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Electromagnetic Scattering in Time Varying, Inhomogeneous Media

by

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ABSTRACT

A finite-difference solution technique is used to solve Maxwell's equations directly in the treatment of electromagnetic pulse scattering in a time-varying inhomogeneous medium. In particular the scattering from a cylindrical rod inside a cylindrical waveguide is considered.

INTRODUCTION

In the theoretical studies of time-varying and inhomogeneous media, it is often found that the solution of Maxwell's equations is extremely difficult to obtain in analytical form. Because of this and recent advances in high-speed digital computer capabilities, more and more theoretical studies employ numerical techniques. Recently Yee [1] demonstrated the finite-difference technique for solving initial-value problems. In this paper that technique has been extended to include time-varying and inhomogeneous media. In particular the technique is applied to the problem of electromagnetic pulse scattering from a perfectly conducting cylindrical rod in a time-varying inhomogeneous lossy medium.

ANALYSIS

Difference Equations

Maxwell's equations are for a lossy medium with no free current density nor charge density

$$\nabla \times \vec{H} = \sigma\vec{E} + \frac{\partial}{\partial t}(\varepsilon\vec{E})$$

$$\nabla \times \vec{E} = -\frac{\partial}{\partial t}(\mu\vec{H})$$

$$\nabla \cdot (\mu\vec{H}) = 0 \qquad (1)$$

$$\nabla \cdot (\varepsilon\vec{E}) = 0$$

where the notation is consistent with Stratton's text [2]. Subsequently cylindrical geometry is to be considered where azimuthal symmetry obtains. Under these considerations, the electromagnetic field components are determined by the following system of partial differential equations.

$$\frac{\partial}{\partial t}(\varepsilon E_r) = -\frac{\partial}{\partial z} H_\phi - \sigma E_r$$

$$\frac{\partial}{\partial t}(\varepsilon E_z) = \frac{1}{r} H_\phi + \frac{\partial}{\partial r} H_\phi - \sigma E_z$$

$$\frac{\partial}{\partial t}(\mu H_\phi) = \frac{\partial}{\partial r} E_z - \frac{\partial}{\partial z} E_r \qquad (2)$$

Here the constitutive properties μ, ε and σ may be functions of time and position. The subscripts r, ϕ and z identify the usual cylindrical components.

The system of equations (2) are particularly suited to solution by the finite difference technique. The method proposed by Lax [3] is used to set up the difference equations. It automatically "centers" the difference formulas thereby reducing the truncation (or round-off) error [4].

The resulting difference equations are

$$H_\phi^{n+1}(I,J) = H_\phi^n(I,J) + \frac{1}{Z}\frac{\Delta T}{\Delta r}\left[E_z^{n+1/2}(I,J+1/2) - E_z^{n+1/2}(I,J-1/2)\right]$$
$$- \frac{1}{Z}\frac{\Delta T}{\Delta z}\left[E_r^{n+1/2}(I+1/2,\ J) - E_r^{n+1/2}(I-1/2,J)\right] \quad (3)$$

$$E_z^{n+3/2}(I,J+1/2) = E_z^{n+1/2}(I,J+1/2) + \frac{\Delta T}{\Delta r}\ \frac{Z}{J}\ H_\phi^{n+1}(I,J+1/2) \quad (4)$$
$$+ Z\frac{\Delta T}{\Delta r}\left[H_\phi^{n+1}(I,J+1) - H_\phi^{n+1}(I,J)\right]$$
$$- Z\,\Delta T\,\sigma^{n+1}(I)\ E_z^{n+1}(I,J+1/2)$$

$$E_r^{n+3/2}(I+1/2,J) = E_r^{n+1/2}(I+1/2,J) - Z\frac{\Delta T}{\Delta z}\left[H_\phi^{n+1}(I+1,J) - H_\phi^{n+1}(I,J)\right]$$
$$- Z\,\Delta T\,\sigma^{n+1}(I+1/2)\ E_z^{n+1}(I+1/2,J) \quad (5)$$

Here the following notation was used

$$T \equiv ct, \text{ c is the speed of light} \quad (6)$$

$$\left[H_\phi(z,r,t)\right]_{\substack{z = I\,\Delta z \\ r = (J-1/2)\Delta r \\ t = n\,\Delta T/c}} \equiv H_\phi^n(I,J) \quad (7)$$

$$\left[\sigma(z,t)\right]_{\substack{z = I\,\Delta z \\ t = n\,\Delta t/c}} = \sigma^n(I) \quad (8)$$

In deriving (3) - (5) the permittivity ε and magnetic permeability μ are considered to be constant and $c = (\mu\varepsilon)^{-1/2}$, $Z = \sqrt{\mu/\varepsilon}$. However the conductivity is considered to be both a function of z and t.

In order to solve the difference equations the indice notation must be consistent. To that end the following approximations are used

$$E_z^{n+1}(I+1/2,J) = (1/2)\left[E_z^{n+3/2}(I+1/2,J) + E_z^{n+1/2}(I+1/2,J)\right] \quad (9)$$

$$E_r^{n+1}(I,J+1/2) = (1/2)\left[E_r^{n+3/2}(I,J+1/2) + E_r^{n+1/2}(I,J+1/2)\right] \quad (10)$$

$$H_\phi^{n+1}(I,J+1/2) = (1/2)\left[H_\phi^{n+1}(I,J+1) + H_\phi^{n+1}(I,J)\right] \quad (11)$$

The substitution of (9) - (11) into (3) - (5) will yield a solvable system of difference equations where the knowledge of the initial field components allows a sequential computation of the field components at selected time intervals.

