos\theta' - j\ \underline{Z}\ \underline{I}^{i} \sin\theta' + \underline{U}(x)$$

$$= (\underline{\mathcal{J}} \cos\theta' + j\ \underline{P}^{i} \sin\theta')\ \underline{S}^{-1}\ \underline{K}(\ell) + \underline{U}(x) \qquad (28\text{-B})$$

$$I = \underline{I}^{i} \cos\theta' - j\,\underline{Y}\,\underline{V}^{i} \sin\theta' + \underline{W}(x)$$

$$= -(\underline{Y}^{i} \cos\theta' + j\,\underline{Y} \sin\theta')\,\underline{S}^{-1}\,K(\ell) + \underline{W}(x)$$

$$= -\underline{Y}(\underline{P}^{i} \cos\theta' + j\,\underline{\mathcal{J}} \sin\theta')\,\underline{S}^{-1}\,\underline{K}(\ell) + \underline{W}(x) \qquad (29\text{-B})$$

$\underline{U}$ and $\underline{W}$ are easily transformed, respectively, to

$$\left.\begin{aligned} \underline{U}(x) &= \int_0^x \left\{\underline{E}^{e}(\xi) \cos\left[\beta(x-\xi)\right] - j\,\underline{Z}\,\underline{H}^{e}(\xi) \sin\left[\beta(x-\xi)\right]\right\} d\xi \\ \underline{W}(x) &= \underline{Y}\int_0^x \left\{\underline{Z}\,\underline{H}^{e}(\xi) \cos\left[\beta(x-\xi)\right] - j\,\underline{E}^{e}(\xi) \sin\left[\beta(x-\xi)\right]\right\} d\xi \end{aligned}\right\} \qquad (30\text{-B})$$

In summary, voltage and current along the line are given by Equations (28-B) and (29-B). The quantities $\underline{S}$, $\underline{K}(\ell)$, $\underline{U}(x)$, and $\underline{W}(x)$ in those equations are defined by Equations (24-B), (27-B) and (30-B), respectively.

Appendix C

General Solution of Multiline Problem with Space-Impulse Vector Voltage Source

Start with Equations (11) of the main text. We are interested in determining various terminal quantities, which are expressed, in the terminology of the present report, and also in the terminology of corresponding quantities in Ref. 1, by the following:

Table I

Quantity	Present Report		Reference 1
Schematic Diagram	Fig. 2(b)		Fig. 2
Left-Hand Terminal Voltage	$\underline{V}(0)$	=	$\underline{V}_-^o$
Left-Hand Terminal Current	$\underline{I}(0)$	=	$-\underline{I}_-^o$
Voltage at Left of Source	$\underline{V}(\theta_o-)$	=	$\underline{V}_-^i$
Current at Left of Source	$\underline{I}(\theta_o-)$	=	$-\underline{I}_-^i$
Voltage at Right of Source	$\underline{V}(\theta_o+)$	=	$\underline{V}_+^i$
Current at Right of Source	$\underline{I}(\theta_o+)$	=	$\underline{I}_+^i$
Right-Hand Terminal Voltage	$\underline{V}(\theta)$	=	$\underline{V}_+^o$
Right-Hand Terminal Current	$\underline{I}(\theta)$	=	$\underline{I}_+^o$
Ground Generator	V_G	=	$-V_g$
N-Conductor Generator	$\underline{V}_o$	=	$\underline{0}$
Left-Hand Length of Line	θ_o	=	θ_-
Right-Hand Length of Line	$\theta - \theta_o$	=	θ_+
Total Length of Line	θ	=	$\theta_- + \theta_+$
Left-Hand Termination	$\underline{Y}^i$	=	$\underline{Y}_-^o$
Right-Hand Termination	$\underline{Y}^o$	=	$\underline{Y}_+^o$
Left Termination Impedance Factor	$\underline{P}^i$	=	$\underline{P}_-$
Right Termination Impedance Factor	$\underline{P}^o$	=	$\underline{P}_+$

