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SCATTERING OF TRANSIENT ELECTROMAGNETIC WAVES FROM CONDUCTING SURFACES IN A TIME VARYING PLASMA: LINEAR APPROXIMATION

By Richard L. Monroe, U.S. Army Mobility Equipment Research and Development Center, Fort Belvoir, Virginia

If a perfectly conducting body is immersed in a cold, lossy, electron plasma containing transient sources of current and ionization, the electromagnetic fields ($\bar{E}$, $\bar{H}$), velocity field $\bar{V}$ and electron density n satisfy the inhomogeneous Maxwell-Euler equations[1]:

$$\nabla \times \bar{H} = -\mu_o\left(\frac{\delta \bar{H}}{\delta t}\right) \tag{1}$$

$$\nabla \times \bar{E} = \epsilon_o\left(\frac{\delta \bar{E}}{\delta t}\right) - en\bar{V} + \bar{J}(\bar{r}, t) \tag{2}$$

$$m\frac{\delta \bar{V}}{\delta t} + \bar{V} \cdot \nabla \bar{V} = -e[\bar{E} + \mu_o \bar{V} \times \bar{H}] - m\nu\bar{V} \tag{3}$$

$$\frac{\delta n}{\delta t} + \nabla \cdot (n\bar{V}) = S(n, \bar{r}, t) \tag{4}$$

In these equations, $\bar{J}$ is the current density consisting of the sum of the induced current $\bar{J}_s$ on the surface of the body and the primary currents $\bar{J}_i$ within the plasma; S is the electron density source function which gives the net rate of production of electrons due to ionization, attachment, recombination and any other processes which may be present; and ν is the electron-neutral particle collision frequency. As usual, the quantities μ_o and ϵ_o are the permeability and permittivity of free space, respectively; and e and m are the charge and mass of the electron. In the general case, equations (1) - (4) are highly nonlinear and, thus, a simple decomposition of the total fields into the sum of incident and scattered components, as employed in classical scattering theory, is not possible. However, if we assume first that the velocity and magnetic fields are such that the nonlinear terms $\bar{V} \cdot \nabla \bar{V}$ and $\bar{V} \times \bar{H}$ can be neglected and second that the electron transport term $\nabla \cdot (n\overline{V})$ in (4) is negligible compared to the source term S, then equations (1) - (4) reduce to

$$\nabla \times \overline{E} = -\mu_o \left(\frac{\delta \overline{H}}{\delta t}\right) \tag{5}$$

$$\nabla \times \overline{H} = \epsilon_o \left(\frac{\delta \overline{E}}{\delta t}\right) - en\overline{V} + \overline{J} \tag{6}$$

$$m\left(\frac{\delta \overline{V}}{\delta t}\right) = -e\overline{E} - m\nu\overline{V} \tag{7}$$

$$\frac{\delta n}{\delta t} = S \tag{8}$$

In this approximation, the electron density variation is completely specified by equation (8) which has been decoupled from the remaining Maxwell-Euler equations. When (8) is solved for n, equations (5), (6), and (7) reduce to a set of nine linear equations for the nine vector quantities $(\overline{E}, \overline{H}, \overline{V})$. Since these equations are linear, we can employ the usual decomposition of vector fields into the sum of incident and scattered components as follows:

$$\overline{E} = \overline{E}_i + \overline{E}_s \tag{9}$$

$$\overline{H} = \overline{H}_i + \overline{H}_s \tag{10}$$

$$\overline{V} = \overline{V}_i + \overline{V}_s \tag{11}$$

Substituting (9), (10), and (11) into (5), (6), and (7) and separating components, we obtain independent sets of equations for the incident fields.

$$\nabla \times \overline{E}_i = -\mu_o \left(\frac{\delta \overline{H}_i}{\delta t}\right) \tag{12}$$

$$\nabla \times \overline{H}_i = \epsilon_o \left(\frac{\delta \overline{E}_i}{\delta t}\right) - en\overline{V}_i + \overline{J}_i \tag{13}$$

$$m\left(\frac{\delta \overline{V}_i}{\delta t}\right) = -e\overline{E}_i - m\nu\overline{V}_i \tag{14}$$

and scattered fields

$$\nabla \times \overline{E}_s = -\mu_o \left(\frac{\delta \overline{H}_s}{\delta t}\right) \tag{15}$$

$$\nabla \times \overline{H}_s = \epsilon_o \left(\frac{\delta \overline{E}_s}{\delta t}\right) - en\overline{V}_s + \overline{J}_s \tag{16}$$

$$m\left(\frac{\delta \overline{V}_s}{\delta t}\right) = -e\overline{E}_s - m\nu\overline{V}_s \tag{17}$$

where we have used the fact that

$$\overline{J} = \overline{J}_i + \overline{J}_s \tag{18}$$

to associate the primary current $\overline{J}_i$ with the incident fields and the induced currents on the surface of the body $\overline{J}_s$ with the scattered fields. Conversely, we can add equations (12-14) and (15-17) using (9-11) to obtain equations (5-6) for the total fields. Thus, if equations (12-14) are solved for the incident fields and (15-17) are solved for the scattered fields then the total fields will satisfy the Maxwell-Euler equations as given by (5-7). In the following, we will assume that equation (8) has been solved for the electron density and that equations (12-14) have been solved for the incident fields. We will attempt to obtain solutions to equations (15-17) in terms of $\overline{J}_s$. Since the two sets of equations are formally identical, it is clear that any method of solution applicable to one set is also applicable to the other.