Stability Considerations

The system of partial differential equations (2) is a hyperbolic system. The hyperbolic system that occurs for constant constitutive properties of the medium has been well studied and documented. In order for the difference equations to be stable and to converge toward the solution, the following relation must be satisfied for spacial and time increments (it is tacitly assumed that $\Delta r = \Delta z$) [5]

$$\frac{\Delta T}{\Delta r} = \frac{\Delta T}{\Delta z} < \frac{1}{\sqrt{2}} \quad (12)$$

For practical reasons, it is best to choose the ratio of the time increment to the spacial increment as large as possible yet satisfying (12). In obtaining the numerical results that are presented

$$\frac{\Delta T}{\Delta r} = \frac{\Delta T}{\Delta z} = \frac{1}{2} \quad (13)$$

It is expected that the stability criterium (12) holds for inhomogeneous and time-varying media provided that these variations are small as compared to the variations in the field components. However it has been proved that if the difference equations are stable then the solution of the difference equations converge to the solution of the differential equations as the time and spacial increments approach zero [6]. Because instability is readily apparent the foregoing result can be used when no convergence criteria exists.

Boundary Conditions

If the scattering from a finite perfectly conducting object is to be treated, then the appropriate boundary conditions are that the tangential component of the electric field and the normal component of the magnetic field both vanish at the surface of the object. Furthermore, the difference equations can only be satisfied over a finite expanse of space. This necessitates defining this expanse and the appropriate boundary conditions. Suppose (3) is satisfied for I = 1,···N and J = 1,···M. From (4) and (5) it is seen that the following components are to be specified

$$\left.\begin{array}{l} E_z^{n+3/2}\ (I,1/2) \\ E_z^{n+3/2}\ (I,M+1/2) \\ E_r^{n+3/2}\ (1/2,J) \\ E_r^{n+3/2}\ (N+1/2,J) \end{array}\right\} \qquad (14)$$

The components may be specified initially or they may computed as a linear extrapolation, i.e.

$$E_z^{n+3/2}(I,1/2) = 2\ E_z^{n+3/2}(I,3/2) - E_z^{n+3/2}\ (I,5/2) \qquad (15)$$

$$E_z^{n+3/2}(I,M+1/2) = 2\ E_z^{n+3/2}\ (I,M-1/2) - E_z^{n+3/2}\ (I,M-3/2) \qquad (16)$$

It is convenient to set the other two components of (14) equal zero, making the computation of the fields to fall within the region between two perfectly conducting plates appearing at z = 1/2Δz and z = (N+1/2)Δz.

Initial Fields

To obtain the initial components of the fields, it is considered that the initial fields are those of a pulse propagating in a cylindrical waveguide. Since the field components of the TM_{01} cylindrical wavequide mode have azimuthal symmetry, it is convenient to consider the initial electromagnetic pulse to be formed by this mode. The field components of the TM_{01} mode are

$$\hat{E}_r(z,r,\omega) = -j\,\frac{h}{\beta}\,E_o(\omega)\,J_o'(\beta r)e^{-jhz} \tag{17}$$

$$\hat{E}_z(z,r,\omega) = E_o(\omega)\,J_o(\beta r)e^{-jhz} \tag{18}$$

$$\hat{H}_\phi(z,r,\omega) = -j\,\frac{k}{Z\beta}\,E_o(\omega)\,J_o'(\beta r)e^{-jhz} \tag{19}$$

where $E_o(\omega)$ is the complex amplitude of the mode for radian frequency ω,

$$\left.\begin{aligned} h &= \sqrt{k^2 - \beta^2} \\ k^2 &= \omega^2\mu\varepsilon \\ \beta &= 2.405/R \\ R &= \text{radius of the waveguide} \end{aligned}\right\} \tag{20}$$

It is most convenient to choose R sufficiently large that

$$h \simeq k \tag{21}$$

for most of the frequency content of the pulse. This allows the pulse to propagate in the wavequide with essentially no dispersion. The shape of the pulse is determined by $E_o(\omega)$, where the field components of the pulse are

$$E_r(z,r,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega\,\hat{E}_r(z,r,\omega)e^{j\omega t} \tag{22}$$

$$E_z(z,r,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega\ \hat{E}_z(z,r,\omega)e^{j\omega t} \tag{23}$$

$$H_\phi(z,r,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega\ \hat{H}_\phi(z,r,\omega)e^{j\omega t} \tag{24}$$

By using (17), (19) and (21) in (22) and (24) the following are easily obtained

$$E_r(z,r,t) = \frac{J_o'(\beta r)}{\beta J_o\ (\beta r)} \frac{\partial}{\partial z} E_z(z,r,t) \tag{25}$$

$$H_\phi(z,r,t) = -\frac{J_o'\ (\beta r)}{Z\beta C\ J_o(\beta r)} \frac{\partial}{\partial t} E_z(z,r,t) \tag{26}$$

The foregoing indicate that the choice of $E_z(z,r,t)$ is arbitrary to a certain extent. From (23) it is seen that

$$E_z(z,r,t) = J_o(\beta r)\ F(z \pm ct) \tag{27}$$

where $F(z \pm ct)$ may be chosen arbitrarily. However the difference equations place a restriction on the initial pulse. It is that the initial field components must all be continuous functions of position. Therefore the choice of $F(z \pm ct)$ must satisfy the restrictions: $F(z \pm ct)$ and $F'(z \pm ct)$ must be continuous for all $z \pm ct$ and the Fourier transform of $F(z \pm ct)$ must have negligible frequency content for $\omega \lesssim \beta c$. A suitable choice for $F(z \pm ct)$ is

$$F(z \pm ct) = \sin^2\left[A(z \pm ct)\right] \qquad o \leq A(z \pm ct) \leq \pi \tag{28}$$

$$= 0 \qquad \text{elsewhere}$$

NUMERICAL RESULTS

The problem treated in obtaining numerical results is the scattering from a perfectly conducting rod inside a cylindrical waveguide. The rod is oriented parallel to the axis of the waveguide and positioned on the axis. The radius of the waveguide is considered to be 2405 meters and the spacial increment is 0.025 meters. For the conducting rod a radius of .1 meters and a length .075 meters are used. And for the incident electromagnetic pulse field components (25) - (28) are used where

$$A = \pi \ (18\Delta Z)^{-1} \qquad (29)$$

The pulse width at half-maximum is 2/3 nanosecond.

