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MISSILE CIRCUMFERENTIAL CURRENT DENSITY FOR PLANE WAVE ELECTROMAGNETIC FIELD ILLUMINATION

Charles W. Harrison, Jr., 2627
Sandia Laboratories, Albuquerque

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ABSTRACT

The asymmetry in current density along the periphery of cylindrical scattering obstacles of both finite and infinite length illuminated by a plane wave electromagnetic field is discussed. In the Appendix the same problem for a spherical scatterer is is considered.

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Introduction

The study of shadowing of plane wave electromagnetic fields by missiles is of importance because RF leakage into a missile through holes and slots in the wall and the RF pickup of cables furrowed into the skin leading to interior circuitry depends on the amplitude and the variation of current density along the circumference of the scattering obstacle. Clearly the azimuthal orientation of the missile with respect to the incident field has a bearing on the RF receiving characteristics of the slots and cables, if indeed electromagnetic field shadowing is significant.[1]

One topic discussed briefly in this memorandum is a theory for shadowing by tubular cylinders of arbitrary dimensions developed by Kao.[2-4] As a practical matter, the coupled integral equations derived by him cannot readily be solved for very long cylinders because of the limited number of storage locations in a computer. To circumvent this, the author presents a theory for shadowing by a cylinder of infinite length. In the entire discussion of both finite and infinite tubes, it is assumed that the electric field is polarized parallel to the axis of the cylinder. This is the case of dominate interest in the field of RF hazards to ordnance. But this specific polarization is an unnecessary restriction. The angle the incident field makes with the axis of the cylinder may be arbitrary. In particular, the magnetic field may be polarized parallel to the conductor.

Discussion of the Shadowing Properties of the Finite Length Hollow Cylinder

Consider a perfectly conducting thin-wall tubular cylinder of complete generality in both length and radius, as portrayed by Figure 1. The axis of the cylinder coincides with the z-axis of a cylindrical coordinate system (r, θ, z). The tube extends over the interval $-h \leq z \leq h$ and is of radius a. The incident electric field E_z^i is directed parallel to the axis of the cylinder and arrives at the angle $\theta = \pi$ radians. This orientation of the electric field, which is termed E polarization, is the one of primary interest in the study of missile shadowing.

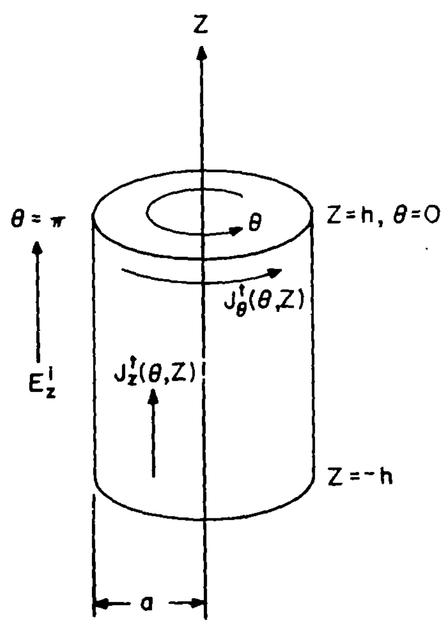

Figure 1.

Finite Tubular Cylinder Illuminated by Electric Field E_z^i. $J_z^t(\theta, z)$ and $J_\theta^t(\theta, z)$ are the coexistent total current densities

For E polarization (as well as for H polarization, i.e., H_z^i parallel to the z-axis), an axial current density $J_z^t(\theta, z)$ and a transverse current density $J_\theta^t(\theta, z)$ are induced. These currents are coupled. The superscript t is used to denote the total current density, i.e., the sum of the currents on the inside and outside surfaces of the tubular cylinder. As will be made plain later for certain values of tube length and electrical radius $k_o a$, where $k_o = 2\pi/\lambda_o$, the inside currents are negligibly small. Under these circumstances the error is not great to consider the cylinder capped at the ends by metal surfaces as is the situation in practical missiles.

Both components of total current density $J_z^t(\theta, z)$ and $J_\theta^t(\theta, z)$ exist simultaneously on a tube of finite length when excited by E_z^i; only the former exists on a tube that is infinitely long. The physical explanation of this fact is not trivial.

Along a tube of finite length or any finite obstacle, standing waves of current and charge obtain. The current and charge are of even and odd symmetry, respectively. Maximum charge concentrations occur at the ends of the tube but there is a distribution of charges all along the tube. The illumination of the tube is asymmetrical, so that different densities of charge exist at different values of θ. These nonuniform concentrations exert forces that cause the charge to flow in the θ direction around the tube, bringing into existence $J_\theta^t(\theta, z)$. The maximum value of this component of total current density is $J_\theta^t(\theta, \pm h)$. The amplitude of $J_\theta^t(\theta, z)$ decreases as one recedes from the ends of the tube toward its center. At that point $J_\theta^t(\theta, 0) \equiv 0$ for reasons of symmetry. Clearly $J_\theta^t(\theta, z)$ is oppositely directed in the regions $0 < z \leq h$ and $-h \leq z < 0$.