From Equations (11)

$$\underline{V}(0) = \underline{V}^{o} = \underline{R}(0)\, \underline{S}^{-1}\, \underline{K}^{e} = \underline{S}^{-1}\, \underline{K}^{e} \qquad (1\text{-C})$$

since

$$\underline{R}(0) = \underline{\mathcal{J}} \text{ by Equations (4)}$$

$$\underline{I}(0) = -\underline{I}^{o} = -\underline{Y}\left\{\underline{T}(0)\, \underline{S}^{-1}\, \underline{K}^{e}\right\} = -\underline{Y}\, \underline{P}^{i}\, \underline{S}^{-1}\, \underline{K}^{e} \qquad (2\text{-C})$$

since

$$\underline{T}(0) = \underline{P}^{i} \text{ by Equations (4)}$$

$$\underline{V}(\theta_{o}) = \underline{V}^{i} = \underline{R}(\theta_{o})\, \underline{S}^{-1}\, \underline{K}^{e}$$

But by Equations (4),

$$\underline{R}(\theta_{o}) = \underline{R}(\theta_{-}) = \underline{\mathcal{J}} \cos\theta_{-} + j\, \underline{P}^{i} \sin\theta_{-}$$

$$= j \sin\theta_{-}(-j \cot\theta_{-} \cdot \underline{\mathcal{J}} + \underline{P}\text{-})^{*}$$

$$= -b_{-}^{-1}(a_{-}\, \underline{\mathcal{J}} + \underline{P}\text{-})^{**}$$

$$= -b_{-}^{-1}\, \underline{M}\text{-} \quad ^{**}$$

Therefore,

$$\underline{V}(\theta_{o}) = \underline{V}^{i} = -b_{-}^{-1}\, \underline{M} - \underline{S}^{-1}\, \underline{K}^{e} \qquad (3\text{-C})$$

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* See Table I

** See Ref. 1

$$\underline{I}(\theta_o^-) = -\underline{I}_-^i = -\underline{Y}\left\{\underline{T}(\theta_o)\ \underline{S}^{-1}\ \underline{K}^e\right\}$$

But by Equations (4),

$$\underline{T}(\theta_o) = T(\theta_-) = \underline{P}^i \cos\theta_- + j\ \underline{\mathcal{J}} \sin\theta_-$$

$$= j \sin\theta_-(\underline{\mathcal{J}} - j\ \underline{P}_- \cot\theta_-)$$

$$= -b_-^{-1}(\underline{\mathcal{J}} + a_-\ \underline{P}_-)$$

$$= -b_-^{-1}\ \underline{N}_- \ ^*$$

Therefore,

$$\underline{I}(\theta_o^-) = -\underline{I}_-^i = b_-^{-1}\ \underline{Y}\ \underline{N}_-\ \underline{S}^{-1}\ \underline{K}^e \qquad (4\text{-C})$$

Passing to the right of the sources (Fig. ?), we have, in Equations (11),

$$u(\theta' - \theta_o) = 1$$

and consequently,

$$\underline{V}(\theta_o^+) = \underline{V}_+^i = \underline{R}(\theta_o)\ \underline{S}^{-1}\ \underline{K}^e + \underline{V}^e$$

$$= \left[-b_-^{-1}\ \underline{M}_-\ \underline{S}^{-1}\ \underline{K}^e(\underline{V}^e)^{-1} + \underline{\mathcal{J}}\right]\ \underline{V}^e \qquad (5\text{-C})$$

$$\underline{I}(\theta_o^+) = \underline{I}_+^i = -\underline{I}_-^i = b_-^{-1}\ \underline{Y}\ \underline{N}_-\ \underline{S}^{-1}\ \underline{K}^e \qquad (6\text{-C})$$

by Equation (4-C).