First the interior of the waveguide is considered to be free space. In figure 1 the E_z component of the pulse is shown after the pulse has propagated 30 spacial increments and 60 spacial increments. It is observed that essentially no dispersion occurs. But after 60 spacial increments and 120 time increments "round-off" error causes a slight disturbance at the tail of the pulse. This disturbance is slightly more noticeable in the other field components of the pulse as shown in figure 2. These data were obtained using an IBM 360 Model 40 computer which carries only 8 significant figures in calculations. Figure 3 shows the buildup of current on the surface of the cylindrical rod. Note that 180 time increments is 7.5 nanoseconds and that the center of the pulse has propagated 50 spacial increments past the end of the rod.

As a second example the waveguide is considered to be filled with a homogeneous, but time-varying, lossy medium. The conductivity is considered to vary in time as

$$\sigma^n = 2n \times 10^{-3} \text{ mhos/meter} \qquad (30)$$

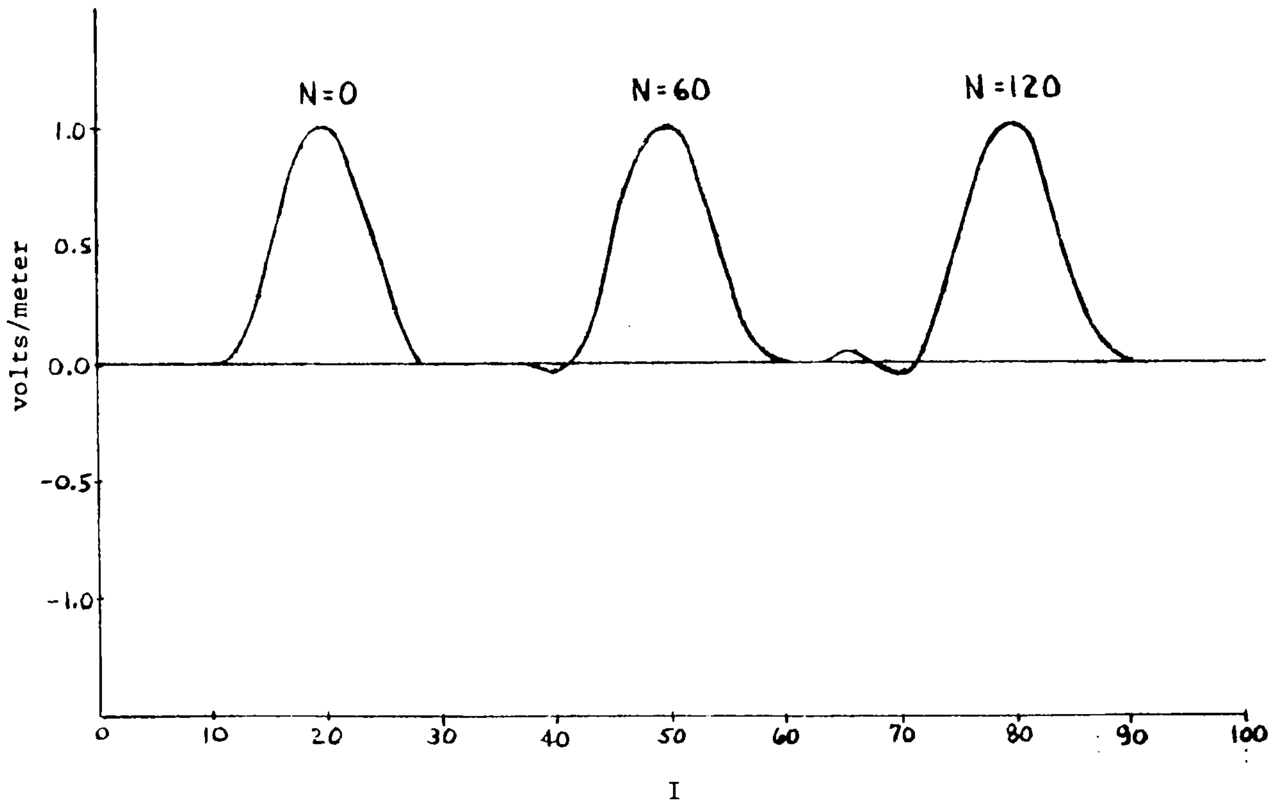

Figure 1: The propagation of the axial component of the electric field evaluated at J = 40. Δz = 0.025m, ΔT = 0.0125m, R = 2405 m.