For an infinite cylinder E_z^i gives rise to axial progressive or traveling waves of current and charge. There can be no axial standing wave of charge or current because there are no discontinuities (ends). Accordingly, there is no source for the current density $J_\theta^t(\theta, z)$ and this component of current density is not excited with the indicated E_z^i polarization of the incident field. Evidently, the total current exists solely on the outside surface of an infinitely long cylinder.

Circumferential symmetry of the current density cannot exist on the periphery of either a finite or infinite tubular conductor because the excitation of the cylinder provided by an incident plain wave is asymmetrical. However, there would be no shadowing effect if one could arrange illumination in the form of a cylindrical wave converging on the obstacle.

In general, one is interested in comparing $J_z^t(\pi, z)$ with $J_z^t(\theta, z)$ and $J_\theta^t(\pi, z)$ with $J_\theta^t(\theta, z)$. This provides a measure of the shadowing of the missile. Remember that the cylinder is first illuminated in the region $\theta \simeq \pi$; a shadow region may occur at several values of θ on the surface of the same cylinder (depending on the value of $k_o a$) and not at $\theta = 0$ radians. It is very important to note that shadowing for a cylinder of finite length is a function of z; i.e., different values of shadowing obtain for $z \simeq 0$ compared to $z \simeq h$.

Kao[2-6] has done some brilliant work in the field of scattering from a thin wall tubular cylinder of completely arbitrary dimensions (length and radius), for E- and H-polarization. Some of his results for E-polarization are shown in Figures 2 through 4. Curves for the shadowing of an infinite cylinder are also given. Evidently the ratio $|J_z^t(0, z)| / |J_z^t(\pi, z)$ is smaller for infinite cylinder shadowing as compared to finite cylinder shadowing for the range of values of $k_o h$ and $k_o a$ considered by Kao. In this sense infinite cylinder theory bounds the problem for $\theta = \pi, 0$.

Tables of data on shadowing by finite length tubes appear in Reference 3, and Reference 2 contains a program in Fortran 4 for the IBM 360/65 computer permitting further computations.

From the Kao formulation of the problem, it is possible to separate $J_z(\theta, z)$ and $J_\theta(\theta, z)$ into inside and outside currents. For the orientation of the field assumed the longest wavelength of the mode that can exist in the tube corresponds to the TM_{01} mode. Thus, if $k_o a < 2.407$, there is essentially no current in the interior of the tube, and $J_z^t(\theta, z)$ and $J_\theta^t(\theta, z)$ are outside currents. This presupposes that the inequality $h > 5a$ is satisfied.

If the incident electric field is tilted with respect to the axis of the tube or H-polarization is under consideration, TE modes may be excited. For cut-off of the TE_{11} mode, $k_o a < 1.843$.

On Estimating the Degree of Nonuniformity of the Current Density and the Total Electric Field About the Periphery of an Infinite Cylinder for Plane-Wave Illumination

The largest cylinders investigated numerically by Kao[3] have an electrical length $k_o h$ of 1.5π, i.e., $h = 0.75\lambda_o$, and electrical radii of $k_o a = 1.0$, 2.0, and 3.0. The following simple analysis, based on infinite cylinder theory for