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*See Ref. 1

Finally,

$$\underline{V}(\theta) = \underline{V}_+^o = \underline{R}(\theta)\ \underline{S}^{-1}\ \underline{K}^e + \underline{V}^e \cos(\theta - \theta_o)$$

$$= \underline{R}(\theta)\ \underline{S}^{-1}\ \underline{K}^e + \underline{V}^e \cos\theta_+$$

$$= \underline{R}(\theta)\ \underline{S}^{-1}\left[\underline{S}\cos\theta_+ + \underline{K}^e(\underline{V}^e)^{-1}\ \underline{R}(\theta)\right]\left[R(\theta)\right]^{-1}\ \underline{V}^e \qquad (7\text{-C})$$

and

$$\underline{I}(\theta) = \underline{I}_+^o = \underline{Y}_+^o\ \underline{V}_+^o \qquad (8\text{-C})$$

where

$$\underline{R}(\theta) = \underline{\mathcal{J}}\cos\theta + j\ \underline{P}^i \sin\theta$$

$$= j\sin\theta\ (-j\ \underline{\mathcal{J}}\cot\theta + \underline{P}\text{-}) \qquad (9\text{-C})$$

$$= -b^{-1}\ (a\ \underline{\mathcal{J}} + \underline{P}\text{-})^*$$

The next step is to transform the quantities $\underline{K}^e$ and $\underline{S}$. First, by Equation (12)

$$\underline{K}^e = -(\underline{P}_+ \cos\theta_+ + j\ \underline{\mathcal{J}}\sin\theta_+)\ \underline{V}^e$$

$$= -j\sin\theta_+\ (\underline{\mathcal{J}} - j\ \underline{P}_+ \cot\theta_+)\ \underline{V}^e$$

$$= b_+^{-1}\ (\underline{\mathcal{J}} + a_+\ \underline{P}_+)\ \underline{V}^e \qquad (10\text{-C})$$

$$= b_+^{-1}\ \underline{N}_+\ \underline{V}^e$$

*See Glossary, Ref. 2.

Transformation of $\underline{S}$ is somewhat more involved. From the second of Equations (4) and Table I we have

$$\underline{S} = (\underline{P}_- + \underline{P}_+) \cos (\theta_+ + \theta_-) + j\, (\underline{\mathcal{J}} + \underline{P}_+ \underline{P}_-) \sin (\theta_+ + \theta_-)$$

$$= (\underline{P}_- + \underline{P}_+) [\cos \theta_+ \cos \theta_- - \sin \theta_+ \sin \theta_-]$$

$$+ j(\underline{\mathcal{J}} + \underline{P}_+ \underline{P}_-) [\sin \theta_+ \cos \theta_- + \cos \theta_+ \sin \theta_-]$$

Expanding and rearranging,

$$S = \left\{j\, \underline{\mathcal{J}} \cos \theta_+ \sin \theta_- + \underline{P}_- \cos \theta_+ \cos \theta_- - \underline{P}_+ \sin \theta_+ \sin \theta_- + j\, \underline{P}_+ \underline{P}_- \sin \theta_+ \cos \theta_-\right\}$$

$$+\left\{j\, \underline{\mathcal{J}} \sin \theta_+ \cos \theta_- - \underline{P}_- \sin \theta_+ \sin \theta_-\right.$$

$$\left. + \underline{P}_+ \cos \theta_+ \cos \theta_- + j\, \underline{P}_+ \underline{P}_- \cos \theta_+ \sin \theta_-\right\}$$

$$= \left\{(\underline{\mathcal{J}} \cos \theta_+ + j\, \underline{P}_+ \sin \theta_+)(j \sin \theta_- + \underline{P}_- \cos \theta_-)\right\}$$

$$+\left\{(j\, \underline{\mathcal{J}} \sin \theta_+ + \underline{P}_+ \cos \theta_+)(\underline{\mathcal{J}} \cos \theta_- + j\, \underline{P}_- \sin \theta_-)\right\}$$

