Interaction Notes

Note 46

January 1970

Electromagnetic Field Penetration of Wire Screens

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ABSTRACT

The study of thin wire antenna systems, particularly large wire arrays or meshes, requires the solution of difficult integral equations. Since it is not feasible to think of a solution in an analytical sense, it is necessary to develop another approach to the problem. Hence, a numerical formulation is developed where, with the use of any of the many standard algorithms, a solution may be obtained. This formulation is applied to two classes of wire structures. The numerical results are presented and compared with results of other investigators.

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CHAPTER I

INTRODUCTION

In general the study of antennas of even the simplest form, involves the solution of integral equations. Consequently, most treatments avoid rigorous analytical approaches to the problem. For example, Jordan and Balmain[1] assume for expedience that the current distribution on a dipole antenna of length 2H is of the form

$$I = I_m \sin k(H - Z) \quad , \qquad (1.1.1)$$

when

$$Z > 0 \quad ;$$

and

$$I = I_m \sin k(H + Z) \quad , \qquad (1.1.2)$$

when

$$Z < 0 \quad .$$

Admittedly this assumption greatly simplifies further investigation and is justified in many cases. However, in rigorous examination of the problem the current distribution must be considered as an unknown and determined either analytically or numerically.

It is the purpose of this thesis, therefore, to develop general integral equations for the currents in an arbitrary thin wire structure; and having developed these equations, a numerical method is presented by which the exact form of the current distribution, although discrete, may be determined.

The aforementioned integral equations are developed in Chapter II for an arbitrary configuration. Chapters III and IV present numerical techniques for solution of the integral equations for the infinite parallel wire array and the infinite wire mesh, respectively.

Chapter V presents a technique for determining the fields around a wire structure, which takes advantage of the discrete form in which the currents are calculated.

CHAPTER II

DERIVATION OF INTEGRAL EQUATIONS FOR CURRENTS IN ARBITRARY THIN WIRE CONFIGURATIONS

It is accepted that once the current distribution in a wire structure is known, the subsequent determination of the fields around that wire is relatively simple. Consequently, in a problem concerning a wire structure, priority is inevitably given to the determination of the current distribution. The problem must be attacked in a somewhat converse manner whereby the fields around the wires are expressed in terms of the unknown current distributions. These expressions for the fields take the form of integrals of the currents; and from these expressions, integral equations of the currents are derived. Once obtained, the integral equations are forced to be satisfied at a finite set of points, thereby generating a system of algebraic equations which are solved numerically using a digital computer.

This method of formulating the problem using integral equations of the currents was originated by Pocklington,[2] who introduced two important concepts which are fundamental to present day thin wire theory. As given by Jones[3] they are (1) the integral equation of the current can be reduced to a one dimensional equation since only the axial currents are significant, and (2) the incident electric field intensity parallel to the wire axis makes the only important contribution to the current.

2.1 Integral Equations for Currents in Open-ended Structures.

Integral equations for the currents in a single wire oriented arbitrarily were developed by Mei.[4] This treatment was further expanded by Taylor[5] to include any number of wires oriented arbitrarily with the possibility of any number of them intersecting. The subsequent discussion is adopted from their presentations, and it extends the limits of their theory to include wire structures that are not open-ended.

In a system of N wires, it has been found[5] that the tangential vector potential at the surface of the n^{th} wire is

$$A_{Sn}(S_n) = \frac{\mu}{4\pi} \sum_{m=1}^{N} \int_{\ell_m} d\hat{S}'_m (\hat{S}'_m \cdot \hat{S}'_n) I_m(S'_m) G_m(S_n, S'_m) \quad , (2.1.1)$$

where

$$G_m(S_n,S'_m) = \exp\left[-jK\sqrt{R^2(S_n,S'_m) + \delta_{mn} a_m^2}\right] / \sqrt{R^2(S_n,S'_m) + \delta_{mn} a_m^2} \quad ,$$

$I_m(S_m')$ is the total axial current, at point S_m' on the m^{th} wire,

a_m is the radius of the m^{th} wire,

$\hat{S}_m$ is the unit vector tangential to the m^{th} wire at point S_m' ,

ℓ_m is the arc length of the m^{th} wire, and

$R(S_n,S_m')$ is the linear distance shown in Figure 2.1 .

The scalar potential can be written in terms of the current as

$$\phi_n(S_n) = j \frac{\eta}{4\pi k}\sum_{m=1}^{N} \int_{\ell_m} dS_m' \frac{d}{dS_m'} I_m(S_m') G_m(S_n,S_m') \quad , \qquad (2.1.2)$$

where η is the free space impedance. It is required that the tangential component of the electric field at the surface of the N^{th} wire be set equal to zero. As a result

$$E_{Sn}(S_n) + E_{Sn}^{i}(S_n) = 0 \quad , \qquad (2.1.3)$$

where $E_{Sn}(S_n)$ is the tangential component of the scattered electric field; and $E_{Sn}^{i}(S_n)$ is the tangential component of the incident electric field.

Assuming harmonic time dependence, a useful relationship is

$$E_{Sn}(S_n) = -\nabla_{Sn}\phi_n(S\ ) - j\omega A_{Sn}(S_n) \quad , \qquad (2.1.4)$$

or

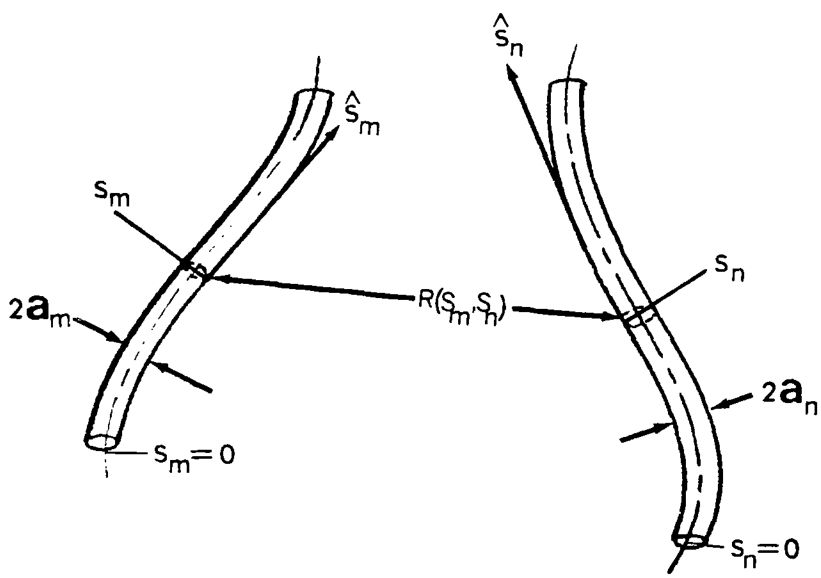

Figure 2.1 Two arbitrarily oriented wires

$$- E^{i}_{Sn}(S_n) = - \nabla_{Sn}\Phi_n(S_n) - j\omega A_{Sn}(S_n) \quad . \qquad (2.1.5)$$

Upon substituting equation (2.1.1) and (2.1.2) into equation (2.1.5), a system of integro-differential equations is obtained for the current induced in the wires by the incident field. Since this system of equations is quite awkward to deal with, it is necessary to consider another approach. Hence, a function is defined by equation (2.1.6) whereby this system of equations is reduced to a system of integral equations. Thus

$$\Phi_n(S_n) = - j \frac{k^2}{\omega} \int_0^{S_n} dS_n \phi_n(S_n) \quad . \qquad (2.1.6)$$

By using equation (2.1.6) in (2.1.5) and by adding $k^2\Phi_n(S_n)$ to both sides, a differential equation is subsequently obtained.

$$\left[\frac{d^2}{dS_n^2} + k^2\right] \Phi_n(S_n)$$

$$= k^2\left[\Phi_n(S_n) - A_{Sn}(S_n)\right] - j \frac{k^2}{\omega} E^{i}_{Sn}(S_n) \quad . \qquad (2.1.7)$$

A formal solution of equation (2.1.7) yields

$$\Phi_n(S_n) = C_n\cos kS_n + D_n\sin kS_n$$

$$+ k\int_0^{S_n} dS_n' \left[\Phi_n(S_n') - A_{Sn}(S_n')\right] \sin k(S_n - S_n')$$

$$- j \frac{k}{\omega} \int_0^{S_n} dS_n' \, E^{i}_{Sn}(S_n') \sin k (S_n - S_n') \quad . \qquad (2.1.8)$$