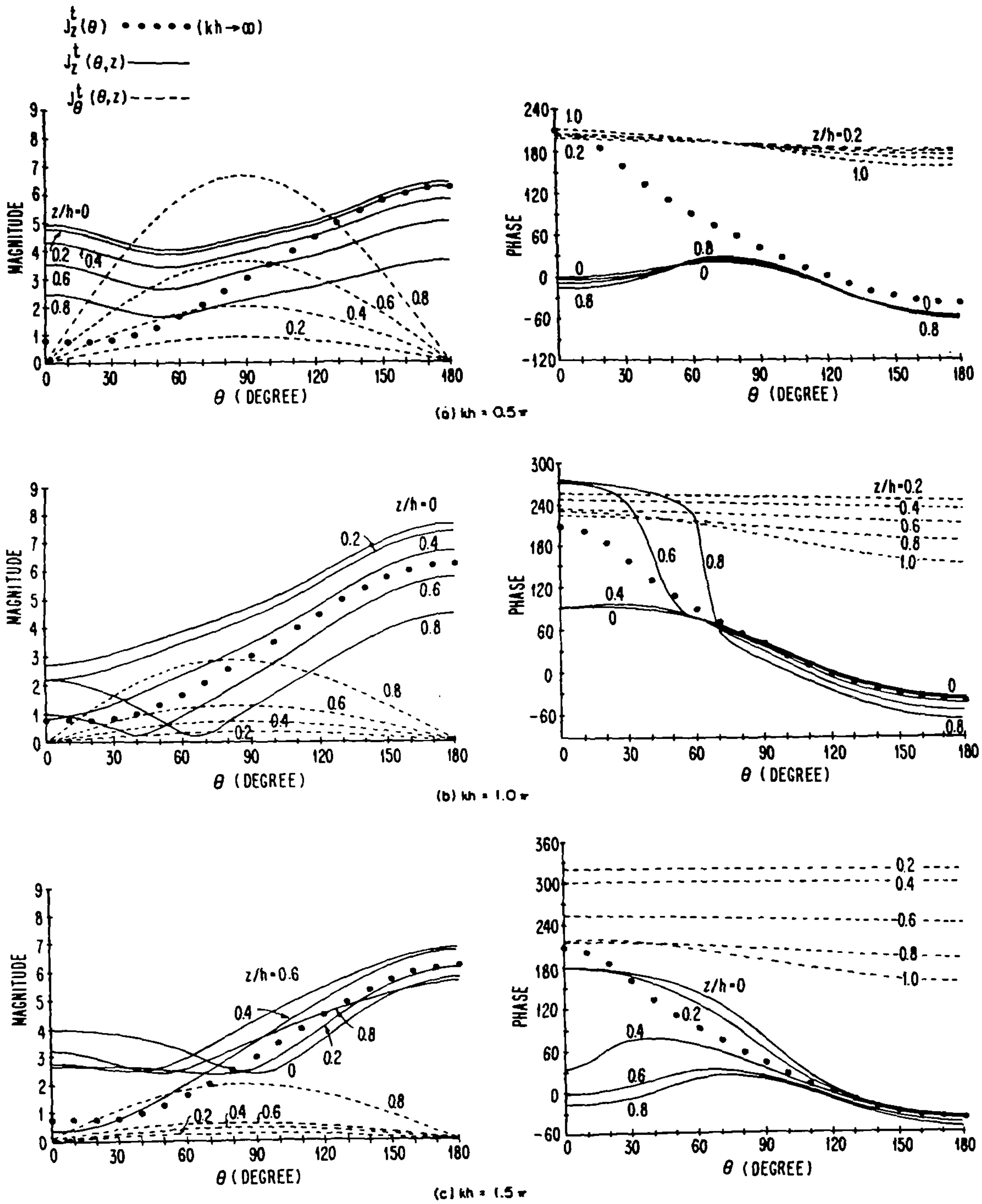

Figure 2. Total Current Density $J_z^t(\theta, z)$ and $J_\theta^t(\theta, z)$ as Functions of θ for Fixed Values of z. The radii of the cylinders are ka = 1.0; the half-lengths are (a) kh = 0.5π, (b) kh = 1.0π, (c) kh = 1.5π. The incident wave is E-polarized. The kh→∞ case is also displayed. (per Kao)