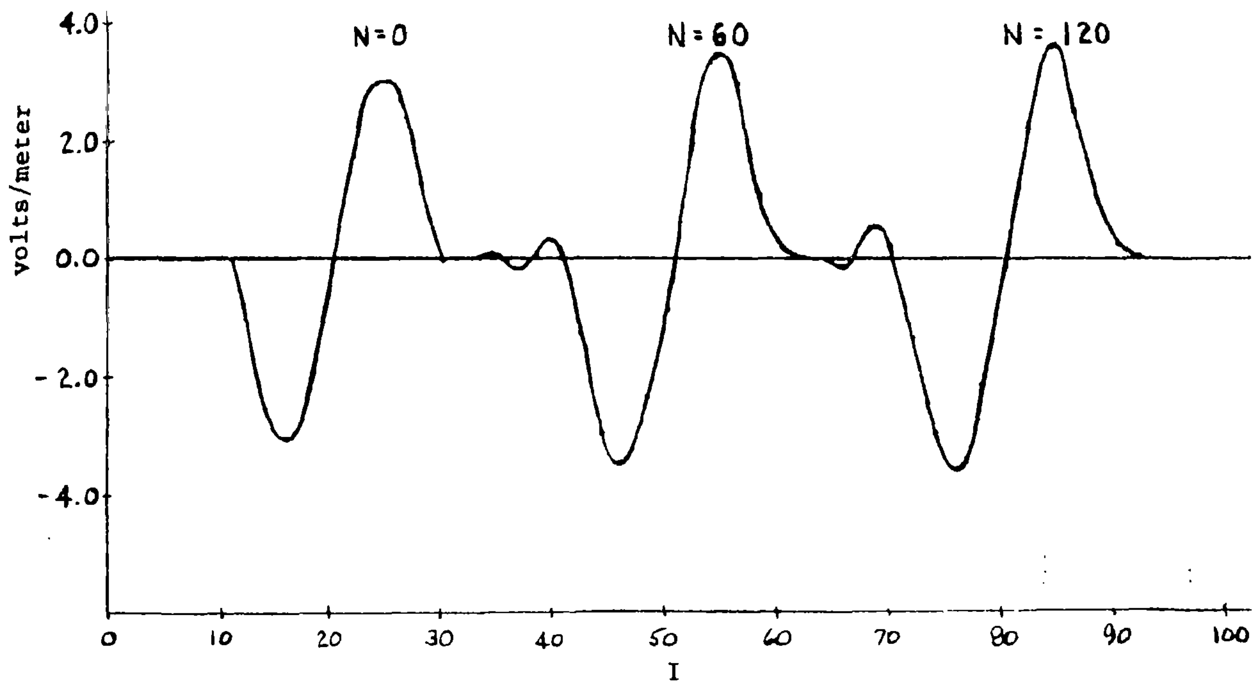

Figure 2: The propagation of the radial component of the electric field evaluated at J = 40.

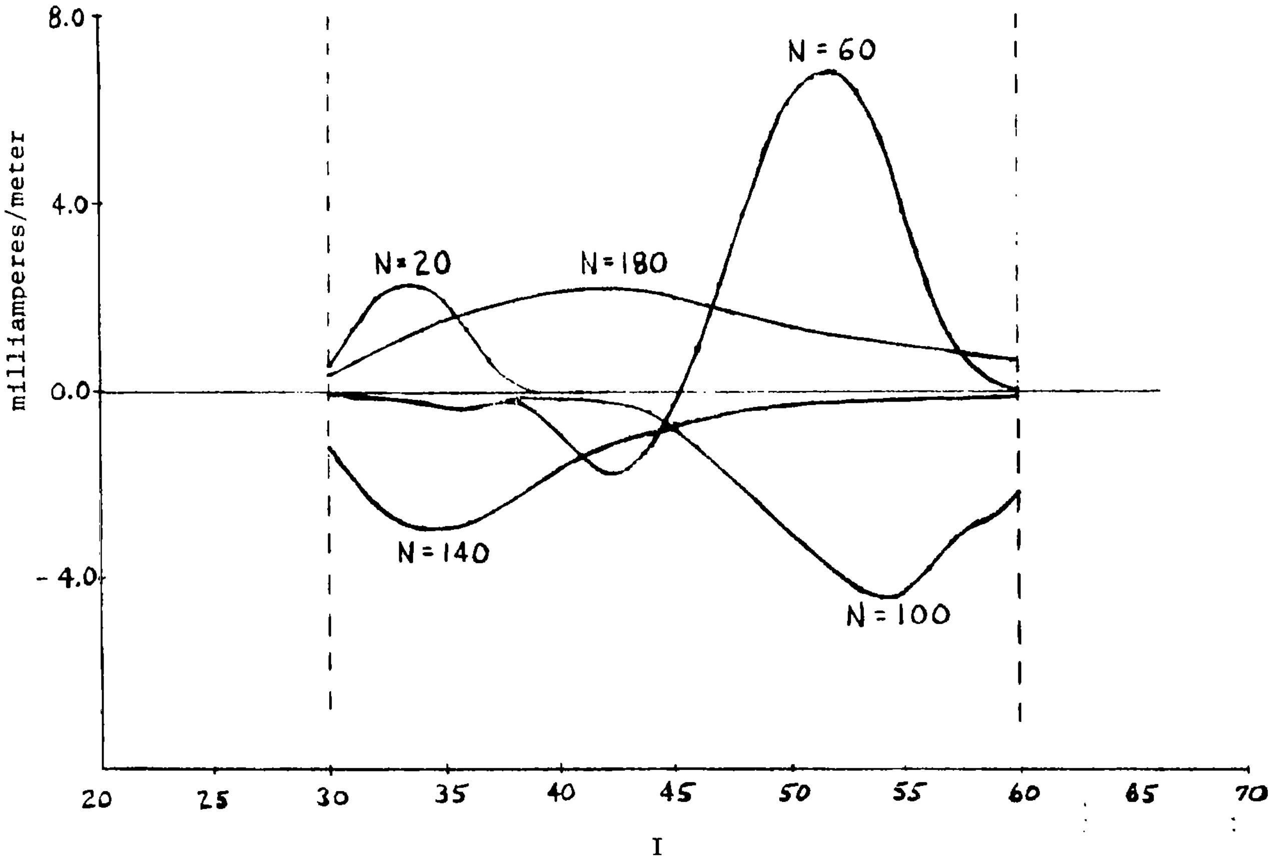

Figure 3: The current distribution induced on the cylindrical scatterer. The peak of the incident pulse at N = 0 is at I = 20. Region of the scatterer is $30 \leq I \leq 60$ and $1 \leq J \leq 4$.

Figure 4 shows the attenuation of the pulse and the dispersion. Again "round-off" error does not seem very pronounced in the computation of the E_z component. In figure 5 the radial component of the electric field is shown for the time-varying conductivity. Figure 6 shows the buildup of current on the conducting rod.

Finally an inhomogeneous, time-varying medium is considered inside the waveguide.

$$\sigma(z,t) = \sigma_o(t)\ f(z-ct) \tag{31}$$

and for t = o, it is shown in Figure 7 along with the other initial field components of the pulse. The factor $\sigma_o(t)$ is assumed to vary in time exactly as (30) until it reaches a chosen maximum value (for the subsequent data this maximum is 0.1 mhos/meter). Figure 8 shows how the pulse propagates after $\sigma_o(t)$ has reached a maximum by showing the z component of the electric field. In Figure 9 the radial component of the electric field is shown. Also in Figure 10 the current distribution on the rod is shown as a function of time. Here the cylindrical rod is considered to have a radius of .075 meters and length .375 meters.