Development of an integral equation of the induced currents from equation (2.1.8) requires expressions for $\Phi_n(S_n)$ and $A_{Sn}(S_n')$ as functions of the currents. To that end, equation (2.1.2) is used in (2.1.6) to arrive at an expression for $\Phi_n(S_n)$. Thus

$$\Phi_n(S_n) = j\frac{k}{\omega}\frac{\eta}{4\pi k}\int_0^{S_n}\sum_{m=1}^{N}\int_{\ell_m}\frac{d}{dS_m'} I_m(S_m')G_m(S_n',S_m')\,dS_n'dS_m' \,, \quad (2.1.9a)$$

and

$$\Phi_n(S_n) = j\frac{\mu}{4\pi}\int_0^{S_n}\sum_{m=1}^{N}\int_{\ell_m}\frac{d}{dS_m'} I_m(S_m')G_m(S_n',S_m')\,dS_n'dS_m' \,. \quad (2.1.9b)$$

Integrating 2.1.9b by parts yields

$$\Phi_n(S_n) = \frac{\mu}{4\pi}\sum_{m=1}^{N}\int_0^{S_n}\Big[I_m(\ell_m)G_m(S_n',\ell_m) - I_m(0)G_m(S_n',0)$$

$$- \int_{\ell_m} I_m(S_m')\,\frac{\partial G_m(S_n',S_m')}{\partial S_m'}\,dS_m'\Big]\,dS_n' \,. \quad (2.1.10)$$

Since this section is concerned with open-ended wire structures, the boundary conditions given by equations (2.1.11a) and (2.1.11b) may be used. Hence

$$I_m(\ell_m) = 0 \,, \quad (2.1.11a)$$

$$I_m(0) = 0 \,. \quad (2.1.11b)$$

In the subsequent section, the case is considered where equations (2.1.11a) and (2.1.11b) are no longer valid. Applying equations (2.1.11a) and (2.1.11b) to equation (2.1.10) yields

$$\Phi_n(S_n) = \frac{\mu}{4\pi} \sum_{m=1}^{N} \int_0^{S_n} \int_{\ell_m} I_m(S_m') \frac{\partial G_m(S_n', S_m')}{\partial S_m'} dS_m' \, dS_n' \quad . \tag{2.1.12}$$

The required expression for the vector magnetic potential is given explicitly by equation (2.1.1). Consequently, by using equations (2.1.12) and (2.1.1) in (2.1.8), and integral equation of the current will be obtained.*

Consider the first term in the first integral of equation (2.1.8). Designating this integral as $F(S_n)$, yields

$$F(S_n) = k \int_0^{S_n} dS_n' \, \Phi_n(S_n') \sin k\,(S_n - S_n') \quad . \tag{2.1.13}$$

Using equation (2.1.12) in (2.1.13) produces

$$F(S_n) = -\frac{\mu k}{4\pi} \sum \int_0^{S_n} dS_n' \int_0^{S_n} d\xi \int_{\ell_m} dS_m' \, I_m(S_m') \frac{\partial G_m(\xi, S_m')}{\partial S_m'}$$

$$\cdot \sin k(S_n - S_n') \quad . \tag{2.1.14}$$

Changing the order of integration of equation (2.1.14) and

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*The procedure for effecting the development of an integral equation of the current was outlined by Mei[4] in his single wire treatment.

adjusting the limits so that the range of integration remains the same, gives

$$F(S_n) = -\frac{\mu k}{4}\int_{\ell_m} dS_m' \int_0^{S_n} d \int_{\xi}^{S_n} dS_n' \; I_m(S_m') \frac{\partial G_m(\xi_n, S_m')}{\partial S_m'} \cdot \sin k(S_n - S_n') \quad . \tag{2.1.15}$$

Equation (2.1.15) is integrated by parts. The result is

$$F(S_n) = \Phi_n(S_n) + \frac{\mu}{4\pi}\sum_{m=1}^{N}\int_{\ell_m} dS_m' \int_0^{S_n} d\xi \; I_m(S_m') \frac{\partial G_m(\xi, S_m')}{\partial S_m} \cdot \Big(1 - \cos k(S_n - \xi)\Big) \quad . \tag{2.1.16}$$

Use of equation (2.1.12) in the foregoing equation gives

$$F(S_n) = \Phi_n(S_n) + \frac{\mu}{4\pi}\sum_{m=1}^{N}\int_{\ell_m} dS_m' \int_0^{S_n} d\xi \; I_m(S_m') \frac{\partial G_m(\xi, S_m')}{\partial S_m'} \cdot \Big[\cos k(S_n - \xi)\Big] . \tag{2.1.17}$$

Next, the second term in the first integral of equation (2.1.8) is considered and is designated as $H(S_n)$. Hence,

$$H(S_n) = -k \int_0^{S_n} dS_n' \; A_{Sn}(S_n') \sin k\,(S_n - S_n') \quad . \tag{2.1.18}$$

Substitution of equation (2.1.1) into equation (2.1.18), yields

$$H(S_n) = -\frac{\mu k}{4\pi}\sum_{m=1}^{N}\int_0^{S_n}\int_{\ell_m} dS_n' \; I_m(S_m') \; G_m(S_n', S_m')(\hat{S}_n' \cdot \hat{S}_m') \cdot \sin k(S_n - S_n') \; dS_m' \quad . \tag{2.1.19}$$

Integrating the foregoing equation by parts gives

$$H(S_n) = -\frac{\mu k}{4\pi}\sum_{m=1}^{N}\Bigg\{\int_{\ell_m} I_m(S_m')\; G_m(S_n,S_m')\hat{S}_n' \cdot \hat{S}_m'\; dS_m'$$

$$-\int_{\ell_m} I_m(S_m')\; G_m(0,S_m')(\hat{0}\cdot\hat{S}_m')\;\cos(k\,S_n)\,dS_m'$$

$$-\int_{\ell_m}\int_{0}^{S_n} I_m(S_m')\left[\frac{\partial G_m(S_n',S_m')}{\partial S_n'}\;\hat{S}_n'\cdot\hat{S}_m' + G_m(S_n',S_m')\frac{\partial(\hat{S}_m'\cdot\hat{S}_n')}{\partial S_n'}\right]$$

$$\cdot\; \cos k\,(S_n - S_n')\; dS_n'\; dS_m'\Bigg\} \quad . \qquad (2.1.20)$$

Finally using equations (2.1.17) and(2.1.20) in equation (2.1.9) produces an integral equation of the induced current. Thus

$$\sum_{m=1}^{\infty}\int_{\ell_m} I_m(S_m')\,\pi(S_n,S_m')\; dS_m' = C_n\,\cos kS_n + D_n\,\sin kS_n$$

$$-\; i\,\frac{4\pi}{\eta}\int_0^{S_n} dS_n'\; E_{S_n'}(S_n')\;\sin k(S_n - S_n') \quad , \qquad (2.1.21)$$

where

$$\pi(S_n,S_m') = G_m(S_n,S_m')\;\hat{S}_n'\cdot\hat{S}_m' - \int_0^{S_m} dS_n'\;\Bigg[\frac{\partial G_m(S_n',S_m')}{\partial S_m'}$$

$$+\frac{\partial G_m(S_n',S_m')}{\partial S_n'}\;\hat{S}_n'\cdot\hat{S}_m' + G_m(S_n',S_m')\frac{\partial(\hat{S}_n'\cdot\hat{S}_m')}{\partial S_m'}$$

$$\cdot\;\cos k(S_n - S_n') \quad . \qquad (2.1.22)$$

2.2 Integral Equations for Currents in Wire Structures that are Not Open-ended.

The foregoing discussion and derivation is sufficient in solving problems where open-ended structures are involved. Such structures may include a crossed wire configuration or an infinite array of finite length wires like the structure that will be studied subsequently.

