SC-R-68-1894

# ON THE RESPONSE OF A MISSILE WITH EXHAUST TRAIL OF TAPERED CONDUCTIVITY TO A PLANE-WAVE ELECTROMAGNETIC FIELD

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December 1968

## ABSTRACT

Using numerical techniques, an integral equation for the current distribution along a missile with plume (ionized trail) is solved when the angle of incidence of the illuminating plane-wave electromagnetic field is arbitrary. In the theory, the conductivity of the plume may taper in any prescribed manner with increasing distance from the exhaust nozzles. A model of cylindrical geometry is assumed.

Missiles of electrical length in the range $1.0 \leq k_o h_m \leq 3\pi/2$ and plume length in the range $1.0 \leq k_o h_m \leq 9\pi/2$ are considered. Two values of the shape parameter $\Omega_m = 2 \ln (2h_m/a)$ are chosen: $\Omega_m = 6$ and $\Omega_m = 10$. Here $k_o$ is the free space wave number, $h_m$ and $h_p$ are the missile and plume lengths, respectively, and a is the radius of both the missile and the plume. The numerical work is carried out at a number of selected frequencies in the range $2.8 \leq f \leq 13.195$ MHz.

A table is provided for the scattering cross sections of missiles with ionized trails when the electric field is directed parallel to the missile — plume axis.

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# ON THE RESPONSE OF A MISSILE WITH EXHAUST TRAIL OF TAPERED CONDUCTIVITY TO A PLANE-WAVE ELECTROMAGNETIC FIELD

## Introduction

The underlying objective of this study is to determine the current distribution along a missile with ionized plume when excited by a plane-wave electromagnetic field. This permits, for example, the currents at a selected point along the missile to be readily compared when the ionized trail is present and when it is absent. Also, a knowledge of the radar cross sections of missiles with plumes is of considerable importance.

The missile is represented in the present theory as an imperfectly conducting, cylindrical, receiving and scattering antenna having a smooth surface; the plume is represented in the same way. Both the missile and plume are assumed to be of the same diameter but not necessarily of the same length.

The current distribution along the missile and plume is obtained in the following way: First the nonhomogeneous differential equation for the vector potential along the missile and its ionized trail is solved. In the solution of the differential equation, a particular integral occurs having the internal impedance per unit length as well as the unknown current distribution under the integral sign. Second, the Helmholtz integral for the vector potential is equated to the solution of the nonhomogeneous differential equation. The resulting expression is then solved for the current distribution subject to appropriate boundary conditions by numerical techniques (piecewise linear zoning).

The scattered field from the missile-plume axis is easily determined numerically as well as the radar cross section of the scattering obstacle.

The scattered field for a DC pulse incident field may be obtained, or the envelope of the reradiated field retrieved for radar pulse excitation.

## Equations for Computing the Current Distribution and Scattering Cross Section of a Missile with Ionized Trail

Consider a missile of length $h_m$, with plume of length $h_p$, both of radius a, as depicted by Figure 1. The origin of a cylindrical system of coordinates is on the missile-plume axis in the midst of the missile propulsion nozzles. The internal impedance per unit length of the missile has the constant value of $z_m^i$. The plume has a variable internal impedance per unit length of $z_p^i(z)$.

The integral equation for the current distribution along the missile and plume is

$$\int_{-h_p}^{h_m} I_z(z')K(z, z')dz' = -j\frac{4\pi}{\zeta_o}\left\{K_1 \cos k_o z + K_2 \sin k_o z + U e^{-jk_o z \cos\theta} - \int_0^z z^i(s)I(s) \sin k_o(z - s)ds\right\}, \quad (1)$$

$z^i(s) = z_p^i(s)$ when $-h_p \leq s \leq 0$; $z^i(s) = z_m^i = \text{const.}$ when $0 \leq s \leq h_m$.