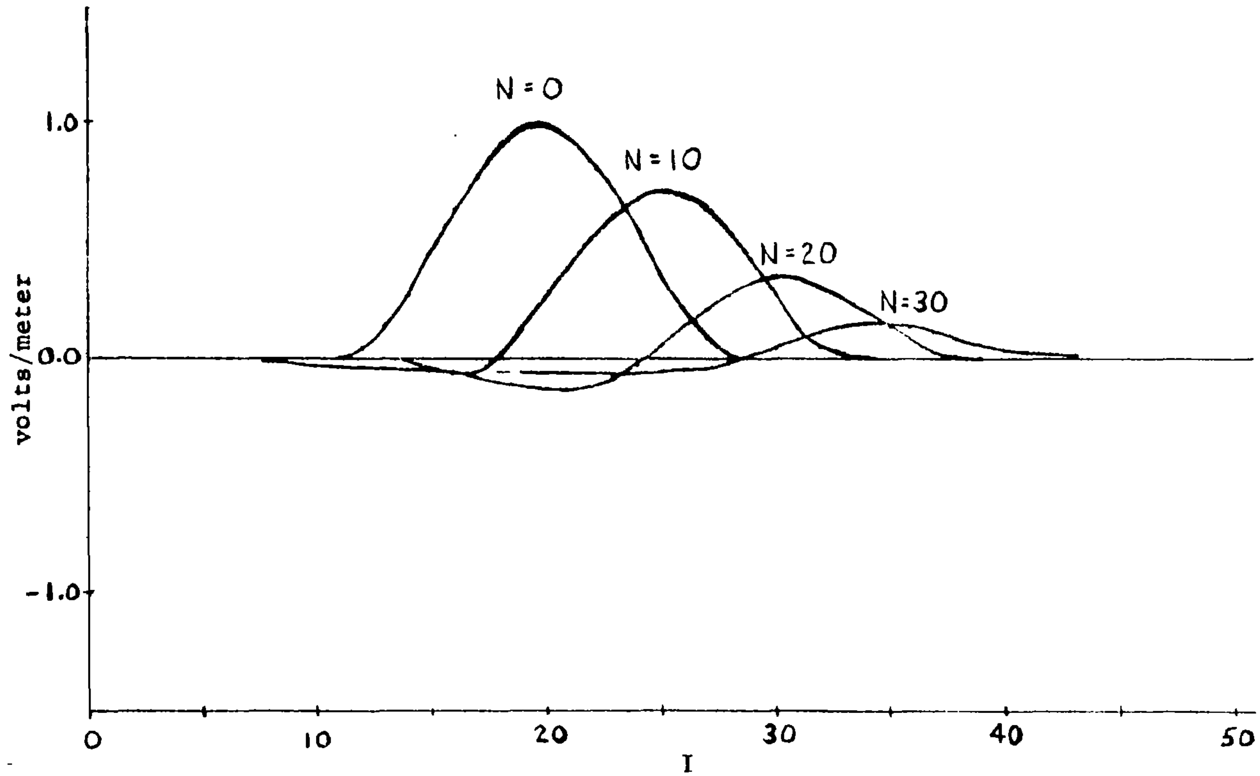

Figure 4: Propagation of the axial component of the electric field in a time-varying homogeneous medium at J = 40.

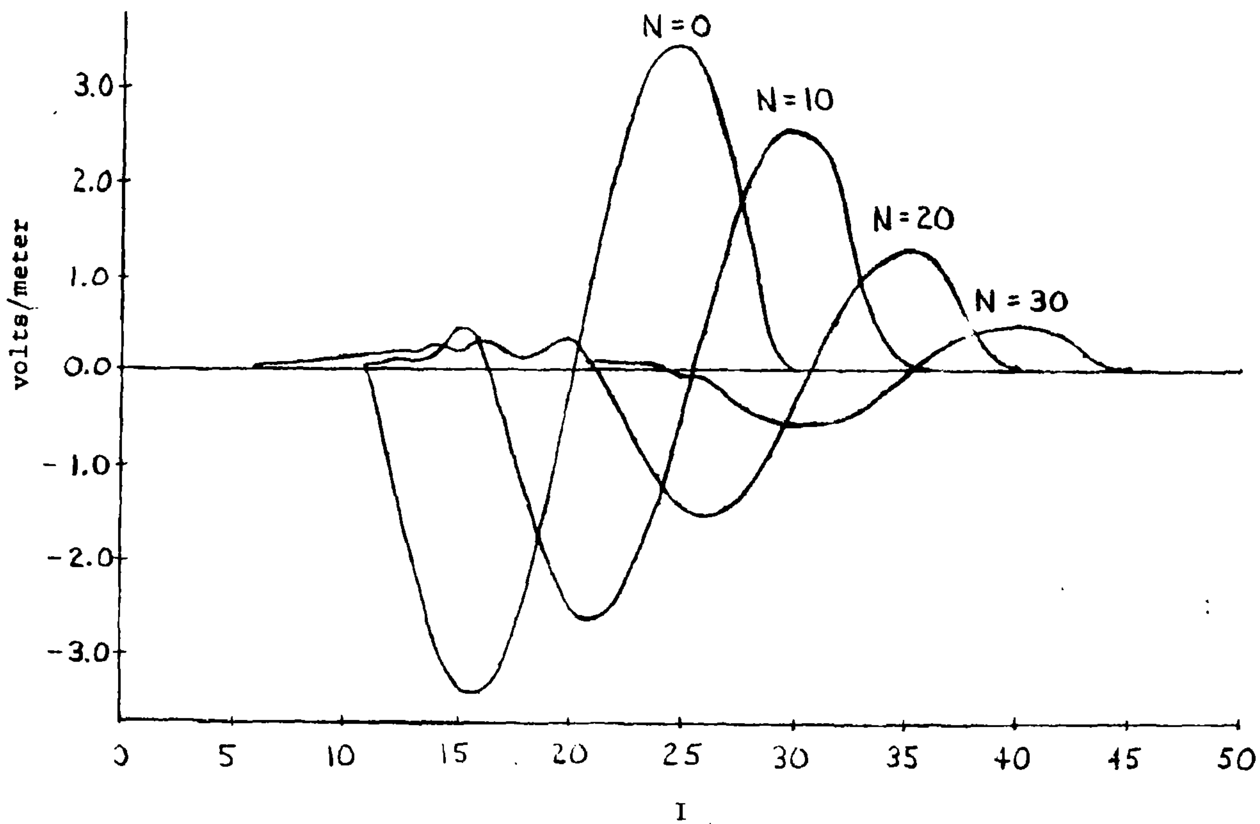

Figure 5: Propagation of the radial component of the electric field in a time-varying homogeneous medium at J = 40.