However, for problems where no open-ended conditions exist the aforementioned derivation may not be sufficiently inclusive. Hence, in studying a problem such as the infinite wire mesh presented in Chapter IV, account must be taken of the fact that the conditions given in equations (2.1.11a) and (2.1.11b) are no longer valid. Consequently, in developing the integral equations for the current, the more general expression for $\Phi_n(S_n)$, equation (2.1.10), must be used. Again, the first integral of equation (2.1.8) is considered; and this term is again designated $F(S_n)$. Hence,

$$F(S_n) = k \int_0^{S_n} dS_n' \; \Phi_n(S_n') \sin k\,(S_n - S_n') \; . \quad (2.2.1)$$

Using equation (2.1.11) in equation (2.2.1) yields

$$\begin{aligned} F(S_n) = & \sum_{m=1}^{N} \frac{\mu k}{4\pi} \int_0^{S_n} dS_n' \int_0^{S_n'} d\xi \Big[I_m(\ell_m)\; G_m(\xi,\ell_m) \\ & - I_m(0)\; G_m(\xi,0) \Big] \cdot \sin k(S_n - S_n') \\ & - \sum_{m=1}^{N} \frac{\mu k}{4\pi} \int_0^{S_n} dS_n' \int_0^{S_n'} d\xi \int_{\ell_m} dS_m' \quad I_m(S_m') \; \frac{\partial G(\xi, S_m')}{\partial S_m'} \\ & \cdot \; \sin k(S_n - S_n') \quad . \end{aligned} \quad (2.2.2)$$

Upon examining equation (2.2.2), it can be seen that the second term on the right hand side is the same as equation (2.1.14). Consequently, only the first term of this expression needs to be considered; and to that end the first term is denoted as $F_1(S_n)$. Therefore

$$F_1(S_n) = \sum_{m=1}^{N} \frac{\mu k}{4\pi} \int_0^{S_n} dS_n' \int_0^{S_n'} d\xi \Big[I_m(\ell_m)\ G_m(\xi, \ell_m)$$
$$- I_m(0)\ G_m(\xi, 0) \Big] \sin k(S_n - S_n') \quad . \qquad (2.2.3)$$

Changing the order of integration of equation (2.2.3) yields

$$F_1(S_n) = \frac{\mu k}{4\pi} \sum_{m=1}^{N} \int_0^{S_n} d\xi \int_\xi^{S_n} dS_n' \Big[I_m(\ell_m)\ G_m(\xi, \ell_m)$$
$$- I_m(0)\ G_m(\xi, 0) \Big] \sin k(S_n - S_n') \quad . \qquad (2.2.4)$$

After integrating the foregoing equation by parts, the result is

$$F_1(S_n) = \frac{\mu}{4\pi} \sum_{m=1}^{N} \int_0^{S_n} d\xi \Big[I_m(\ell_m)\ G_m(\xi, \ell_m) - I_m(0)\ G_m(\xi, 0) \Big]$$
$$- \frac{\mu}{4\pi} \sum_{m=1}^{N} \int_0^{S_n} d\xi \Big[I_m(\ell_m)\ G_m(\xi, \ell_m) - I_m(0)\ G_m(\xi, 0) \Big]$$
$$\cdot \cos k(S_n - \xi) \quad . \qquad (2.2.5)$$

By using equation (2.1.10) in equation (2.2.5) a useful expression is obtained. Hence

$$F_1(S_n) = \Phi_n(S_n) + \sum_{m=1}^{N} \frac{\mu}{4\pi} \int_0^{S_n} \int_{\ell m} I_m(S_m') \frac{\partial G_m(S_n', S_m')}{\partial S_m'} dS_m' dS_n'$$
$$- \frac{\mu}{4\pi} \sum_{m=1}^{N} \int_0^{S_n} d\xi \Big[I_m(\ell_m) G_m(\xi, \ell_m) - I_m(0)\ G_m(\xi, 0) \Big]$$
$$\cdot \cos k(S_n - \xi) \quad . \qquad (2.2.6)$$

At this point it is appropriate to recombine equations 2.2.6) and (2.1.16) which give

$$F(S_n) = \Phi_n(S_n) - \frac{\mu}{4\pi} \sum_{m=1}^{N} \int_0^{S_n} dS_n' \left[I_m(\ell_m)\; G_m(S_n',\ell_m) - I_m(0)\; G_m(S_n',0)\right] \cdot \cos k(S_n - S_n')$$
$$+ \frac{\mu}{4\pi} \sum_{m=1}^{N} \int_{\ell_m} dS_m' \int_0^{S_n} dS_n'\; I_m(S_m')\; \frac{\partial G_m(S_n',S_m')}{\partial S_m'} \cdot \cos k(S_n - S_n') \quad . \qquad (2.2.7)$$

To obtain the final form of the integral equation of the current, equations (2.2.7) and (2.1.20) are used in equation (2.1.8) to produce

$$\int_{\ell_m} dS_n'\; I_n(S_n')\; F(S_m,S_n') = C_m \cos k\, S_m + D_m \sin k\, S_m - j\, \frac{4\pi}{\eta} \int_0^{S_m} dS_m'\; E_{S_n}' (S_m') \sin k(S_m - S_m') \quad , \qquad (2.2.8)$$

where

$$F(S_m,S_n') = G_n(S_m,S_m')\; \hat{S}_m \cdot \hat{S}_n' - \int_0^{S_m} dS_m' \left[\frac{\partial G_m(S_m',S_n')}{\partial S_m'} + \frac{\partial G_n(S_m',S_n')}{\partial S_n'}\; \hat{S}_m' \cdot \hat{S}_n' + G_n(S_m',S_n')\; \frac{\partial(\hat{S}_m' \cdot \hat{S}_n')}{\partial S_n'} \right] \cdot \cos k(S_m - S_m') \quad . \qquad (2.2.9)$$

Here it should be noted that as a consequence of all the boundary condition terms being absorbed by the constants C_m and D_m, the final integral equation is identical to that of the foregoing section. Hence, equation (2.1.21) may be used in both cases considered in Chapters III and IV.

CHAPTER III

CURRENT DISTRIBUTION IN WIRES OF AN INFINITE PARALLEL ARRAY

In the analysis of the interaction of wire structures and electromagnetic fields, it is necessary to know the currents induced in the wires of the structures.

One particular structure that is often encountered in theory is an infinite array of parallel finite length wires. The investigation of the electrical characteristics of this structure is considered in this chapter. Since this structure can be included in the general category described in Chapter II, the equations that were developed there can be used in acquiring a solution.

3.1 Integral Equation for a Parallel Wire Array.

In determining the induced current distribution Equation (2.1.21) is used. By requiring equation (2.1.21) to conform to the particular geometry and boundary conditions of the structure in Figure 3.1, the integral equation for a parallel wire array is obtained.

Before introducing equation (2.1.21), several relationships between the arbitrary configuration in Figure 1.1 and the particular configuration (in Figure 3.1)