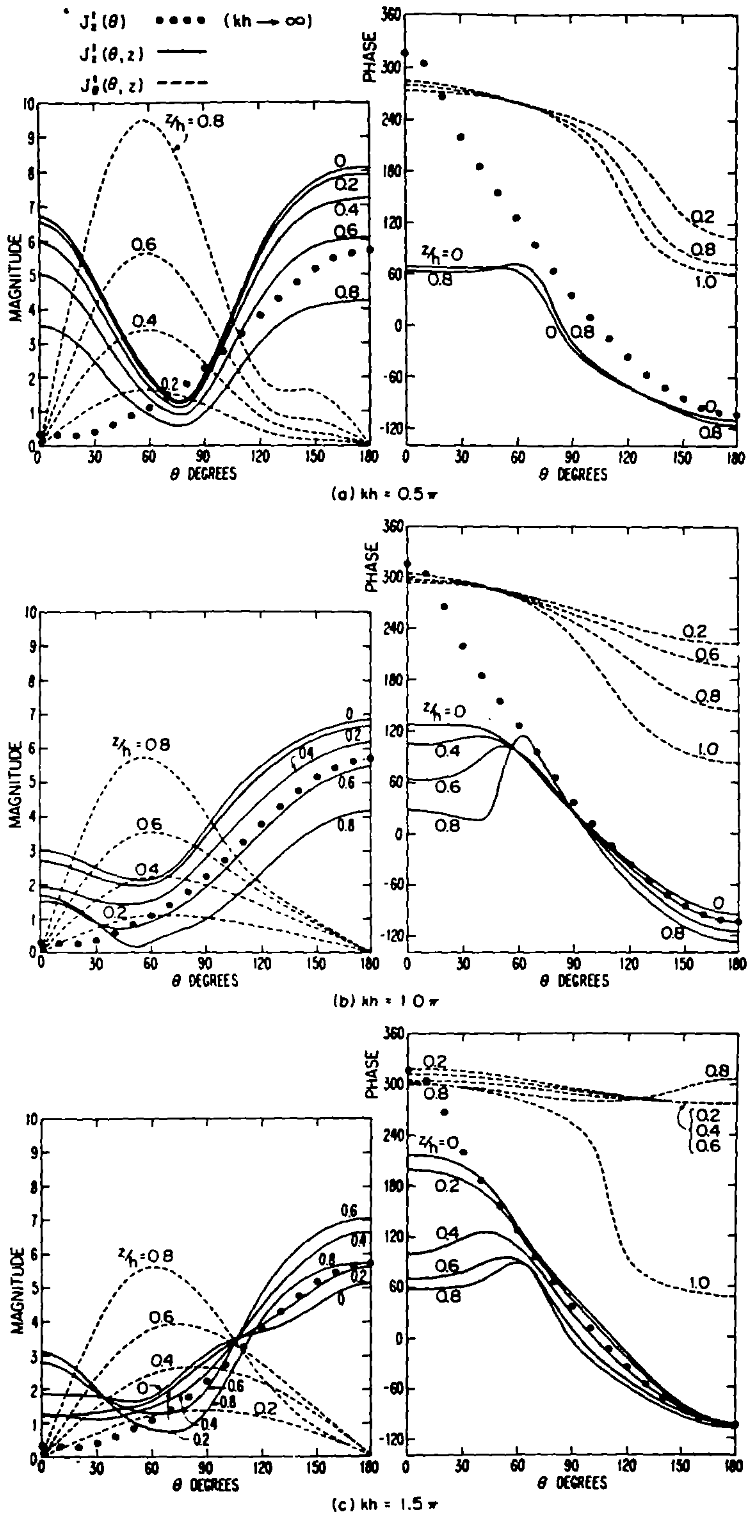

Figure 3.

Total Current Density $J_z^t(\theta, z)$ and $J_\theta^t(\theta, z)$ as Functions of θ for Fixed Values of z. The radii of the cylinders are ka = 2; the half-lengths are (a) kh = 0.5π, (b) kh = 1.0π, (c) kh = 1.5π. The incident field is E-polarized. The kh→∞ case is also displayed. (per Kao)