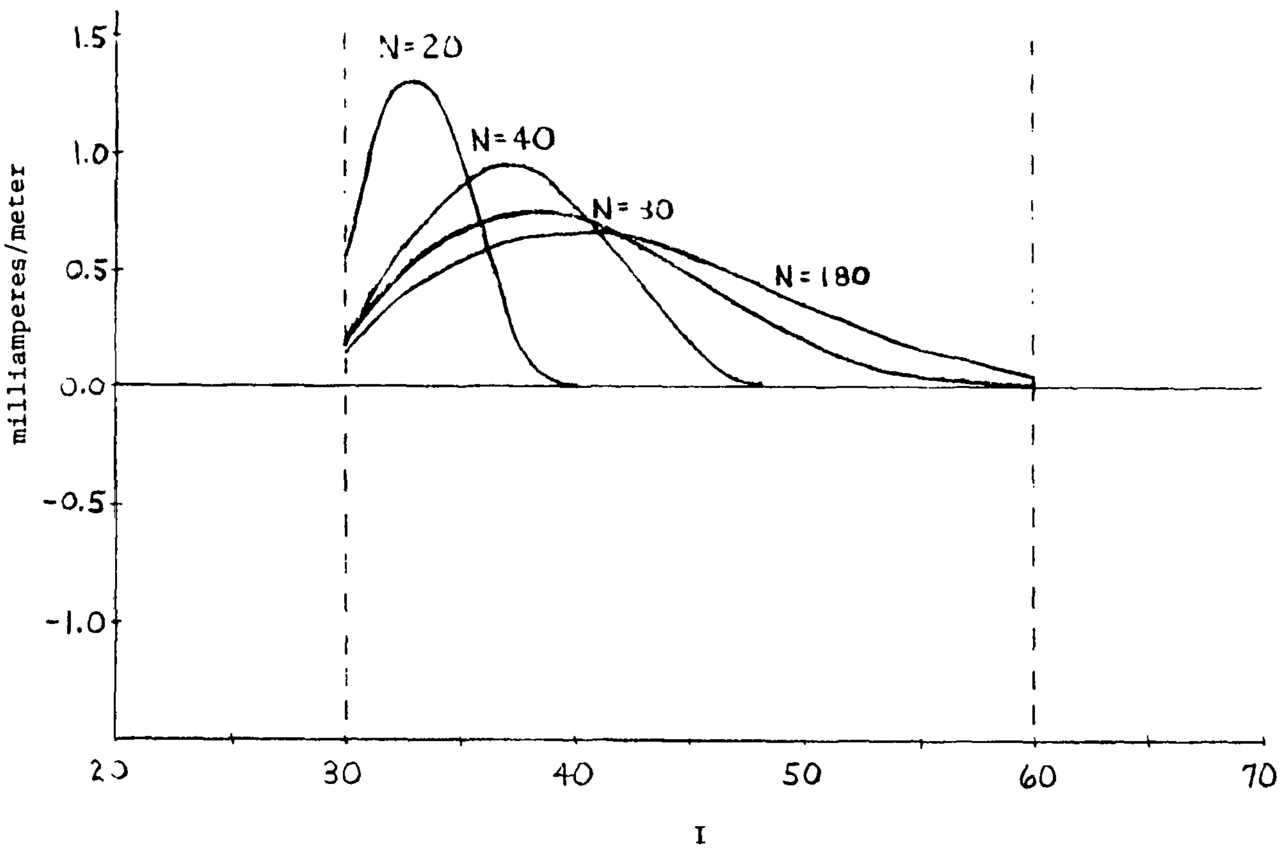

Figure 6: Current distribution induced on the cylindrical scatterer. The region of the scatter is $30 \leq I \leq 60$ and $1 \leq J \leq 4$. The peak of the incident pulse at N = o is at I = 22.