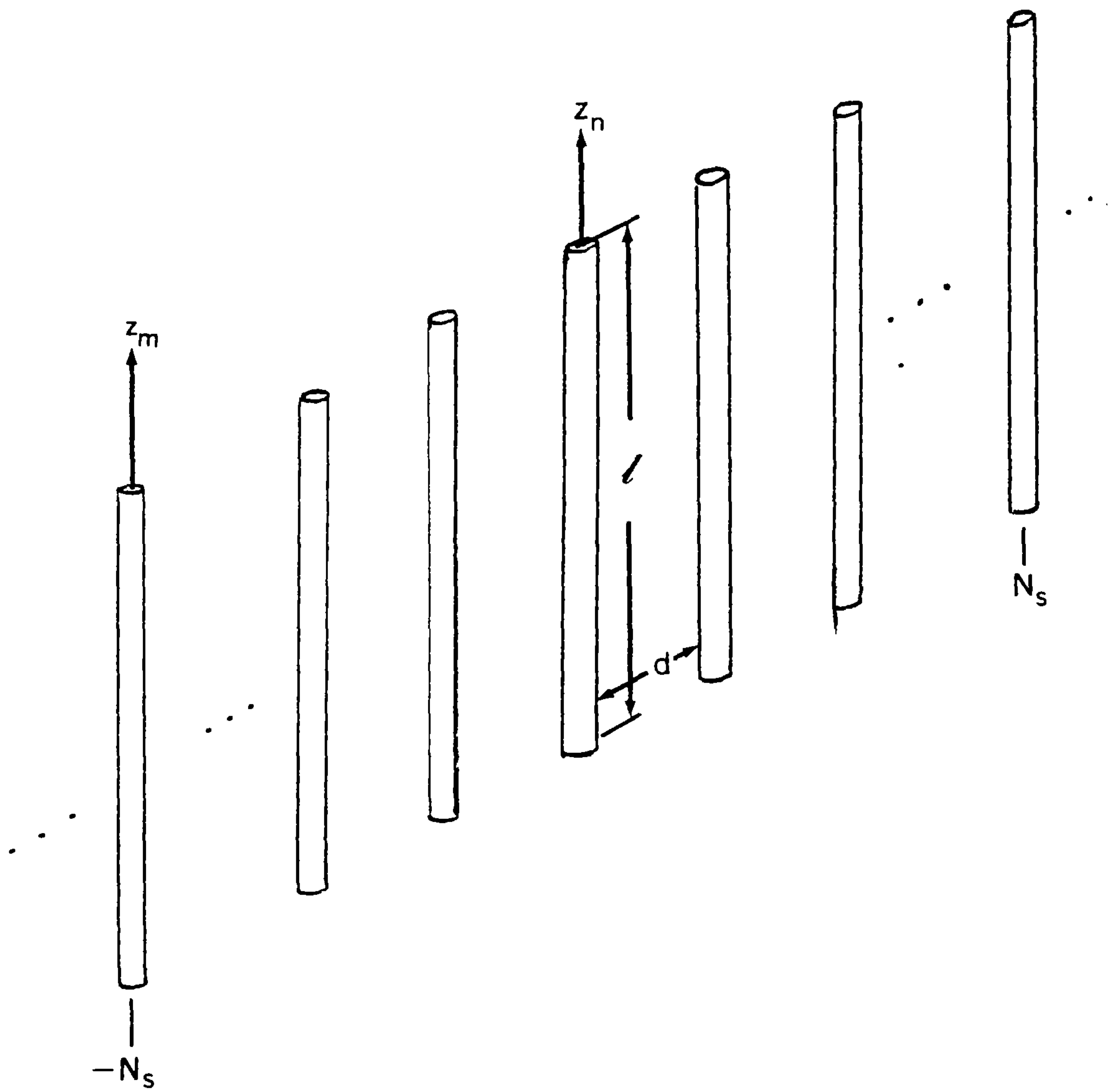

Figure 3.1 Infinite parallel wire array

should be established. First, in considering the respective coordinate systems, it can be seen that

$$\hat{s}_n = \hat{z}_n \quad , \qquad (3.1.1a)$$

$$\hat{s}_m = \hat{z}_m \quad , \qquad (3.1.1b)$$

and

$$\hat{z}_n \cdot \hat{z}_m = 1 \quad . \qquad (3.1.1c)$$

It is evident that from equation (3.1.1c) the unit vectors tangential to the m^{th} and n^{th} wires at any point z'_n or z'_m are equivalent. This fact is also obvious from Figure 3.1.

By using the foregoing relations in equation (2.1.21), integral equations of the current in an infinite parallel array are obtained. Thus,

$$\sum_{n=\infty}^{\infty} \int_0^{\ell} dz'_n I_n(z'_n) \Pi_{mn}(z_m, z'_n) = C_m \cos k_o z_m + D_m \sin k_o z_m - j \frac{4\pi}{\eta} \int_0^{z_m} dz'_m E^i_{sn}(z'_m) \sin k_o (z_m - z'_m) \quad , \qquad (3.1.2)$$

where

$$\Pi_{mn}(z_m, z'_n) = G_{mn}(z_m, z'_n) - \int_0^{z_m} dz'_m \left[\frac{\partial}{\partial z_m} G_{mn}(z'_m, z'_n) + \frac{\partial}{\partial z'_n} G_{mn}(z'_m, z'_n) \right] \cos k_o (z_m - z'_m) \quad . (3.1.3)$$

In straight wire orthogonal systems it is true that

$$\frac{\partial}{\partial z'_n} G_m(z'_m, z'_n) = -\frac{\partial}{\partial z'_n} G_{mn}(z_m, z'_n) \quad . \qquad (3.1.4)$$

Consequently,

$$\Pi_{mn}(z_m, z'_n) = G_{mn}(z_m, z'_n) \quad . \qquad (3.1.5)$$

Further, from equation (3.1.1c) and Figure 3.1, it is evident that

$$\hat{z}_m = \hat{z}_n = \hat{z} \quad . \qquad (3.1.6)$$

Thus the integral equations (3.1.2) may be written in the following manner;

$$\sum_{n=-\infty}^{\infty} \int_0^{\ell} dz' \, I_n(z') G_{mn}(z, z') = C_n \cos k_o z + D_n \sin k_o z$$

$$- j \frac{4\pi}{\eta} \int_0^{z} dz' \, E'_{sn}(z') \sin k_o (z - z') \quad , \qquad (3.1.7)$$

where

$$G_{mn}(z, z') = \exp\left[- j k R_{mn}\right] / R_{mn} \qquad (3.1.8)$$

and

$$R_{mn} = \mathrm{SQRT}\left[(n-m)^2 d^2 + (z-z')^2 + \delta_{mn} a^2\right] \quad . \qquad (3.1.9)$$

Considering the infinite extent and symmetry of the array it is obvious that

$$I_o(z') = I_n(z') \quad , \qquad (3.1.10)$$

where $-\infty \leq n \leq \infty$

Therefore,

$$\int_o^{\ell} dz' \, I_o(z') \sum_{n=-\infty}^{\infty} G_{no}(z,z') = C_o \cos k_o z$$
$$+ D_o \sin k_o z - j \frac{4\pi}{\eta} \int_o^z dz' \, E^i_{sn}(z') \sin k_o(z-z') \; . \quad (3.1.11)$$

Symmetry conditions further require that

$$G_{no} = - G_{-no} \quad \text{as} \quad -\infty \leq n \leq \infty \qquad (3.1.12)$$

Thus,

$$\int_o^{\ell} dz' \, I(z') \sum_{n=0}^{\infty} \varepsilon_n G_n(z,z') = C \cos kz + D \sin kz$$
$$- j \frac{4\pi}{\eta} \int_o^z dz' \, E^i_{sn}(z') \sin k(z - z') \quad , \qquad (3.1.13)$$

where

$$\varepsilon_n = \begin{cases} 1 \; , & n = 0 \\ 2 \; , & n \neq 0 \end{cases} \; .$$

In this analysis $E^i_{so}(z')$ will be considered to be a plane wave, hence

$$E_z(z') = E_o \quad , \qquad (3.1.14)$$

and

$$- j \frac{4\pi}{\eta} \int_o^z dz' \, E_o \sin k_o(z - z') = - j \frac{4\pi}{\eta} \frac{E_o}{k_o} \; . \quad (3.1.15)$$

With equation (3.1.15), the final form of the integral equation is

$$\int_0^{\ell} dz' \; I(z') \sum_{n=0}^{\infty} \epsilon_n \; G_n(z,z') = C \cos k_o z$$

$$+ D \sin k_o z \frac{4\pi F_o}{\eta \; k_o} \quad . \qquad (3.1.16)$$

It should be noted that as a consequence of the symmetry and assumed infinite extent of the array, the system of integral equations in (3.1.7) has been reduced to a single integral equation, (3.1.16).

3.2 Numerical Solution of the Integral Equation.

Since the integral equation for the wire array has now been developed, it is important to consider a technique whereby a solution can be obtained. The particular numerical method to be utilized here is the flat-zoning technique outlined by Aronson and Taylor.[6] The attractiveness of this procedure is a result of its ability to transform an otherwise difficult integral equation into a system of linear algebraic equations whose attendant solution may be obtained using any of the many standard algorithms.

Following through with this technique, it is appropriate to define I(z) as a piecewise constant function. Hence,

$$I(z) = \sum_{p=1}^{M} g_p x(z_p, z_{p+1}) \quad , \qquad (3.2.1)$$

where

$$X(z_p, z_{p+1}) = \begin{cases} 1 \quad , & z_p \leq z \leq z_{p+1} \\ 0 \quad , & \text{otherwise} \quad . \end{cases}$$

Equation (3.2.1) is introduced into the integral equation (3.1.16) with the result that

$$\sum_{p=1}^{M} g_p \int_{(p-1)\Delta z}^{p\Delta z} dz' \sum_{n=0}^{\infty} \varepsilon_n \, G_n(z\ , z') = \cos kz_m$$
$$+ D \sin kz_m - j \frac{4\pi E_o}{\eta \ k_o} \ , \qquad (3.2.2)$$

where

$$\Delta z = \ell/M = \ell_m/M$$

Equation (3.2.2) is used to generate a system of linear algebraic equations by forcing the equation to be satisfied at a finite set of M points. These points are chosen with regard for the function $G_n(z_m, z')$ to be

$$z_m = (m - \tfrac{1}{2}) \ \Delta z \qquad (3.2.3)$$

where

$$m = 1, 2, \ldots , M \quad .$$

Thus, the system of linear algebraic equations to be solved is, from equations (3.2.2) and (3.2.3),

$$\sum_{p=1}^{M+2} g_p \, \Pi_{mp} = \Gamma_m \ , \text{ when } m = 1, 2, \ldots , M \quad , \qquad (3.2.4)$$

and where

$$g_{m+1} = C \quad ,$$

$$g_{m+2} = D \quad ,$$

$$\Gamma p \quad = -j \frac{4}{\eta} \frac{E_o}{k_o} \; ;$$

and

$$\Pi_{mp} = \int_{(p-1)\Delta z}^{p\Delta z} dz' \sum_{n=o}^{\infty} \varepsilon_n \, G_n(z_m, z') \quad .$$

From equation (3.2.4), it is evident that there are M + 2 unknowns and only M equations. Hence, two more equations are needed to determine a unique solution. From the boundary conditions,

$$g_1 = g_M = 0 \quad ;$$

or

$$I(0) = I(\ell) = 0 \quad , \qquad (3.2.5)$$

which states that the current is zero at both ends of the wires. Therefore, M + 2 equations are available; and a unique solution can be obtained.