In Equation 1, $U = -E^i \cos\psi/k_o \sin\theta$.[1] When $\psi = 0$ and $\theta = \pi/2$, $U = E_z^i/k_o$, where $E_z^i$ is the incident electric field directed parallel to the axis of the missile and plume, $k_o$ is the free space wave number, and $\zeta_o$ is the characteristic resistance of space.

Introducing the notation $H_m = k_o h_m$, $H_p = k_o h_p$, $Z = k_o z$, and $A = k_o a$, and setting $U = 1$, Equation 1 may be recast into a form suitable for computation. This equation is

$$\int_{-H_p}^{H_m} I\left(\frac{\xi}{k_o}\right) K(Z, \xi) d\xi - j \frac{4\pi}{\zeta_o k_o} \int_0^Z z^i\left(\frac{\xi}{k_o}\right) I\left(\frac{\xi}{k_o}\right) \sin (Z - \xi) d\xi$$

$$= C_1 \cos Z + C_2 \sin Z - j \frac{4\pi}{\zeta_o} e^{jZ \cos \theta} , \tag{2}$$

where

$$K(Z, \xi) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{-jR}}{R} d\eta , \tag{3}$$

and

$$R = \sqrt{\left(2A \sin \frac{\eta}{2}\right)^2 + (Z - \xi)^2} . \tag{4}$$

Equation 2 is solved by a linear piecewise zoning technique.[2] A discussion of this technique for constant $z_p^i$ has been given.[3] In the present application $z_p^i\left(\frac{\xi}{k_o}\right)$ is taken as piecewise linear over each zone.

Evidently the integral equation for the current distribution along a perfectly conducting unloaded receiving and scattering antenna is

$$\int_{-H}^{H} I\left(\frac{\xi}{k_o}\right) K(Z, \xi) d\xi = C_1 \cos Z + C_2 \sin Z - j \frac{4\pi}{\zeta_o} , \tag{5}$$

for U = 1 and $\theta = \pi/2$. To the accuracy of the work reported here, $C_2 \simeq C_1 \times 10^{-9}$, thus forcing the current to zero at each end of the conductor by use of the constants $C_1$ and $C_2$ establishes the fact that the current is an even function when the incident electric field is directed parallel to the axis of the rod ($\theta = \pi/2$). When $\theta \neq \pi/2$, $C_2 \neq 0$. The important point here is that it is unnecessary to solve different integral equations for the even and odd currents separately, and then employ superposition to obtain the total current in the scatterer. The matrix method of solving the problem performs all of these operations in concert. However, from the viewpoint of computer speed and memory size it may be more economical to solve two integral equations each over an interval H than one over the interval 2H.

The radar cross section of the scattering obstacle for parallel polarization of the electric field ($\theta = \pi/2$, $\psi = 0$) is defined by the relation

$$\sigma_{||} = 4\pi R^2 \left| \frac{E^r}{E^i} \right|^2 . \tag{6}$$

The magnitude of the reradiated far-zone electric field $E^r$, again directed parallel to the axis of the scatterer is given by the formula

$$|E_r| = \left| \frac{k_o \zeta_o}{4\pi R} \int_{-h_p}^{h_m} I_z(z')dz' \right| . \tag{7}$$

Substituting Equation 7 into Equation 6, remembering that U = 1 so that $E_z^i = k_o$, the formula for the normalized monostatic radar cross section $\sigma_{||}/\lambda_o^2$ of the missile plume configuration in a form suitable for computation is obtained. It is

$$\frac{\sigma_{||}}{\lambda_o^2} = \frac{\zeta_o^2}{16\pi^3} \left| \int_{-H_p}^{H_m} I\left(\frac{\xi}{k_o}\right) d\xi \right|^2 . \tag{8}$$