$$S = \sin \theta_+ \sin \theta_- \left\{(-j\, \underline{\mathcal{J}} \cot \theta_+ + \underline{P}_+)(\underline{\mathcal{J}} - j\, \underline{P}_- \cot \theta_-)\right.$$

$$\left. +(\underline{\mathcal{J}} - j\, \underline{P}_+ \cot \theta_+)(-j\, \underline{\mathcal{J}} \cot \theta_- + \underline{P}_-)\right\}$$

$$= (b_+ b_-)^{-1}\left\{(a_+ \underline{\mathcal{J}} + \underline{P}_+)(\underline{\mathcal{J}} + a_- \underline{P}_-) + (\underline{\mathcal{J}} + a_+ \underline{P}_+)(a_- \underline{\mathcal{J}} + \underline{P}_-)\right\}$$

$$\underline{S} = (b_+ b_-)^{-1} (\underline{M}_+ \underline{N}_- + \underline{N}_+ \underline{M}_-) \qquad \text{(11-C)}$$

Further manipulation of $\underline{S}$ yields its relation to $\underline{\Lambda}$ [1]. We have

$$\underline{S} = (b_+ b_-)^{-1} \underline{N}_+ (\underline{N}_+^{-1} \underline{M}_+ + \underline{M}_- \underline{N}_-^{-1}) \underline{N}_-$$

$$= (b_+ b_-)^{-1} \underline{N}_+ (\underline{M}_+ \underline{N}_+^{-1} + \underline{M}_- \underline{N}_-^{-1}) \underline{N}_-$$

$$= (b_+ b_-)^{-1} \underline{N}_+ \underline{\Lambda} \underline{N}_- \quad (12\text{-C})$$

conversely,

$$\underline{\Lambda} = b_+ b_- \underline{N}_+^{-1} \underline{S} \underline{N}_-^{-1} \quad (13\text{-C})$$

In Equation (5-C) we need to reduce the quantity

$$\underline{Q} = -b_-^{-1} \underline{M}_- \underline{S}_-^{-1} \underline{K}^e (\underline{V}^e)^{-1} + \underline{\mathcal{J}}$$

$$= \underline{M}_- \underline{S}^{-1} \left[-b_-^{-1} \underline{K}^e (\underline{V}^e)^{-1} + \underline{S} \underline{M}_-^{-1} \right]$$

Using Equations (10-C) and (11-C),

$$\underline{Q} = \underline{M}_- \underline{S}^{-1} \left[-b_-^{-1} (b_+^{-1} \underline{N}_+) + (b_+ b_-)^{-1} (\underline{M}_+ \underline{N}_- + \underline{N}_+ \underline{M}_-) \underline{M}_-^{-1} \right]$$

$$= (b_+ b_-)^{-1} \underline{M}_- \underline{S}^{-1} \left[-\underline{N}_+ + \underline{M}_+ \underline{N}_- \underline{M}_-^{-1} + \underline{N}_+ \right]$$

$$= (b_+ b_-)^{-1} \underline{M}_- \underline{S}^{-1} \underline{M}_+ \underline{N}_- \underline{M}_-^{-1}$$

$$= (b_+ b_-)^{-1} \left[\underline{M}_- \underline{N}_-^{-1} \underline{M}_+^{-1} \underline{S} \underline{M}_-^{-1} \right]^{-1}$$

Using Equation (11-C),

$$\underline{Q} = \left[\underline{M}_- \underline{N}_-^{-1} \underline{M}_+^{-1} (\underline{M}_+ \underline{N}_- + \underline{N}_+ \underline{M}_-) \underline{M}_-^{-1}\right]^{-1}$$

$$= (\underline{\mathcal{J}} + \underline{M}_- \underline{N}_-^{-1} \underline{M}_+^{-1} \underline{N}_+)^{-1}$$

$$= \left[(\underline{N}_+^{-1} \underline{M}_+ + \underline{M}_- \underline{N}_-^{-1})(M_+^{-1} N_+)\right]^{-1}$$

$$= (\underline{\Lambda} \underline{N}_+ \underline{M}_+^{-1})^{-1}$$

$$= \underline{M}_+ (\underline{\Lambda} \underline{N}_+)^{-1} \qquad (14\text{-C})$$