3.3 Numerical Results.

Using techniques developed in the foregoing sections, a computer program was developed to solve for the current distribution in the wires of the array. (See Appendix)

In order to verify the accuracy of the solution technique, a special case involving only one wire was used. From a paper by Harrison, Taylor, O'Donnel and Aronson,[7] tabulated results of several numerical methods treating a single wire are available. In Table 3.1 the results obtained from the approach presented in this thesis are compared with highly accurate results of their paper.

The current distribution given by the iterative method was obtained after eighty-two iterations, and the current distribution obtained by the series solution required one hundred terms. On the other hand, the current distribution obtained by using the integral equation piecewise zoning technique required forty-one zones. Although the results presented for this method appear less accurate, two points should be made regarding this approach to the solution. First, the flat zone technique calculates current values for entire zones, not discrete points. Consequently, a shift takes place when these zones are represented as points. In fact, in the computation of these zone values, a point may be lost as is evident in Table 3.1 for point z/h = 0.1. By increasing the number of zones, this technique will yield an accuracy comparable to the other methods. This is a result of zone values approaching point values with a subsequent reduction in the previously mentioned shift.

Table 3.1

CURRENT DISTRIBUTION COMPARISON

Kh = 6.2831 I(z) (milliamperes per volt)

	ITERATIVE SOLUTION [7]	FOURIER SERIES [7]	INTEGRAL ZONE METHOD
1.0	3.3459- j8.4079	3.3455- j8.4089	3.37518- j8.3799
0.9	3.2914- j8.1888	3.2910- j8.1898	3.33393- j8.1531
0.8	3.1315- j7.5530	3.1311- j7.5539	3.19743-j7.6908
0.7	2.8762- j6.5625	2.8758- j6.5634	2.95758- j6.7899
0.6	2.5414- j5.3150	2.5412- j5.3157	2.63320- j5.62045
0.5	2.1478- j3.9342	2.1477- j3.9348	2.23762- j4.2652
0.4	1.7188- j2.5591	1.7187- j2.5595	1.80178- j2.9183
0.3	1.2787- j1.3322	1.2786- j1.3326	1.34222- j1.6606
0.2	0.8504- j0.3895	0.8503- j0.3899	0.89199- j0.6647
0.1	0.4515+ j0.1442	0.4509+ j0.1430	0.46035- j0.0388

In Figure 3.2 curves of the current distribution on a single wire and on the middle wire in an eighty-one wire array are given for $k\ell = 2\pi$. It is observed that the curves in both cases are basically of the same form. However, the imaginary curves decrease in amplitude substantially from the one wire case to that of the mid-wire in an eighty-one wire array. It was found and is demonstrated here that the points where the curve crosses the axis move toward the center as the number of wires increases. The solid curves representing the large array approximates the limiting form of the current distribution as the number of wires approach infinity. The current distributions for the array are in effect a compromise between a more exact solution and existing computer facilities. Although the distributions are approximate, it is sufficient to say that they prove to be quite adequate in a subsequent study of the fields around the array. Figures 3.3 and 3.4 give comparisons between the single wire current distribution and that for a wire in a large array for frequencies where $k\ell = \pi$ and 3π respectively. This gives an indication of the effect of increasing the number of wires in an array has on the current distributions. Further, these three graphs also show the effect of increasing the frequency of the incident wave on the current distribution.

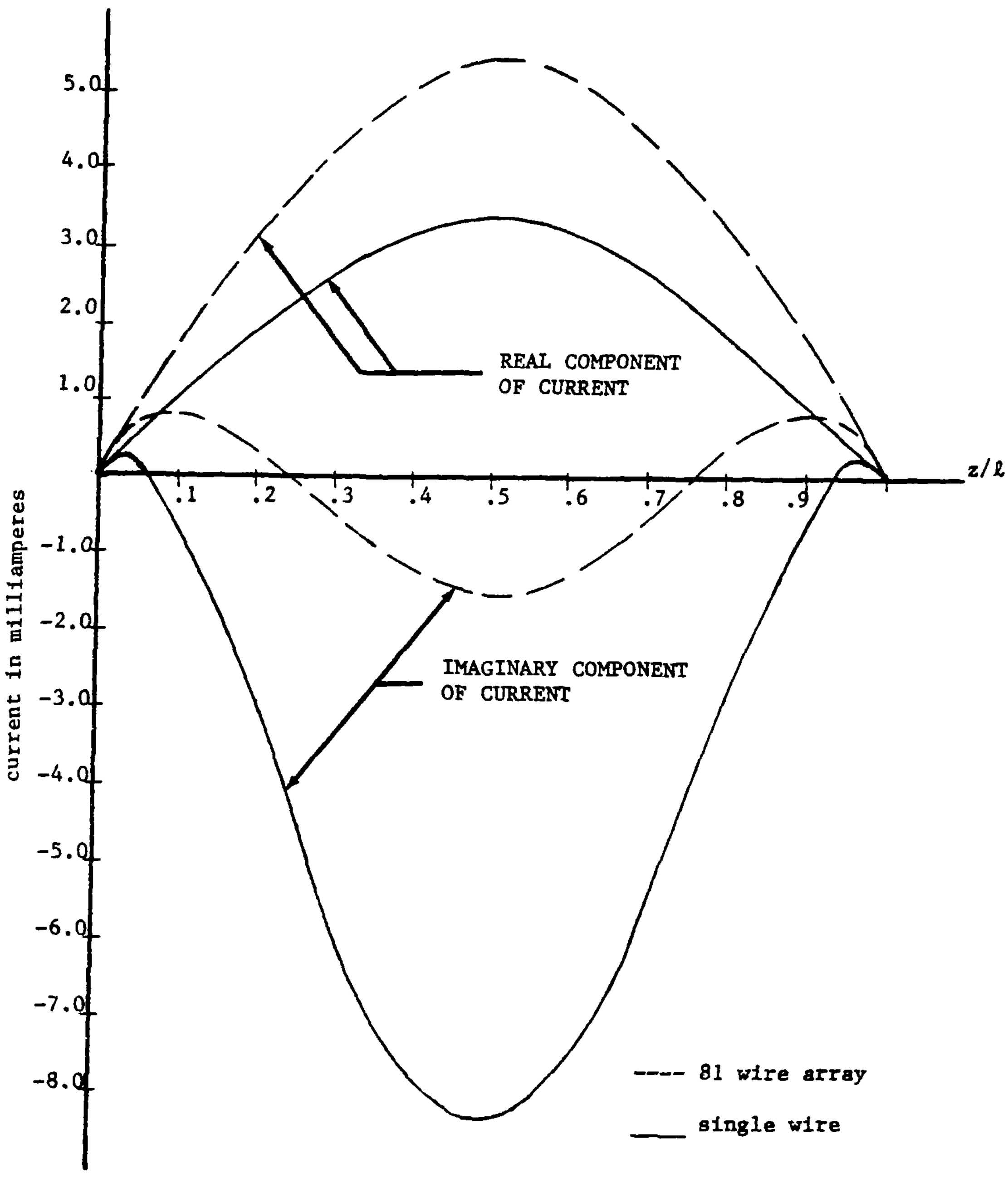

Figure 3.2 CURRENT DISTRIBUTION IN WIRES OF ARRAY
$k\ell=2\pi$, $\Omega = 2\ln(\ell/a) = 10.0$