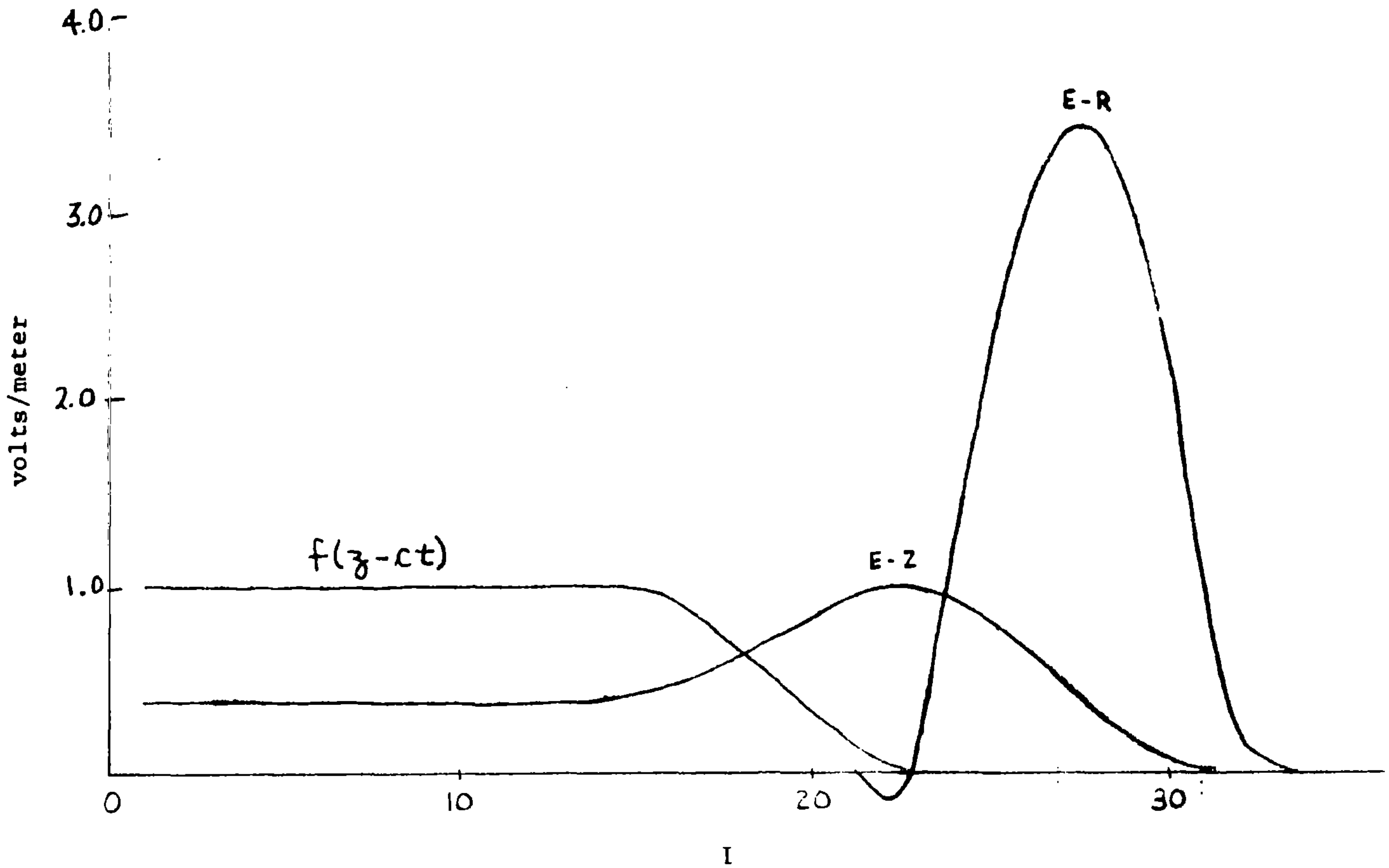

Figure 7: The electric field components at J = 40, after $\sigma(t)$ has reached maximum value (0.1 v/m). The unitless quantity f(z-ct) is the spacial-time variation of the conductivity.

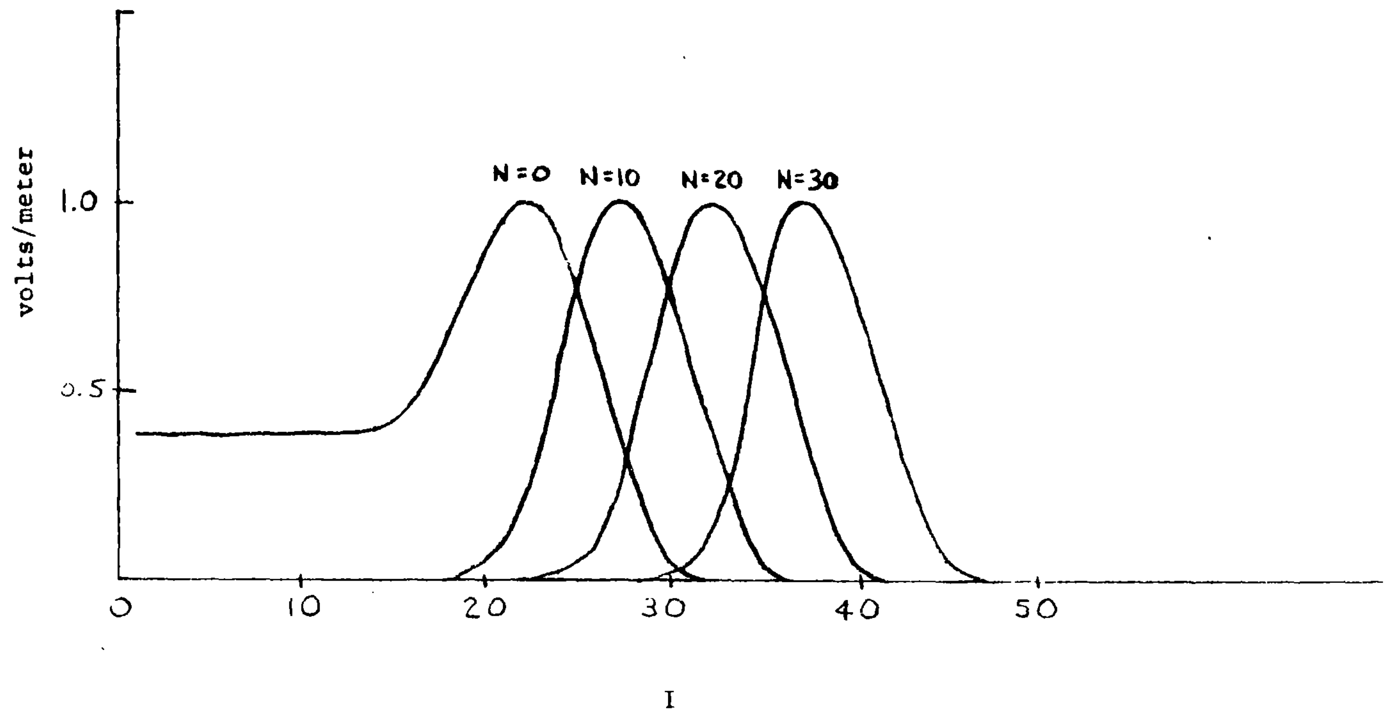

Figure 8: Propagation of the axial component of the electric field at J = 40 in a time-varying, inhomogeneous medium.