In Equation (7-C) we need to reduce the quantity

$$\underline{X} = \underline{S} \cos \theta_+ + \underline{K}^e (\underline{V}^e)^{-1} \underline{R}(\theta)$$

Using Equations (4) and (12), and Table I, we get

$$\underline{X} = \cos \theta_+ \left[(\underline{P}_- + \underline{P}_+) \cos \theta + j(\underline{\mathcal{J}} + \underline{P}_+ \underline{P}_- \sin \theta)\right]$$

$$- \left[\underline{P}_+ \cos \theta_+ + j \underline{\mathcal{J}} \sin \theta_+\right] \left[\underline{\mathcal{J}} \cos \theta + j \underline{P}_- \sin \theta\right]$$

$$= \underline{P}_- (\cos \theta_+ \cos \theta + \sin \theta_+ \sin \theta) + j \underline{\mathcal{J}} (\cos \theta_+ \sin \theta - \sin \theta_+ \cos \theta)$$

$$= \underline{P}_- \cos (\theta - \theta_+) + j \underline{\mathcal{J}} \sin (\theta - \theta_+)$$

$$= \underline{P}_- \cos \theta_- + j \underline{\mathcal{J}} \sin \theta_-$$

$$\therefore \underline{X} = -b_-^{-1} \underline{N}_- \qquad (15\text{-C})$$

$\therefore$ We can now proceed to further transformation of Equations (1-C) to (8-C). Using Equations (10-C) to (15-C) where appropriate we get

$$\underline{V}(0) = \underline{V}_-^{o} = (b_+ \, b_-)(\underline{N}_+ \, \underline{\Lambda} \, \underline{N}_-)^{-1} \, (b_+^{-1} \, \underline{N}_+ \, \underline{V}^e)$$

$$= b_-(\underline{N}_-^{-1} \, \underline{\Lambda}^{-1} \, \underline{N}_+^{-1})(\underline{N}_+ \, \underline{V}^e)$$

$$= b_- \, \underline{N}_-^{-1} \, \underline{\Lambda}^{-1} \, \underline{V}^e$$

$$= b_- \, (\underline{\Lambda} \, \underline{N}_-)^{-1} \, \underline{V}^e \qquad (16\text{-C})$$

$$\underline{I}(0) = -\underline{I}_-^{o} = \underline{Y} \, \underline{P}_- \, \underline{S}^{-1} \, \underline{K}^e$$

$$= -\underline{Y}_-^{o} \, \underline{V}_-^{o}$$

$$= -b_- \, \underline{Y}_-^{o} \, (\underline{\Lambda} \, \underline{N}_-)^{-1} \, \underline{V}^e \qquad (17\text{-C})$$

$$\underline{V}(\theta_o^-) = \underline{V}_-^{i} = -b_-^{-1} \, \underline{M}_- \left[b_- \, (\underline{\Lambda} \, \underline{N}_-)^{-1} \, \underline{V}^e \right]$$

$$= -\underline{M}_- \, (\underline{\Lambda} \, \underline{N}_-)^{-1} \, \underline{V}^e \qquad (18\text{-C})$$

$$\underline{I}(\theta_o^-) = -\underline{I}_-^{i} = b_-^{-1} \, \underline{Y} \, \underline{N}_- \left[b_- \, (\underline{\Lambda} \, \underline{N}_-)^{-1} \, \underline{V}^e \right]$$

$$= \underline{Y} \, \underline{N}_- \, (\underline{N}_-^{-1} \, \underline{\Lambda}^{-1}) \, \underline{V}^e$$

$$= \underline{Y} \, \underline{\Lambda}^{-1} \, \underline{V}^e$$

$$= (\underline{\Lambda} \, \underline{Z})^{-1} \, \underline{V}^e \qquad (19\text{-C})$$