Solving equation (17) for $\overline{V}_s$ in terms of $\overline{E}_s$, we obtain

$$\overline{V}_s(\bar{r},t) = -\frac{e}{m}\int_0^t e^{-\nu(t-\lambda)}\,\overline{E}_s(\bar{r},\lambda)\,d\lambda \tag{19}$$

where we have assumed $\overline{V}_s(\bar{r},o) = 0$. Eliminating $\overline{V}_s$ from (16) using (19), we obtain the set

$$\nabla\times\overline{E}_s = -\mu_o\left(\frac{\delta\overline{H}_s}{\delta t}\right) \tag{20}$$

$$\nabla\times\overline{H}_s = \epsilon_o\left(\frac{\delta\overline{E}_s}{\delta t}\right) + n\left(\frac{e^2}{m}\right)\int_0^t e^{-\nu(t-\lambda)}\,\overline{E}_s(\bar{r},\lambda)d\lambda + \overline{J}_s \tag{21}$$

We can simplify (20) and (21) by introducing a vector potential $\overline{A}$ and scalar potential ϕ satisfying the Lorentz condition

$$\nabla\cdot\overline{A} + \epsilon_o\frac{\delta\phi}{\delta t} = 0. \tag{22}$$

In the usual way, we let

$$\overline{H}_s = \nabla\times\overline{A} \tag{23}$$

$$\bar{E}_s = -\mu_o \frac{\delta \bar{A}}{\delta t} - \nabla \phi \tag{24}$$

and introduce (23) and (24) into (20) and (21). With the aid of (22), we obtain

$$\left(\frac{1}{c^2}\right) \frac{\delta^2 \phi}{\delta t^2} - \nabla^2 \phi = \frac{\rho_s}{\epsilon_o} \tag{25}$$

$$\left(\frac{1}{c^2}\right) \frac{\delta^2 \bar{A}}{\delta t^2} - \nabla^2 \bar{A} + \left(\frac{e^2 \mu_o}{m}\right) n \int_o^t e^{-\nu(t-\lambda)} \left(\frac{\delta \bar{A}}{\delta \lambda}\right) d\lambda$$

$$= \left(\frac{e^2}{m}\right) n \int_o^t e^{-\nu(t-\lambda)} \nabla\phi\,(\bar{r}, \lambda) d\lambda + \bar{J}_s \tag{26}$$

where ρ_s, the charge density on the surface of the conductor, is related to $\bar{J}_s$ through the continuity equation

$$\nabla \cdot \bar{J}_s = -\frac{\delta \rho_s}{\delta t} \tag{27}$$

Equations (25) and (26) together with (8) constitute a set of five equations with five potential functions (n, ϕ, $\bar{A}$) as unknowns. If solutions to these equations can be obtained, the scattered electromagnetic and velocity fields can be determined from (23), (24) and (19). Examining these equations, we find that equation (25) for the scalar potential is identical to the corresponding equation for a current source in free space. The exact solution to (25) is well-known.

$$\phi = \int_V \frac{\rho_s\left(t - \frac{R}{c}\right)}{4\pi\epsilon_o R} dV \tag{28}$$

With n determined from (8) and ϕ given by (28), $\bar{A}$ can be determined by solving equation (26). The presence of integral operators in (26) immediately suggests perturbation techniques as possible methods of solution: If the initial electron density is zero, then, for small values of t, equation (27) reduces to the standard Helmholtz equation for the vector potential

$$\frac{1}{c^2} \frac{\delta^2 \bar{A}_o}{\delta t^2} - \nabla^2 \bar{A}_o = \bar{J}_s \tag{29}$$

since the remaining terms involve products of small quantities in the neighborhood of $t = 0$. Hence, we can write a solution to (26) as follows

$$\bar{A} = \bar{A}_o + \bar{A}_1 + \bar{A}_2 + \ldots \tag{30}$$

where the zeroth order term

$$\bar{A}_o = \int_V \frac{\bar{J}_s(t - \frac{R}{c})}{4\pi R}\, dV \tag{31}$$

is the solution to (29) and the higher order terms are solutions to the recursive equation

$$\frac{1}{c^2}\frac{\delta^2 \bar{A}i}{\delta t^2} - \nabla^2 \bar{A}i = -\left(\frac{e^2 \mu_o}{m}\right) n \int_o^t e^{-\nu(t-\lambda)} \left(\frac{\delta \bar{A}i\text{-}1}{\delta\lambda}\right) d\lambda$$

$$- \left(\frac{e^2}{m}\right) n \int_o^t e^{-\nu(t-\lambda)} \nabla\phi d\lambda \;, \quad i=1,2,3 \ldots \tag{32}$$

Since the terms on the right side of (32) are equivalent to known current sources at each step, this equation is identical in form to (29) and, in principle, can always be solved for $\bar{A}_1$, $\bar{A}_2$, $\bar{A}_3$ . . . in succession.