## Conductivity Profiles and Formulas for the Internal Impedance per Unit Length of an Ionized Trail

Some plume conductivity profiles of interest are

$$\sigma(z) = \sigma(-h_p) + \left[\sigma(0) - \sigma(-h_p)\right] \sin^2\left\{\frac{\pi}{2}\left[\frac{e^{\frac{\nu(h_p+z)}{h_p}} - 1}{e^{\nu} - 1}\right]\right\} , \tag{9}$$

$$\sigma(z) = \sigma(-h_p) - \left[\sigma(-h_p) - \sigma(0)\right]\left|\frac{z + h_p}{h_p}\right|^{\nu} . \tag{10}$$

In these expressions $-h_p \leq z \leq 0$ and $\nu > 0$ but need not be an integer. Figures 2 and 3 present plots of Equations 9 and 10, respectively. Evidently $\sigma(0) = 3.54 \times 10^7$ mhos/m (aluminum) and $\sigma(-h_p) = 10$ mhos/m. Also $h_p = 17$ meters in these figures, but the same curves are obtained for $h_p = 17n$ meters, where n is an integer.

The exponential conductivity profile is given by the relation

$$\sigma(z) = \sigma(0)\ e^{\nu z} \qquad (-h_p \leq z \leq 0) . \tag{11}$$

Evidently fixing the values of $\sigma(0)$ and $\sigma(-h_p)$ determines $\nu$. Equation 11 was used in the numerical work reported in this paper.

The complete formula for the internal impedance per unit length of a solid cylindrical conductor is

$$z^i = \frac{k}{2\pi a(\sigma + j\omega\epsilon)} \frac{J_o(ka)}{J_1(ka)} , \tag{12}$$

subject to the inequalities $h \gg a$ and $|k| \gg k_o$. Here

$$k = \sqrt{\omega^2 \mu \epsilon - j\sigma\omega\mu_o} \quad . \tag{13}$$

In Equation 13 $\epsilon = \epsilon_o \epsilon_r = 8.85\, \epsilon_r \times 10^{-12}$ farads/m, and $\mu_o = 4\pi \times 10^{-7}$ henry/m.

The parameters $\sigma$ and $\epsilon$ may be functions of z provided $\sigma(z)$ and $\epsilon(z)$ do not change too rapidly with z. Additionally, if $\sigma \gg \omega\epsilon$, one may write

$$z^i(z) = \frac{k(z) J_o[k(z)a]}{2\pi a \sigma(z) J_1[k(z)a]} , \tag{14}$$

with

$$k(z) = (1 - j)\sqrt{\frac{\omega\mu_o \sigma(z)}{2}} \quad . \tag{15}$$

If desired the ionized trail may be considered a thin-wall tubular conductor. In this case[4]

$$z^i(z) = \frac{1}{2\pi a \sigma(z) t} \quad . \tag{16}$$

Here a, the missile and plume radius, is determined from the relation

$$\Omega_m = 2 \ln\left(\frac{2h_m}{a}\right) , \tag{17}$$

and the wall thickness t should be computed at the point of greatest plume conductivity by setting

$$t = 0.1 d_s \, , \tag{18}$$

where $d_s$, the skin depth, is given by

$$d_s = \sqrt{\frac{2}{\omega \mu_0 \sigma_{max}(z)}} \; . \tag{19}$$

It is of importance to observe that Equation 16 is frequency independent.

## Current Distributions Along Missiles and Plumes

A number of curves are provided for the current distributions along missiles and plumes of various lengths at selected frequencies. These are exhibited in Figure 4. By definition, $\Omega_m = 2 \ln (2h_m/a)$. In the numerical work, $\Omega_m = 6.0$ or $10.0$. At any specified value of $k_o z$ the total current is found by taking the square root of the sum of the squares of the real and imaginary parts of $I_z(z)$. In four drawings the effect on the current distributions brought about by allowing $\theta$ to change, is shown. In the table some monostatic scattering cross sections for missiles with ionized trails are presented.

## Concluding Remarks

Readers are reminded that in all of the numerical work presented in this paper $\sigma(-h_p) = 10\omega\epsilon_o$. Truncation of the plume at some other conductivity level may make a substantial difference in the current distributions and scattering cross sections of missiles with ionized trails.