Using Equation (14-C),

$$\underline{V}(\theta_o^+) = \underline{V}_+^{i} = \underline{M}_+ \, (\underline{\Lambda} \, \underline{N}_+)^{-1} \, \underline{V}^e \qquad (20\text{-C})$$

$$\underline{I}(\theta_o^{\pm}) = \underline{I}_+^i = -\underline{I}_-^i = (\underline{\Lambda}\ \underline{Z})^{-1}\ \underline{V}^e \tag{21-C}$$

by Equation (19-C).

Finally, to transform $\underline{V}(\theta)$ and $\underline{I}(\theta)$ it is desirable to cast $\underline{S}$ and $\underline{R}(\theta)$ into other forms.

Let

$$\left.\begin{aligned} \underline{M}_-^{\ell} &= a\ \underline{\mathcal{J}} + \underline{P}_- \\ \underline{N}_-^{\ell} &= \underline{\mathcal{J}} + a\ \underline{P}_- \end{aligned}\right\} \tag{22-C}$$

Then Equation (9-C) is written

$$\underline{R}(\theta) = -b^{-1}\ \underline{M}_-^{\ell} \tag{23-C}$$

while, from the second of Equations (4) we have

$$\begin{aligned} \underline{S} &= j \sin\theta \left[a(\underline{P}_- + \underline{P}_+) + (\underline{\mathcal{J}} + \underline{P}_+\ \underline{P}_-)\right] \\ &= -b^{-1}\left[(\underline{\mathcal{J}} + a\ \underline{P}_-) + \underline{P}_+(a\ \underline{\mathcal{J}} + \underline{P}_-)\right] \\ &= -b^{-1}\ (\underline{N}_-^{\ell} + \underline{P}_+\ \underline{M}_-^{\ell}) \end{aligned} \tag{24-C}$$

Using Equation (15-C) in (7-C)

$$\begin{aligned} \underline{V}(\theta) = \underline{V}_+^o &= \underline{R}(\theta)\ \underline{S}^{-1}\ (-b_-^{-1})\ \underline{N}_-\ \left[R(\theta)\right]^{-1}\ \underline{V}^e \\ &= -b_-^{-1}\ R(\theta)\ \underline{S}^{-1}\ \underline{N}_-\ \left[R(\theta)\right]^{-1}\ \underline{V}^e \end{aligned}$$

Then substituting Equations (23-C) and (24-C) yields

$$\underline{V}(\theta) = \underline{V}_+^o = -b_-^{-1} (-b^{-1} \underline{M}_-^{\ell})(-b)(\underline{N}_-^{\ell} + \underline{P}_+ \underline{M}^{\ell})^{-1} \underline{N}_-(-b)(\underline{M}_-^{\ell})^{-1} \underline{V}^e$$

$$= b_-^{-1} b \underline{M}_-^{\ell} (\underline{N}_-^{\ell} + \underline{P}_+ \underline{M}_-^{\ell})^{-1} \underline{N}_- (\underline{M}_-^{\ell})^{-1} \underline{V}^e$$

$$= b_-^{-1} b \left[\underline{M}_-^{\ell} \underline{N}_-^{-1} (\underline{N}_-^{\ell} + \underline{P}_+ \underline{M}_-^{\ell}) (\underline{M}_-^{\ell})^{-1}\right]^{-1} \underline{V}^e$$

Now since $\underline{M}_-^{\ell}$ and $\underline{N}_-$ are each linear functions only of $\underline{\mathcal{J}}$ and $\underline{P}_-$, the products of which commute, we have

$$\underline{M}_-^{\ell} \underline{N}_- = \underline{N}_- \underline{M}_-^{\ell}$$

Whence

$$\underline{N}_-^{-1} \underline{M}_-^{\ell} = \underline{M}_-^{\ell} \underline{N}_-^{-1}$$

By a similar argument

$$\underline{M}_-^{\ell} \underline{N}_-^{\ell} = \underline{N}_-^{\ell} \underline{M}_-^{\ell} \qquad (25\text{-C})$$

etc.