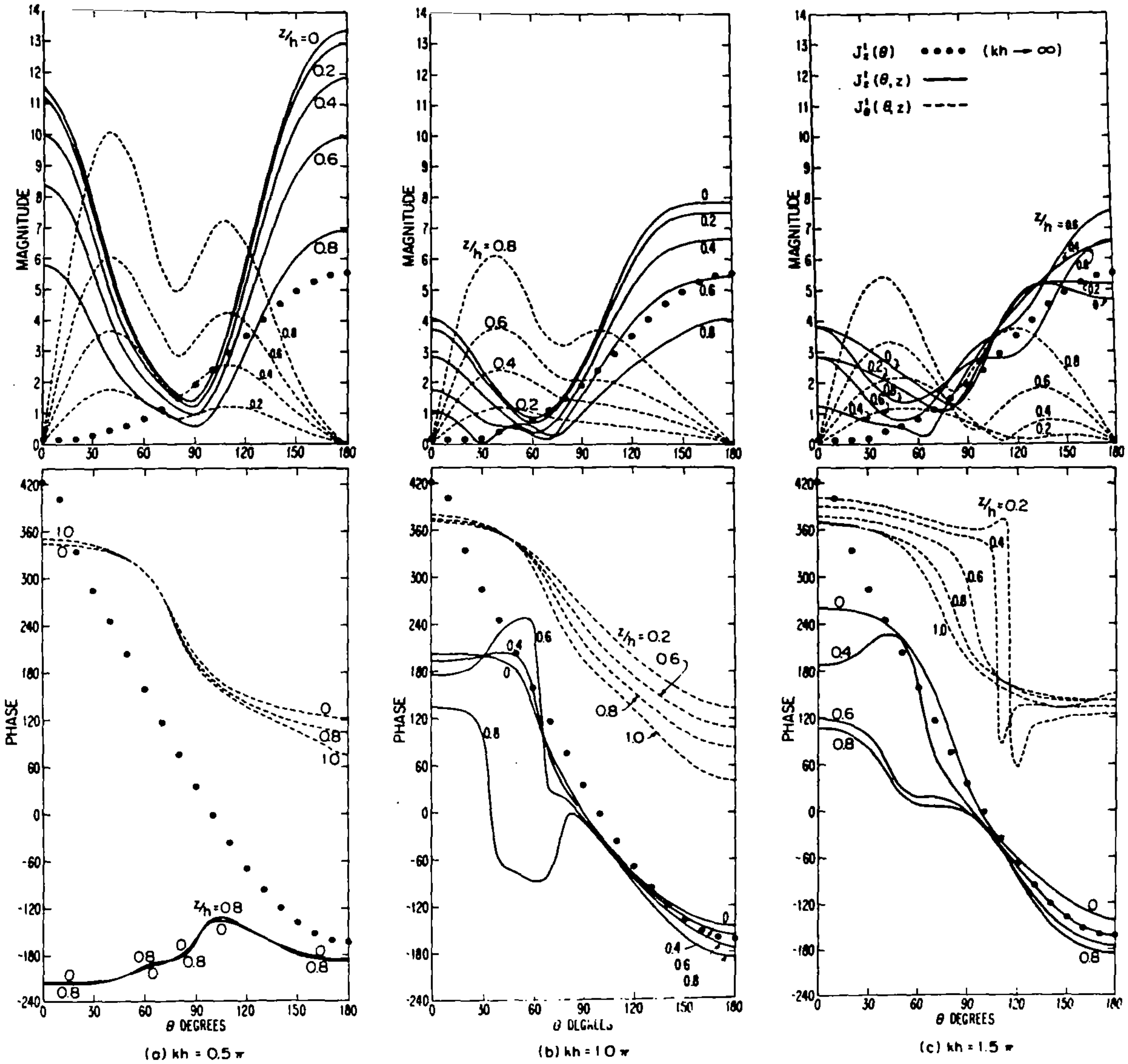

Figure 4. Total Current Density $J_z^t(\theta, z)$ and $J_\theta^t(\theta, z)$ as Functions of θ for Fixed Values of z. The radii of the cylinders are ka = 3; the half-lengths are (a) kh = 0.5π, (b) kh = 1.0π, (c) kh = 1.5π. The incident wave is E-polarized. The kh→∞ case is also displayed. (per Kao)

E-polarization, may be of utilitarian value in estimating shadowing effects for missiles when $k_o h >> 1$ and $h/a >> 1$.

The total field in the z-direction in the vicinity of an infinite cylindrical scatterer illuminated at θ = π by a plane wave is[7]

$$E_z^t(r,\theta) = E_o \sum_{n=-\infty}^{\infty} j^{-n} \frac{\left[J_n(k_o r)H_n^{(2)}(k_o a) - J_n(k_o a)H_n^{(2)}(k_o r)\right]e^{jn\theta}}{H_n^{(2)}(k_o a)} \,. \tag{1}$$

Since interest centers in the case $(k_o r)^2 \ll 1$, i.e., in the low-frequency region of the radio spectrum, it is sufficient to retain only three terms in the above series.

Using the relations

$$\left.\begin{aligned} J_{-n}(z) &= (-1)^n J_n(z) \\ H_{-n}^{(2)}(z) &= (-1)^n H_n^{(2)}(z) \end{aligned}\right\} , \tag{2}$$

Equation (1) may be written

$$E_z^t(r,\theta) = E_o \left\{ \frac{\left[J_o(k_o r)H_o^{(2)}(k_o a) - J_o(k_o a)H_o^{(2)}(k_o r)\right]}{H_o^{(2)}(k_o a)} - j2\, \frac{\left[J_1(k_o r)H_1^{(2)}(k_o a) - J_1(k_o a)H_1^{(2)}(k_o r)\right]}{H_1^{(2)}(k_o a)} \cos\theta \right\} \tag{3}$$

For small arguments $(k_o r)^2 \ll 1$ this expression reduces to

$$E_z^t(r,\theta) = j\, \frac{\frac{2}{\pi} E_o \ln\left(\frac{r}{a}\right)}{1 + j\frac{2}{\pi} \ln\left(\frac{2}{\gamma k_o a}\right)} F(r,\theta) \,, \tag{4}$$

where

$$F(r,\ \theta) = 1 - \frac{k_o r\left[1 - \frac{a^2}{r^2}\right]\left[1 + j\,\frac{2}{\pi}\,\ell n\left(\frac{2}{\gamma k_o a}\right)\right]\cos\theta}{\frac{2}{\pi}\,\ell n\,\frac{r}{a}} \; . \qquad (5)$$