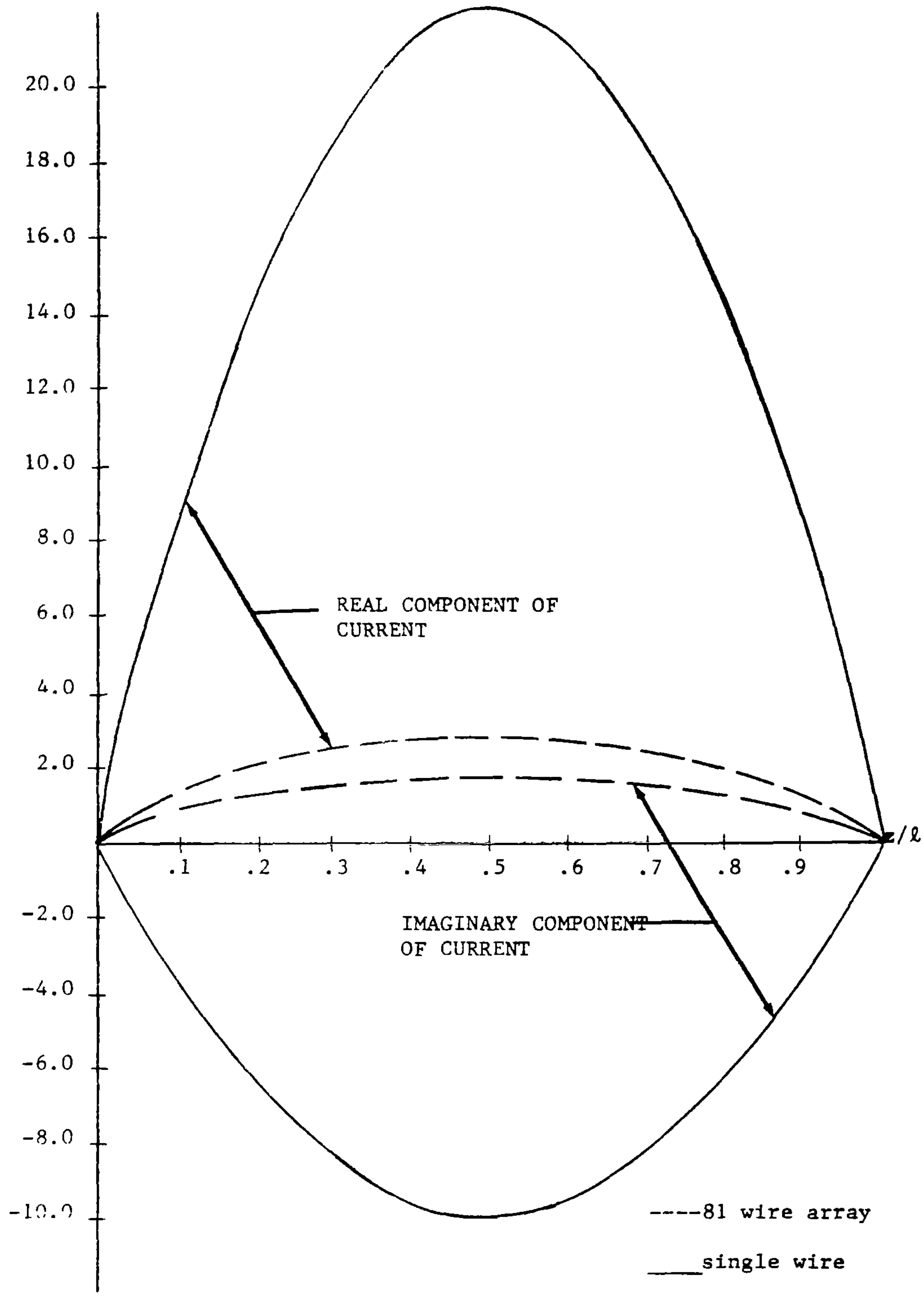

Figure 3.3 CURRENT DISTRIBUTION IN WIRES OF ARRAY
$k\ell = \pi$, $\Omega = 2 \ln(\ell/a) = 10.0$

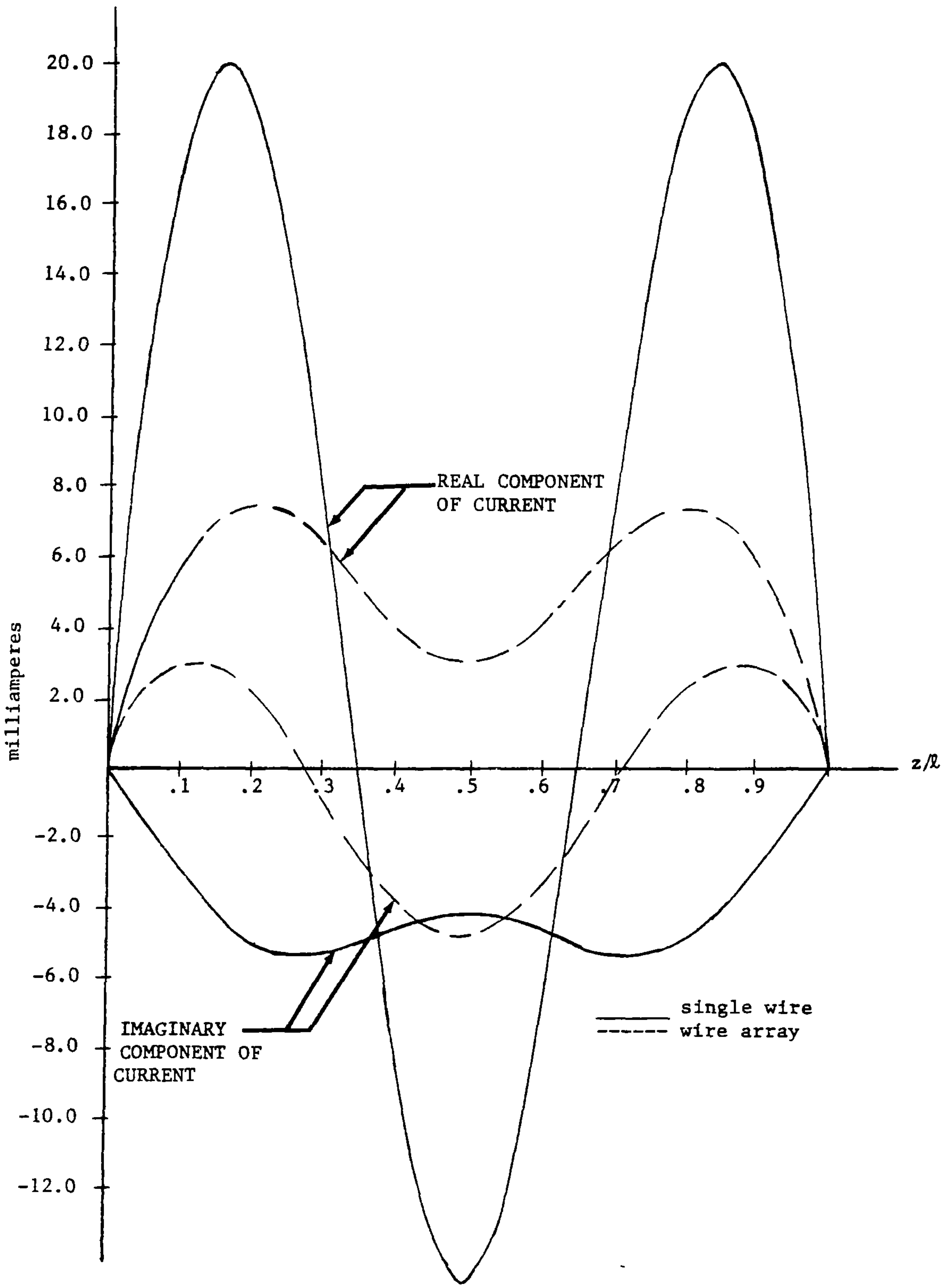

FIGURE 3.4 CURRENT DISTRIBUTION IN WIRE
$k\ell = 3\pi$, $\Omega = 2\ \ln(\ell/a) = 10.0$

CHAPTER IV

FIELDS AROUND WIRE ARRAYS

In Chapter III the current distribution in wires of a large array were determined numerically. Of greater interest, however, is a knowledge of the fields surrounding this array. Therefore, it is the prupose of this chapter to present a method by which these fields may be computed and to examine the behavior of these fields around the structure.