Therefore we may write

$$\underline{V}(\theta) = \underline{V}_+^o = b_-^{-1} b \left[\underline{N}_-^{-1} \underline{M}_-^{\ell} (\underline{N}_-^{\ell} + \underline{P}_+ \underline{M}_-^{\ell})(\underline{M}_-^{\ell})^{-1}\right]^{-1} \underline{V}^e$$

$$= b_-^{-1} b \left\{\underline{N}_-^{-1} \left[\underline{M}_-^{\ell} \underline{N}_-^{\ell} (\underline{M}_-^{\ell})^{-1} + \underline{M}_-^{\ell} \underline{P}_+\right]\right\}^{-1} \underline{V}^e$$

$$= b_-^{-1} b \left[\underline{N}_-^{-1} (\underline{N}_-^{\ell} + \underline{M}_-^{\ell} \underline{P}_+)\right]^{-1} \underline{V}^e \qquad (26\text{-C})$$

In virtue of Equation (25-C).

Consider the quantity

$$\underline{A} = \underline{N}^{\ell}_{-} + \underline{M}^{\ell}_{-}\ \underline{P}_{+}$$

$$= (\underline{\mathcal{J}} + a\ \underline{P}_{-}) + (a\ \underline{\mathcal{J}} + \underline{P}_{-})\ \underline{P}_{+}$$

$$= (\underline{\mathcal{J}} + \underline{P}_{-}\ \underline{P}_{+}) + a(\underline{P}_{-} + \underline{P}_{+}) \qquad (27\text{-C})$$

But

$$a = -j \cot\theta = -j \cot(\theta_{+} + \theta_{-})$$

$$= j\ \frac{1 - \cot\theta_{+} \cot\theta_{-}}{\cot\theta_{+} + \cot\theta_{-}} = \frac{1 + a_{+}\ a_{-}}{a_{+} + a_{-}} \qquad (28\text{-C})$$

Substituting (28-C) in (27-C)

$$\underline{A} = (\underline{\mathcal{J}} + \underline{P}_{-}\ \underline{P}_{+}) + \frac{1 + a_{+}\ a_{-}}{a_{+} + a_{-}}\ (\underline{P}_{-} + \underline{P}_{+})$$

$$= \frac{1}{a_{+} + a_{-}} \left\{(a_{+} + a_{-})\ (\underline{\mathcal{J}} + \underline{P}_{-}\ \underline{P}_{+}) + (1 + a_{+}\ a_{-})(\underline{P}_{-} + \underline{P}_{+})\right\}$$

$$= \frac{1}{a_{+} + a_{-}} \left\{(\underline{\mathcal{J}} + a_{-}\ \underline{P}_{-})(a_{+}\ \underline{\mathcal{J}} + \underline{P}_{+}) + (a_{-}\ \underline{\mathcal{J}} + \underline{P}_{-})(\underline{\mathcal{J}} + a_{+}\ \underline{P}_{+})\right\}$$

$$\underline{A} = \frac{1}{a_{+} + a_{-}}\ (\underline{N}_{-}\ \underline{M}_{+} + \underline{M}_{-}\ \underline{N}_{+}) = \underline{N}^{\ell}_{-} + \underline{M}^{\ell}_{-}\ \underline{P}_{+} \qquad (29\text{-C})$$