In Equations (4) and (5), $\gamma = 1.781$ so that $\ell n\ \gamma = c = 0.5772\ \ldots$ is Euler's constant. Use was made of the small argument form of the Bessel functions:

$$\left.\begin{aligned} J_0(z) &\simeq 1 \\ J_1(z) &\simeq z/2 \\ H_0^{(2)}(z) &\simeq 1 + j\,\frac{2}{\pi}\,\ell n\left(\frac{2}{\gamma z}\right) \\ H_1^{(2)}(z)' &= \frac{z}{2} + j\,\frac{2}{\pi z} \end{aligned}\right\} . \qquad (6)$$

Note that the function $F(r,\ \theta)$ is the factor by which the principal rotationally symmetrical part of the resultant field near the cylinder must be multiplied in order to obtain the first-order correction for small departures from rotational symmetry.

Numerical Example

$F(\pi)$ and $F(0)$ are to be computed 0.5 cm away from the skin of the missile. Let $f = 8.485$ MHz and $a = 53.377$ cm. It follows that $\lambda_o = 35.357$ m, and $r = 53.877$ cm. Then $k_o r = 9.574 \times 10^{-2}$, $(2/\pi)\ \ell n\ (r/a) = 5.9634 \times 10^{-3}$, $[1 + j(2/\pi)\ \ell n\ (2/\gamma k_o a)] = 1 + j1.5733$, and $(1 - a^2/r^2) = 1.8476 \times 10^{-2}$. (In evaluating $\ell n\ (r/a)$ it was found convenient to use the approximation $\ell n\ (1 + z) \simeq z$.) Substituting these values into (5), $|F(0)| = 0.8441$ and $|F(\pi)| = 1.3781$. Hence the shadowing ratio is $\left|\frac{F(0)}{F(\pi)}\right| = 0.6125$ for the total electric field 0.5 cm away from the surface of the cylinder.

The current density $J_z^t(a, \theta)$ on the cylinder is given by

$$J_z^t(a, \theta) = H_\theta^t(a, \theta) \ . \tag{7}$$

Now,

$$\nabla \times \overline{E} = -j\omega\mu\overline{H} \ , \tag{8}$$

so that

$$J_z(a, \theta) = \frac{1}{j\omega\mu} \left. \frac{\partial E_z^t(r, \theta)}{\partial r} \right|_{r=a} \ . \tag{9}$$

Applying (9) to (1) it is found that

$$J_z^t(a, \theta) = \frac{2E_o}{\pi\zeta_o k_o a} \sum_{n=-\infty}^{\infty} \frac{j^{-n} e^{jn\theta}}{H_n^{(2)}(k_o a)} \ . \tag{10}$$

In deriving (10) the Wronskian relation

$$J_n(z)Y_n'(z) - Y_n(z)J_n'(z) = 2/\pi z \ , \tag{11}$$

was employed. Note that this formula is valid for all z; in particular for $z = k_o a$. Observe, too, that $\frac{d}{dx} J_n(u) = J_n'(u) \frac{du}{dx}$ and $\frac{d}{dx} Y_n(u) = Y_n'(u) \frac{du}{dx}$. The substitution $\omega\mu = k_o \zeta_o$ was also made in obtaining (10). Again, for $(k_o a)^2 \ll 1$, (10) may be written

$$J_z^t(a, \theta) = \frac{2E_o}{\pi\zeta_o k_o a} \left[\frac{1}{H_o^{(2)}(k_o a)} - j \frac{2 \cos \theta}{H_1^{(2)}(k_o a)} \right] \tag{12}$$

Using (6) in (12) yields

$$J_z^t(a, \theta) = \frac{2E_o}{\pi\zeta_o k_o a\left[1 + j\,\frac{2}{\pi}\,\ell n\left(\frac{2}{\gamma k_o a}\right)\right]}\,G(a, \theta) \,, \tag{13}$$

where

$$G(a, \theta) = 1 - \pi k_o a\left[1 + j\,\frac{2}{\pi}\,\ell n\left(\frac{2}{\gamma k_o a}\right)\right]\cos\theta \,. \tag{14}$$

Numerical Example

For the practical situation previously described one need only determine $\pi k_o a$ to complete the problem. This factor has the value $\pi k_o a = 0.2980$. It follows that $|G(0)| = 0.8441$ and $|G(\pi)| = 1.3781$. Hence $\left|\frac{G(0)}{G(\pi)}\right| = 0.6125$.

On comparing the total field and current density shadowing ratios, it is observed that they are the same. However, as r becomes larger, less perturbation of E_z^i by the presence of the scattering obstacle should be expected, i.e., $E_z^t \to E_z^i$. Shadowing probably is best defined in terms of the current density on the periphery of the missile, because it is this current that flows into holes and slots that may exist in the missile skin.