The turbulence generated in the plume by the shock wave, the degree of inhomogeneity in the ionization of the plume in the radial direction is ignored, as well as discontinuities in the surface of the ionized trail that may give rise to reflections in the current flowing along the plume and missile, i.e., bring about partial resonances at certain frequencies. The supposition is made that the hypothetical missile with ionized trail applies during the missile launch phase while the vehicle is at relatively low altitudes and fuel burning is in progress.

At low altitudes electromagnetic coupling exists between the missile-plume and its collinear image. Evidently, at early times the plume even makes contact with the earth. (At broadcast frequencies and longer wavelengths, when the earth acts essentially like a perfect conductor, the total length of the scatterer is doubled when the image is taken into account.) These effects are ignored in the theoretical development. This is tantamount to making the assumption that immediately following launch the missile-ionized trail configuration is in free flight, and all other conductors and dielectrics in the universe are in the far zone.

It appears that the maximum enhancement of missile current would occur when the plume is very highly conducting and of sufficient length to effect resonance at the frequency of the incident electromagnetic field. If the over-all length of the missile and its plume is less than one-half wavelength at the frequency of the incident field, as the length of the plume or its conductivity is decreased, one would expect the amplitude of the missile current distribution to decrease. Thus the plume may be regarded as a missile tuning element. And even if the electrical properties of the plume should turn out to be those of an essentially pure dielectric at some given frequency (an unlikely circumstance), it is still possible that the missile current might be increased somewhat over the current that would exist if the plume were absent. This is true because a dielectric rod may also act as a tuning element. If the plume is considered to possess homogeneous electrical properties, have no discontinuities in its surface, and to be infinitely long, no resonances are possible. Particular attention is invited to the fact that if a reactive junction impedance is developed between the missile and plume, resonances can be obtained for missiles and plumes of almost any length. Also, for nonwavey surface (i.e., not corrugated) plumes of finite length having appropriately tapered conductivity resonances can be eliminated.

Evidently, the current at the center of the missile when the plume is considered to be perfectly conducting will be larger in magnitude than the resulting current at the same point when the plume has some uniform finite conductivity. Also, the magnitude of this current will be even smaller if the conductivity decreases along the plume axis rather than remains constant.

In this study the missile is considered to be made of commercial grade aluminum so that the internal impedance per unit length along the missile is constant, and is essentially of zero value. It is assumed that immediately to the rear of the exhaust nozzles the conductivity of the gases equals that of the missile and decreases in a prescribed manner with distance from the nozzles. Accordingly, no charge builds up at the junction of the missile and plume.

Figures 2 and 3 are plots of Equations 9 and 10, respectively. Although these conductivity profiles are not used in the present study, they may be of interest to the reader as an aid in further missile-plume studies.