Substituting (29-C) in (26-C)

$$\underline{V}(\theta) = \underline{V}_+^o = b_-^{-1} b \left[\underline{N}_-^{-1} (a_+ + a_-)^{-1} (\underline{N}_- \underline{M}_+ + \underline{M}_- \underline{N}_+)\right]^{-1} \underline{V}^e$$

$$= b_-^{-1} b(a_+ + a_-) \left[(\underline{M}_+ \underline{N}_+^{-1} + \underline{M}_- \underline{N}_-^{-1})\underline{N}_+\right]^{-1} \underline{V}^e$$

$$= b_-^{-1} b(a_+ + a_-) (\underline{\Lambda}\, \underline{N}_+)^{-1} \underline{V}^e \tag{30-C}$$

The scalar coefficient is easily simplified. We have

$$b_-^{-1} b(a_+ + a_-) = \frac{j \csc \theta(-j \cot \theta_+ - j \cot \theta_-)}{j \csc \theta_-}$$

$$= -j \frac{\csc \theta}{\csc \theta_-} \left(\frac{\cos \theta_+}{\sin \theta_+} + \frac{\cos \theta_-}{\sin \theta_-}\right)$$

$$= -j \frac{\csc \theta}{\csc \theta_-} \left(\frac{\sin \theta_- \cos \theta_+ + \cos \theta_- \sin \theta_+}{\sin \theta_+ \sin \theta_-}\right)$$

$$= -j \frac{\csc \theta}{\sin \theta_+} \sin (\theta_+ + \theta_-)$$

$$= -j \csc \theta_+ = -b_+$$

Thus (30-C) becomes, finally,

$$\underline{V}(\theta) = \underline{V}_+^o = -b_+ (\underline{\Lambda}\, \underline{N}_+)^{-1} \underline{V}^e \tag{31-C}$$

while the current is simply

$$\underline{I}(\theta) = \underline{I}_+^o = -b_+ \underline{Y}_+^o (\underline{\Lambda}\, \underline{N}_+)^{-1} \underline{V}^e \tag{32-C}$$

Collecting the results for convenience, we have

$$\left.\begin{aligned}
\underline{V}(0) &= \underline{V}_{-}^{o} = b_{-}(\underline{\Lambda}\ \underline{N}_{-})^{-1}\ \underline{V}^{e} && \text{(a)}\\
\underline{I}(0) &= -\underline{I}_{-}^{o} = -b\ \underline{Y}_{-}^{o}\ (\underline{\Lambda}\ \underline{N}_{-})^{-1}\ \underline{V}^{e} && \text{(b)}\\
\underline{V}(\theta_{o}^{-}) &= \underline{V}_{-}^{i} = -\underline{M}_{-}\ (\underline{\Lambda}\ \underline{N}_{-})^{-1}\ \underline{V}^{e} && \text{(c)}\\
\underline{I}(\theta_{o}^{-}) &= -\underline{I}_{-}^{i} = (\underline{\Lambda}\ \underline{Z})^{-1}\ \underline{V}^{e} && \text{(d)}\\
\underline{V}(\theta_{o}^{+}) &= \underline{V}_{+}^{i} = \underline{M}_{+}\ (\underline{\Lambda}\ \underline{N}_{+})^{-1}\ \underline{V}^{e} && \text{(e)}\\
\underline{I}(\theta_{o}^{+}) &= \underline{I}_{+}^{i} = -\underline{I}_{-}^{i} = (\underline{\Lambda}\ \underline{Z})^{-1}\ \underline{V}^{e} && \text{(f)}\\
\underline{V}(\theta) &= \underline{V}_{+}^{o} = -b_{+}\ (\underline{\Lambda}\ \underline{N}_{+})^{-1}\ \underline{V}^{e} && \text{(g)}\\
\underline{I}(\theta) &= \underline{I}_{+}^{o} = -b_{+}\ \underline{Y}_{+}^{o}\ (\underline{\Lambda}\ \underline{N}_{+})^{-1}\ \underline{V}^{e} && \text{(h)}
\end{aligned}\right\}\quad (33\text{-C})$$

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