Conclusion and Desiderata

Kao has developed a theory for determining the current density inside and outside of a tubular cylinder of arbitrary dimensions for both E- and H-polarization and has written a computer program for studying such scattering obstacles numerically. It has been pointed out that if $k_o a < 1.843$ the tube is below cut-off for both TE and TM modes, and the total current density is essentially the current density on the outside surface of the tube. In such cases, the tube may be considered as capped at the ends. The Kao program fails for very long tubes because of computer storage capacity. When $k_o h >> 1$ and $h/a >> 1$, the method proposed by the author for investigating shadowing, based on infinite cylinder theory, will yield the upper bound for shadowing in finite length structures ($\theta = \pi$, 0). Accordingly, it may be a satisfactory approximation to base all missile shadowing studies on (14) if the frequency is sufficiently low. This simple formula is amenable to slide-rule evaluation for obtaining the current density on the surface of the cylinder at various azimuth angles θ.

Evidently no information on the value of $\left| J_z^t(0, z) / J_z^t(\pi, z) \right|$ with changes in z is available nor are any data on the values of $\left| J_\theta^t(0, z) / J_\theta^t(\pi, z) \right|$. The latter currents do not exist on infinite structures for E-polarization but could be obtained for H-polarization with no z-dependence.

It appears that, if the Kao computer program[2] is rewritten for use on the computer available, missile shadowing can be investigated completely. But since interest centers on missiles having radii small compared to the wavelength, it may be that the problem of shadowing of an incident plane wave electromagnetic field by a missile simply fades away.

APPENDIX

SHADOWING BY PERFECTLY CONDUCTING SPHERES

Perhaps the prolate spheroid more nearly conforms to the shape of a missile than any other body of finite dimensions. The spheroid has the advantage over the tubular cylinder of finite length in that its ends are closed. Taylor[8] has obtained exactly the total axial current distribution along a prolate spheroidal antenna. This work is of great importance in the field of radio frequency hazards to ordnance inasmuch as RF leakage into a missile containing an access door is related to the total surface current.

Unfortunately, it appears a formidable undertaking to determine the current densities on the surface of a spheroid, but this can be accomplished for the specialized spheroid -- a sphere. The author now undertakes to investigate shadowing by spheres for plane wave illumination for the case $k_o a << 1$. This inequality implies a sphere of small radius, or long wavelength of the incident field.

The current densities on a perfectly conducting sphere for a plane wave propagating in the z-direction with electric field parallel to the x-axis (Figure 5) are[9]

$$J_\theta = \frac{jE_o \cos \phi}{\zeta_o k_o a} \sum_{n=1}^{\infty} \frac{j^{-n}(2n+1)}{n(n+1)} \left[\frac{\sin \theta \, P_n^{I'}(\cos \theta)}{\left[k_o a h_n^{(2)}(k_o a)\right]'} + j \frac{P_n^{I}(\cos \theta)}{\sin \theta \, k_o a h_n^{(2)}(k_o a)} \right] \qquad (A1)$$

$$J_\phi = \frac{jE_o \sin \phi}{\zeta_o k_o a} \sum_{n=1}^{\infty} \frac{j^{-n}(2n+1)}{n(n+1)} \left[\frac{P_n^{I}(\cos \theta)}{\sin \theta \left[k_o a h_n^{(2)}(k_o a)\right]'} + \frac{j \sin \theta \, P_n^{I'}(\cos \theta)}{k_o a h_n^{(2)}(k_o a)} \right] \qquad (A2)$$