# TABLE

## Monostatic Scattering Cross Section of a Missile with Ionized Trail for Parallel Incidence of the Electric Field

| $k_o h_m$ | $k_o h_p$ | $\Omega_m$ | f (MHz) | $\sigma_{\|\|}/\lambda_o^2$ | $k_o h_m$ | $k_o h_p$ | $\Omega_m$ | f (MHz) | $\sigma_{\|\|}/\lambda_o^2$ |
|---|---|---|---|---|---|---|---|---|---|
| 1.0 | 3.0 | 10 | 2.8 | 0.20094 | 1.0 | 3.0 | 6 | 2.8 | 0.27755 |
| 1.0 | 2.0 | 10 | 2.8 | 0.14862 | 1.0 | 2.0 | 6 | 2.8 | 0.51764 |
| 1.0 | 1.0 | 10 | 2.8 | 0.00441 | 1.0 | 1.0 | 6 | 2.8 | 0.03552 |
| 1.0 | 0.5 | 10 | 2.8 | 0.00067 | 1.0 | 0.5 | 6 | 2.8 | 0.00426 |
| $\pi/2$ | $3\pi/2$ | 10 | 4.398 | 0.12193 | $\pi/2$ | $3\pi/2$ | 6 | 4.398 | 0.30534 |
| $\pi/2$ | $\pi$ | 10 | 4.398 | 0.14297 | $\pi/2$ | $\pi$ | 6 | 4.398 | 0.30131 |
| $\pi/2$ | $\pi/2$ | 10 | 4.398 | 0.41608 | $\pi/2$ | $\pi/2$ | 6 | 4.398 | 0.56174 |
| $\pi/2$ | $\pi/4$ | 10 | 4.398 | 0.03286 | $\pi/2$ | $\pi/4$ | 6 | 4.398 | 0.27105 |
| 2.0 | 6.0 | 10 | 5.6 | 0.13568 | 2.0 | 6.0 | 6 | 5.6 | 0.38740 |
| 2.0 | 4.0 | 10 | 5.6 | 0.13427 | 2.0 | 4.0 | 6 | 5.6 | 0.34921 |
| 2.0 | 2.0 | 10 | 5.6 | 0.20324 | 2.0 | 2.0 | 6 | 5.6 | 0.38019 |
| 2.0 | 1.0 | 10 | 5.6 | 0.51416 | 2.0 | 1.0 | 6 | 5.6 | 0.68848 |
| $\pi$ | $3\pi$ | 10 | 8.796 | 0.62829 | $\pi$ | $3\pi$ | 6 | 8.796 | 1.8698 |
| $\pi$ | $2\pi$ | 10 | 8.796 | 0.20181 | $\pi$ | $2\pi$ | 6 | 8.796 | 1.5535 |
| $\pi$ | $\pi$ | 10 | 8.796 | 0.16260 | $\pi$ | $\pi$ | 6 | 8.796 | 0.46941 |
| $\pi$ | $\pi/2$ | 10 | 8.796 | 0.17400 | $\pi$ | $\pi/2$ | 6 | 8.796 | 0.45255 |
| $3\pi/2$ | $9\pi/2$ | 10 | 13.195 | 1.6044 | $3\pi/2$ | $9\pi/2$ | 6 | 13.195 | 5.6192 |
| $3\pi/2$ | $3\pi$ | 10 | 13.195 | 0.79039 | $3\pi/2$ | $3\pi$ | 6 | 13.195 | 2.9459 |
| $3\pi/2$ | $3\pi/2$ | 10 | 13.195 | 0.38238 | $3\pi/2$ | $3\pi/2$ | 6 | 13.195 | 2.2283 |
| $3\pi/2$ | $3\pi/4$ | 10 | 13.195 | 0.19663 | $3\pi/2$ | $3\pi/4$ | 6 | 13.195 | 0.63343 |

NOTES:

1. The conductivity of the ionized trail is exponentially tapered.
2. The conductivity of the missile and the starting point of the plume is $\sigma(0) = 3.54 \times 10^7$ mhos/m.
3. The conductivity at the point of truncation of plume is $\sigma(-h_p) = 10\omega\epsilon_o$ where $\omega = 2\pi f$ and $\epsilon_o = 8.85 \times 10^{-12}$ farads/m. Hence, $\sigma(-h_p) < \sigma(0)$. $\lambda_o$ is the free-space wavelength.
4. The internal impedance per unit length $z^i$ is evaluated at each zone boundary from the exponentially decaying $\sigma(z)$. The function $z^i(z)$ is then taken to be piecewise linear across each zone in the integration.
5. The defining relationship for $z^i$ implies that the missile and plume are both solid conductors.
6. $\sigma_{||}$ is the scattering cross section for parallel incidence of the electric field; $\lambda_o$ is the free-space wavelength.