Following the steps indicated by (29-32), we can obtain a representation of the vector potential in the form

$$\bar{A} = B[\bar{J}_s(\tau)] \tag{33}$$

where $B[\cdot]$ is a linear operator which depends on the retarded time τ as well as the spatial variables. In general, B will consist of an infinite sum of operators involving iterates, of the zeroth order operator

$$B_o = \int_V dv \, \frac{[\cdot]}{4\pi R} \tag{34}$$

In practice, due to the rapidly increasing complexity of these operators, it will be necessary to truncate this sum after two or three terms. A two-term approximation to B can be written explicitly as follows:

$$B = B_o[\cdot] + B_1[\cdot] \tag{35}$$

where B_o is given by (34) and B_1 is given by the following:

$$B_1[\cdot] = -\frac{\mu_o}{m}\left(\frac{e}{4\pi}\right)^2 \int_V \frac{dVn(t-\frac{R}{c})}{R} \int_{V'} \frac{dV'}{R'} \int_o^{t-\frac{R'}{c}} d\lambda e^{-\nu(t-\frac{R'}{c}-\lambda)} \frac{\delta[\cdot]}{\delta\lambda}$$

$$+\frac{1}{\epsilon_o}\left(\frac{e}{4\pi}\right)^2 \int_V \frac{dVn(t-\frac{R}{c})}{R} \nabla \int_{V'} \frac{dV'}{R'} \int_o^{t-\frac{R'}{c}} d\lambda\, e^{-\nu(t-\frac{R'}{c}-\lambda)} \int^{\lambda} d\lambda' \nabla\cdot[\cdot] \tag{36}$$

Using (23) and (24) with A given by (33) and ϕ given by (28), we obtain the following expressions for the scattered electromagnetic fields in terms of the induced current density

$$\bar{H}_s = \nabla \times B[\bar{J}_s(\tau)] \tag{37}$$

$$\bar{E}_s = -\mu_o \frac{\delta B[\bar{J}_s(\tau)]}{\delta\tau} + \nabla \int_V dV \int_o^{\tau} d\tau' \frac{\nabla\cdot \bar{J}_s(\tau')}{4\pi\epsilon_o R} \tag{38}$$

where $\tau = t - \frac{R}{c}$ is the retarded time. In the final term of (38), we have used the continuity equation for the current (27) to eliminate ρ_s. If $\bar{J}_s(\tau)$ is known, the scattered fields can be determined from (37) and (38); thus, the scattering problem reduces to the problem of determining the induced current density on the surface of the conductor.

With the aid of (37) and (38), we can obtain two equivalent integro-differential equations for $\bar{J}_s$ by applying the following boundary conditions on the surface of the conductor:[2]

$$\hat{n} \times \bar{E} = \hat{n} \times (\bar{E}_i + \bar{E}_s) = 0 \tag{39}$$

$$\bar{J}_s = \hat{n} \times \bar{H} = \hat{n}(\bar{H}_i + \bar{H}_s) \tag{40}$$

($\hat{n}$ = unit vector normal to the surface)

where the incident fields $\bar{E}_i$ and $\bar{H}_i$ are assumed known. Equation (39) together with (38) leads to

$$\hat{n} \times \bar{E}_i = \mu_o \frac{\delta\, \hat{n} \times B[\bar{J}_s(\tau)]}{\delta\tau} + \hat{n} \times \nabla \int_V dV \int_o^{\tau} d\tau' \frac{\nabla\cdot \bar{J}_s(\tau')}{4\pi R} \tag{41}$$

while equation (40) with (39) leads to

$$\bar{J}_s(\tau) = 2\hat{n}\times \bar{H}_i(\tau) + 2\hat{n}\times \nabla \times B[\bar{J}_s(\tau)] \tag{42}$$

where it is understood that all quantities are to be evaluated on the conductor. Equations (41) and (42) are two time dependent integro-differential equations for the current density. When one or the other of these equations is solved for $\bar{J}_s$, the scattering problem is essentially solved. Bennett[3, 4] has recently developed a direct time domain method of solving integro-differential equations of the same general type as (42). There appears to be no reason why his method could not be applied to the present problem. This approach is currently being explored for the specific case of a perfectly conducting body in the form of a circular cylinder of finite length.

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