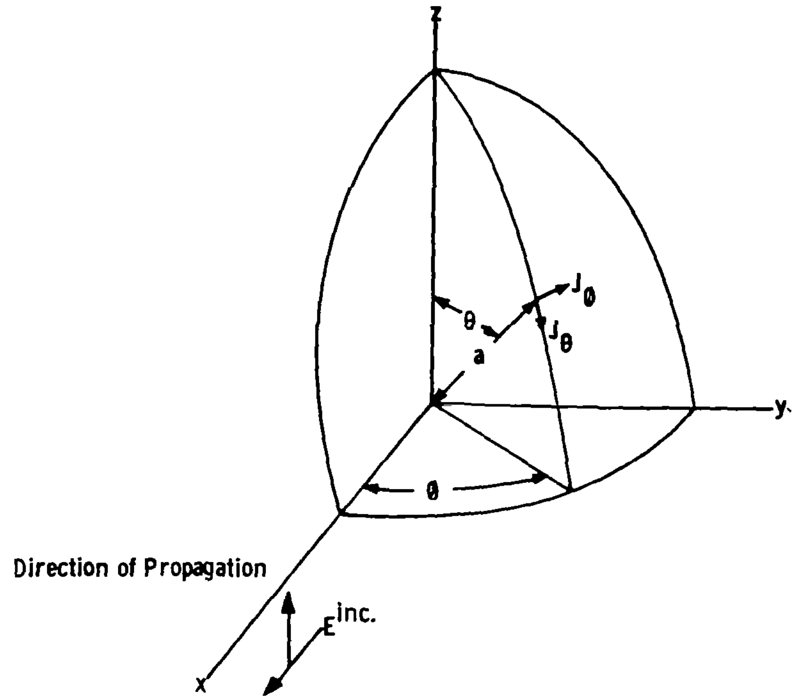

Figure 5. Geometry of the Problem for Determining the Current Densities J_θ and J_ϕ Generated by Plane-Wave Illumination of a Perfectly Conducting Sphere

In these expressions, which are exact for any value of $k_o a$, E_o is the amplitude of the incident field. The objective is to determine approximately J_θ and J_ϕ for n = 1. The result obtained should be useful when $k_o a << 1$.

To evaluate (A1) and (A2) for n an arbitrary positive integer, the following relations are needed:

$$P_o^I(x) = 0 \qquad \text{(A3)}$$

$$P_I^I(x) = -\left(1 - x^2\right)^{1/2} \qquad \text{(A4)}$$

$$P_{n+1}^I(x) = \frac{1}{n}\left[x(2n + 1)P_n^I(x) - (n + 1)P_{n-1}^I(x)\right] \qquad \text{(A5)}$$

$$P_n^{I'}(x) = \frac{1}{1 - x^2}\left[-nxP_n^{(1)}(x) + (n + 1)P_{n-1}^I(x)\right] \qquad \text{(A6)}$$

$$h_n^{(2)}(z) = j^{n+1} z^{-1} e^{-jz} \sum_{k=o}^{k=n} \frac{(n + k)!}{k!\Gamma(n - k + 1)}(j2z)^{-k} \tag{A7}$$

$$h_n^{(2)'}(z) = h_{n-1}^{(2)}(z) - \left(\frac{n + 1}{z}\right)h_n^{(2)}(z) \tag{A8}$$

For example, since $P_o^I(x)$ and $P_I^I(x)$ are known, $P_2^I(x)$ may be found from (A5), and $P_2^{I'}(x)$ from (A6). From (A7) it is determined that

$$h_o^{(2)}(z) = je^{-jz}/z \tag{A9}$$

and

$$h_1^{(2)}(z) = -\frac{1}{z}\left(1 - \frac{j}{z}\right)e^{-jz} \tag{A10}$$

With (A9) and (A10) it may prove convenient to employ

$$h_{n-1}^{(2)}(z) + h_{n+1}^{(2)}(z) = (2n + 1)z^{-1}h_n(z) \tag{A11}$$

to generate the spherical Hankel functions of order n.

In the present instance $x = \cos\theta$ and $z = k_o a$. It is found using (A3) to (A8) that

$$\left.\begin{aligned}
&P_1^I(\cos\theta) = -\sin\theta \\
&P_1^{I'}(\cos\theta) = \cot\theta \\
&h_1^{(2)}(k_o a) = \left(-\frac{1}{k_o a} + \frac{j}{(k_o a)^2}\right)e^{-k_o a} \\
&\left[k_o a h_1^{(2)}(k_o a)\right]' = je^{-jk_o a} + \left(\frac{1}{k_o a} - \frac{j}{\left(k_o a\right)^2}\right)e^{-jk_o a}
\end{aligned}\right\} \tag{A12}$$

Also,

$$\frac{j^{-n}(2n + 1)}{n(n + 1)} = \frac{3}{j2}\Bigg|_{n=1} \qquad \text{(A13)}$$

Since $k_o a << 1$ the only terms that count are the ones involving $(k_o a)^2$ in the denominator. Also, one sets $e^{-jk_o a} = 1$. With these substitutions, (A1) and (A2) become

$$J_\theta \simeq - \frac{3}{2\zeta_o} E_o \cos \phi \qquad \text{(A14)}$$

$$J_\phi \simeq - \frac{3}{2\zeta_o} E_o \sin \phi (jk_o a - \cos \theta)$$

$$\simeq - \frac{3}{2\zeta_o} E_o \sin \phi \cos \theta \qquad \text{(A15)}$$

provided $k_o a << 1$.

Note that $J_\theta = 0$ when $\phi = \pi/2$ as it should. Observe also that J_ϕ is analogous to $J_z(\theta, z)$ for the cylinder. Both expressions are essentially independent of $k_o a$ when $k_o a << 1$.

Evidently, it is no marvel that two components of current density exist along cylinders of finite length and also on the surface of spheres.

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