4.1 Fitting Current Distributions

Since the information concerning the current distribution is discrete, a method outlined by Otto and Richmond[8] proves appropriate in computing the fields from the currents. Using the expressions as developed by Schelkunoff and Friis[9] for the fields of a time harmonic electric line source where the current is of the form

$$I(z) = A \cos kz + B \sin kz \quad , \qquad (4.1.1)$$

the expression for the Ez field is

$$Ez(\rho, z) = \frac{j}{4\pi\omega\varepsilon}\left[I'(z')\frac{e^{-jkr}}{r} + I(z')\frac{\partial}{\partial z}\frac{e^{-jkr}}{r}\right]_{z'=z_1}^{z'=z_2} \quad , \qquad (4.1.2)$$

where

$$r = \text{SQRT}\left[\rho^2 + (z - z')^2\right] .$$

and

$$I'(z') = \frac{d}{dz'} I(z') \quad .$$

This gives the z component of the electric field associated with the line source in Figure 4.1. In the present problem where N discrete values of current are available, it is considered that there is a series of line sources (see Figure 4.2). Thus by superposition the Ez field is given by

$$E_z(\rho, z) = \frac{j}{4\pi\omega\varepsilon} \sum_{i=1}^{N-1} \left[I'(z') \frac{e^{-jkr_i}}{r_i} + I(z') \frac{\partial}{\partial z} \frac{e^{-jkr_i}}{r_i} \right]_{z'=z_i}^{z'=z_i+1} \qquad (4.1.3)$$

Here it is assumed that the current has the form given in (4.1.1) between each consecutive pair of points. Furthermore, if I(z') is continuous across the junctions z_i and is zero at the endpoints z_1 and z_N , then Equation (4.1.3) becomes

$$E_z(\rho, z) = \frac{j}{4\pi\omega\varepsilon} \sum_{i=1}^{N} I'_i \frac{e^{-jkR_i}}{r_i} \quad . \qquad (4.1.4)$$

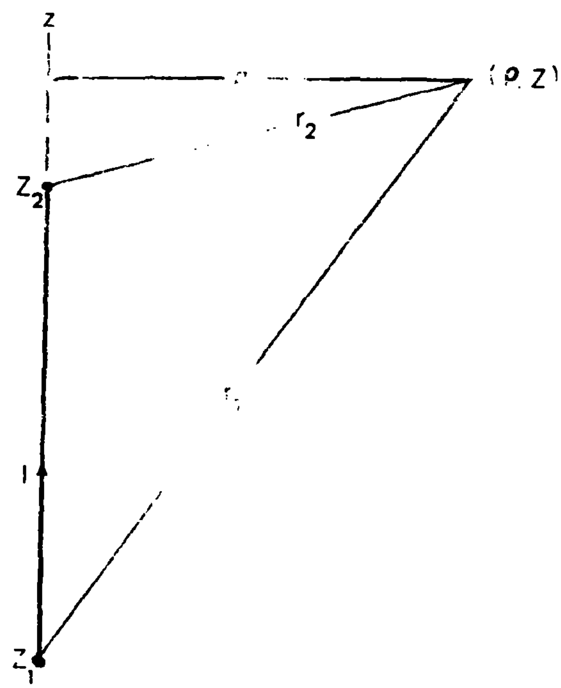

Figure 4.1 Electric line source located on z axis with its field observed at (ρ, z)

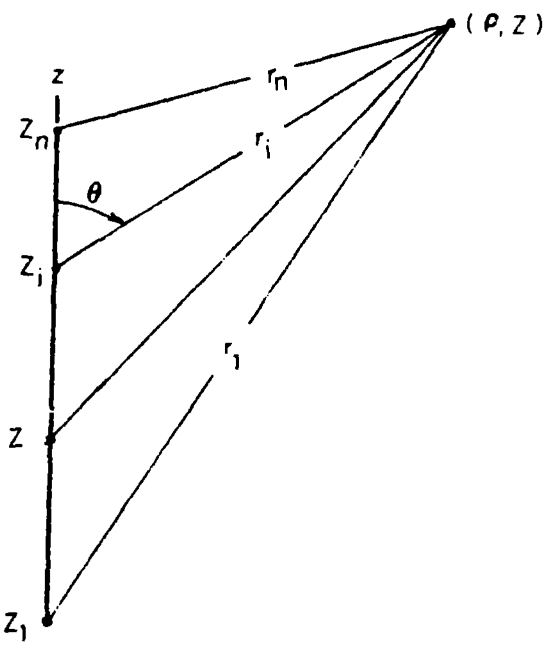

Figure 4.2 Line source with peicewise-sinusoidal distribution

By a similar procedure it is found that

$$E(\rho, z) = -j \frac{j}{4\pi\omega\varepsilon} \sum_{i=1}^{N} \Delta I' \, e^{-jkR_i} \cos\theta_i \quad , \qquad (4.1.5)$$

where

$$R_i = \text{SQRT}\left[\rho^2 + (z - z_i)^2\right] , \qquad (4.1.6a)$$

with

$$z_i = (i - 0.5)\,\Delta_z \qquad (4.1.6b)$$

and

$$\cos\theta_i = (z - z_i)/R_i \quad . \qquad (4.1.6c)$$

From Equations (4.1.3) and (4.1.4), it can be seen that

$$\Delta I_i' = I'_{i-1}(z_i) - I'_i(z_i) \quad . \qquad (4.1.7)$$

Since $I'_i = d/dz_i \; I_i'(z_i)$, it follows from Equation (4.1.1) that

$$I'_i = k\left[- A_i \sin kz_i + B_i \cos kz_i\right] \quad . \qquad (4.1.8)$$

All that is now required to completely define the fields are expressions for the coefficients A_i and B_i . Therefore, consider Equation (4.1.1).

$$I_i(z) = A_i \cos kz + B_i \cos kz \qquad (4.1.9)$$

where,

$$z_i \leq z \leq z_{i+1} \quad .$$

Then

$$I_i(z_i) = A_i \cos kz_i + B_i \sin kz_i \quad , \qquad (4.1.10)$$

and

$$I_i(z_{i+1}) = A_i \cos kz_{i+1} + B \sin kz_{i+1} \quad . \qquad (4.1.11)$$

Multiplying Equation (4.1.10) by $\sin kz_{i+1}$ and Equation (4.1.11) by $\sin kz_i$ and subtracting the two yields an expression for A_i .

$$A_i = \frac{I(z_i) \sin Kz_{i+1} - I(z_{i+1}) \sin kz_i}{\sin k\Delta z} \quad . \qquad (4.1.12)$$

In a similar manner the expression for B_i is

$$B_i = \frac{I(z_{i+1}) \cos kz_i - I(z_i) \cos kz_{i+1}}{\sin k\Delta z} \quad . \qquad (4.1.13)$$

It should be noted that as a consequence of the manner in which A_i , B_i , and $\Delta I'_i$ are expressed, $\Delta I'_1$ and $\Delta I'_N$ require special definitions. From Equation (4.1.7) it is seen that

$$\Delta I'_1 = - I'_1(z_1) \quad , \qquad (4.1.14)$$

and

$$\Delta I'_N = I_{N-1}(z_N) \quad . \tag{4.1.15}$$

Finally with $\Delta I'_i$ completely defined with Equations (4.1.7), (4.1.14), and (4.1.15); and A_i and B_i defined with Equations (4.1.12) and (4.1.13) respectively, the expressions given by (4.1.4) and (4.1.5) become fully meaningful in describing the fields.

The foregoing technique provides a rigorous determination of field quantities around a wire structure. In the subsequent section, the facility of this method will be demonstrated by application to the large wire array, utilizing the results of Chapter III.

4.2 Numerical Results

Using the formulation of the foregoing section, a computer program was developed which calculated the fields at a number of points around the structure for three different frequencies, $k\ell = \pi$, $k\ell = 2\pi$, and $k\ell = 3\pi$. Figures 4.3, 4.4, 4.5, and 4.6 show graphical representations of the radial and z fields at various distances from the plane of the array. It is evident that the fields assume a plane wave mode at some distance from the structure. This distance varies with frequency from 0.2682 meters for $k\ell = \pi$ to 1.724 meters for $k\ell = 3\pi$.