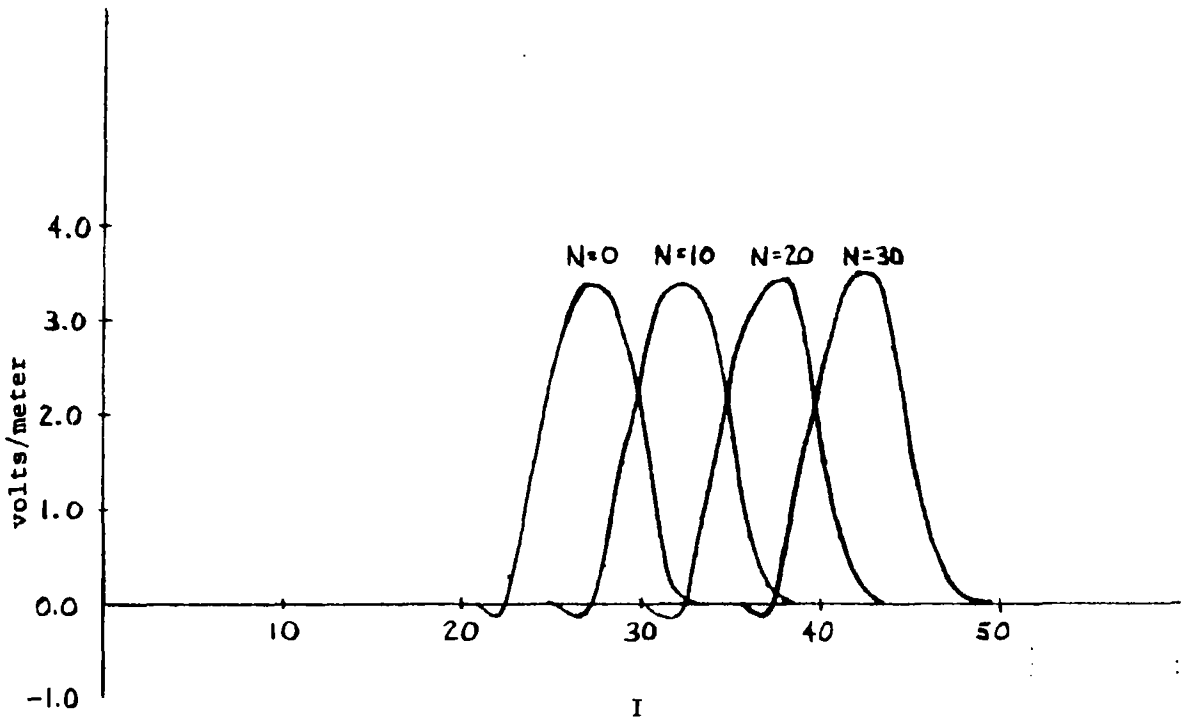

Figure 9: Propagation of the radial component of the electric field at J = 40 in a time-varying inhomogeneous medium. The peak of the incident pulse at N = o is at I = 22.

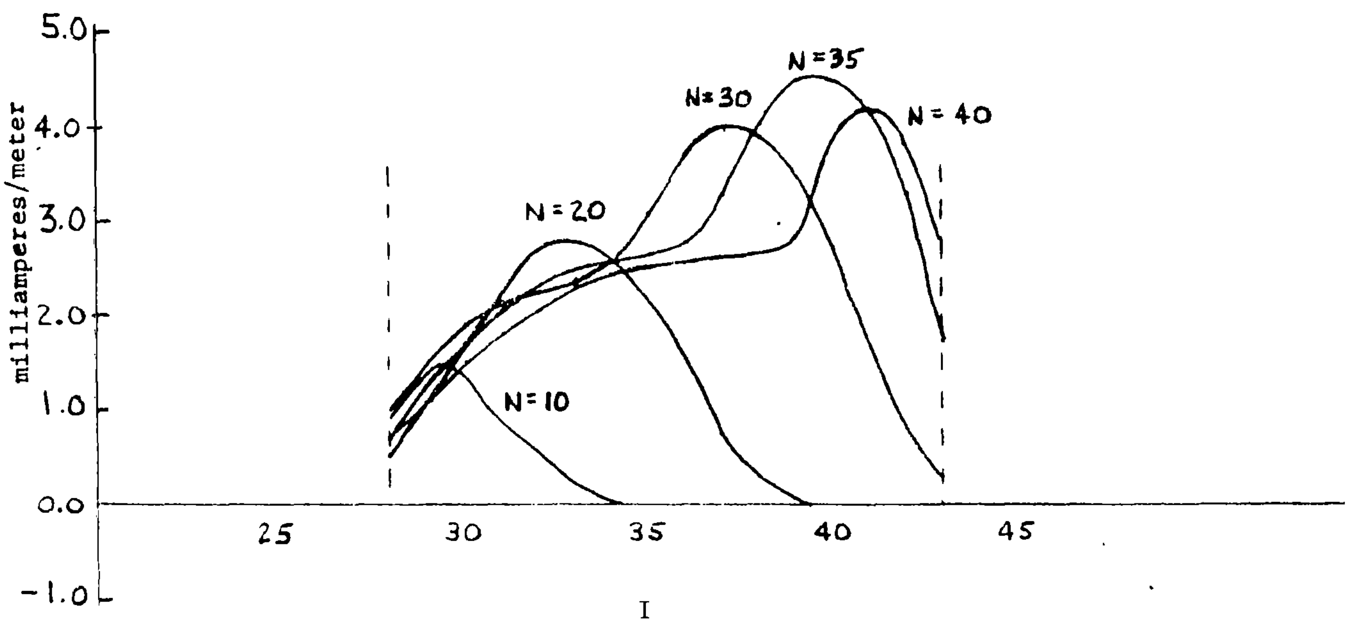

Figure 10: Current distribution induced on the cylindrical scatterer. The region of the scatterer is 28 ≤ I ≤ 43 and 1 ≤ J ≤ 4. The peak of the incident pulse at N = o is at I = 22.

CONCLUSION

A very general technique is presented that treats initial value problems in inhomogeneous, time-varying media. Extensive numerical data which are obtained demonstrate the usefulness and the applicability of the presented solution technique. The results obtained should apply to the problem of electromagnetic scattering from objects immersed in a plasma such as a space capsule during re-entry.

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