By using the magnitude of the Ez field after it becomes a plane wave, transmission coefficients for the three cases may be calculated. It is not necessary to

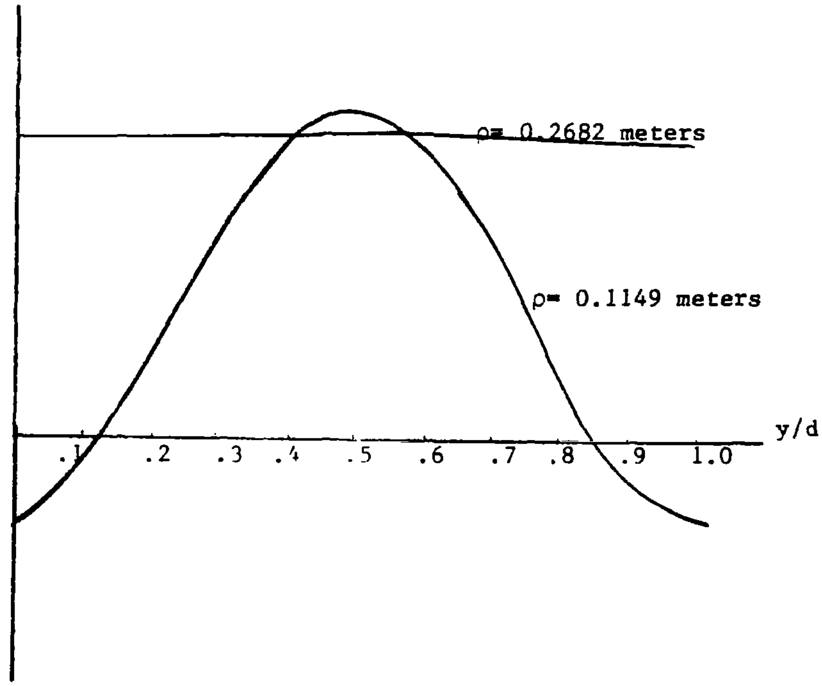

Figure 4.3 E_r field between wires at ρ distance from array plane. $kl=\pi$, $\Omega = 2\ln(h/a)$.

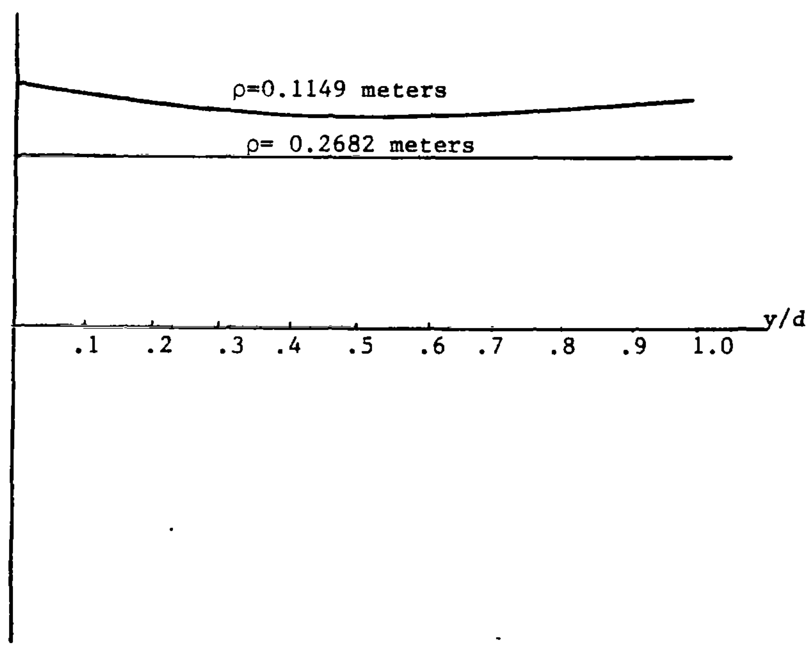

Figure 4.4 E_z field between wires at ρ distance from array plane. $kl=\pi$, $\Omega = 2\ln(h/a) = 10.0$

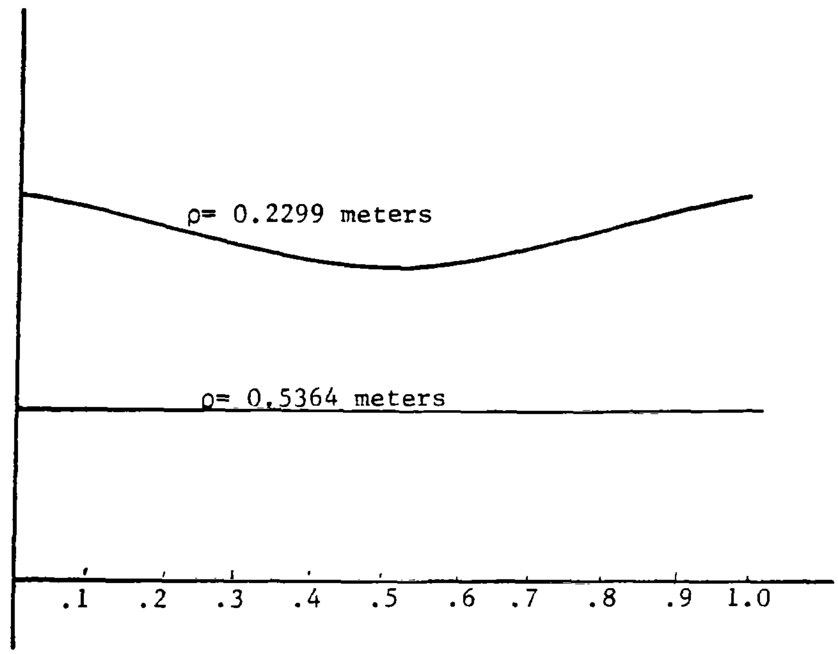

Figure 4.5 E_z field between wires a ρ distance from array plane. kl= 2π, Ω =2ln(h/a) =10.0

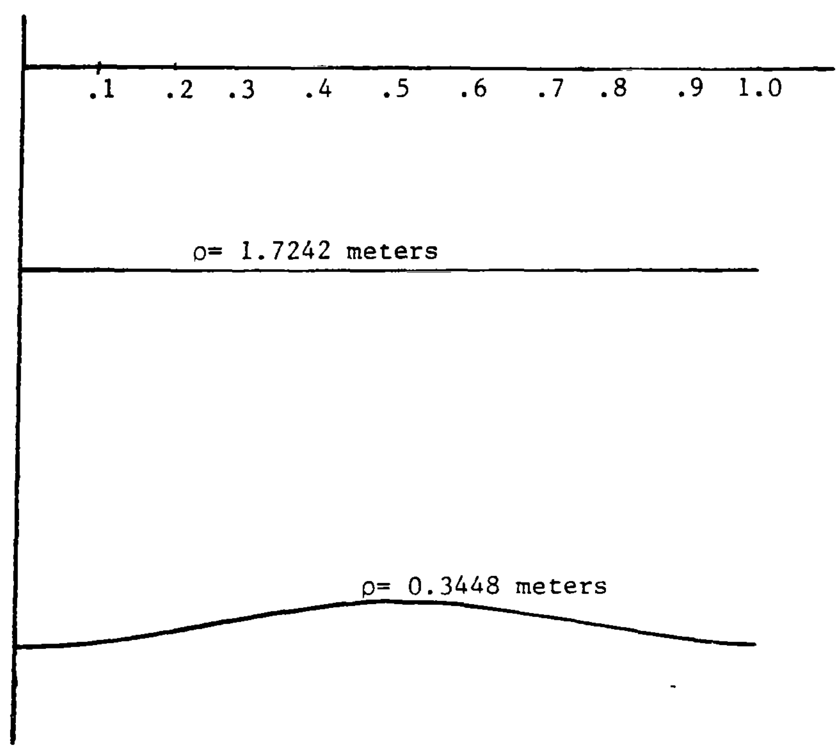

Figure 4.6 E_z field between wires at ρ distance from array plane. kl= 2π, Ω = 2ln(h/a) =10.0

include the radial field since it is more than two orders of magnitude below the z-component. Thus the transmission coefficient can be written as

$$T = \frac{E_t}{E_i} \quad , \tag{4.2.1}$$

where

$$E_t = E_s + E_i = E_z + E_i \quad .$$

Thus,

$$T = \frac{E_z + E_i}{E_i} \quad . \tag{4.2.2}$$

Using Equation (4.2.2) the transmission coefficients can be quickly calculated. However, because of time limitations imposed by existing computer facilities, only the coefficients for the cases already described could be determined. They are

$$T = 0.332$$

$$\text{For } k\ell = \pi \quad ,$$

$$T = 0.596 \quad ,$$

$$\text{For } k\ell = 2\pi \text{ , and}$$

$$T = 0.523$$

$$\text{For } k\ell = 3\pi \quad .$$

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