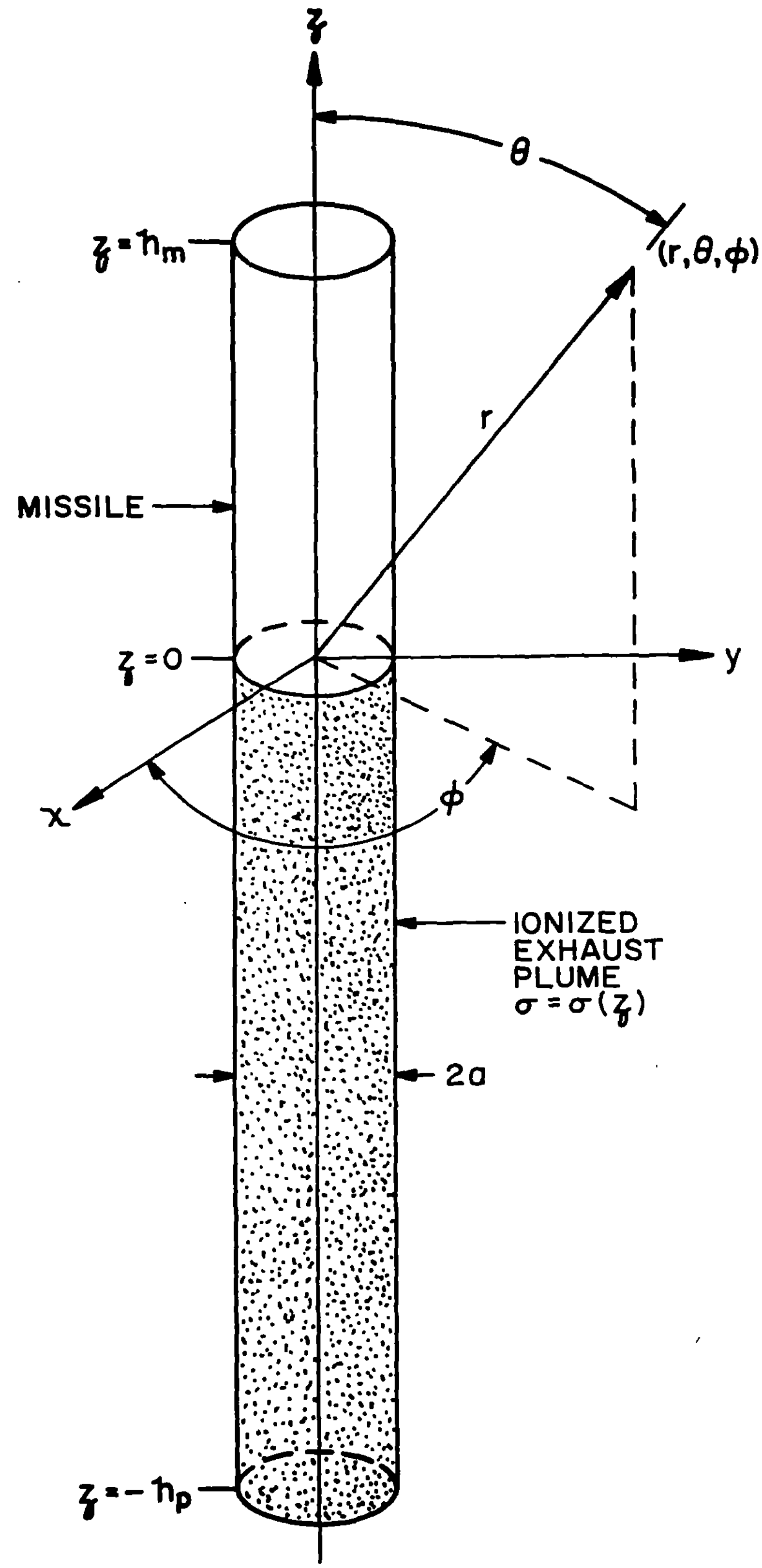



Figure 1. Missile with its Ionized Exhaust Plume Represented by an Inhomogeneous Circular Cylinder of Length $h_m + h_p$

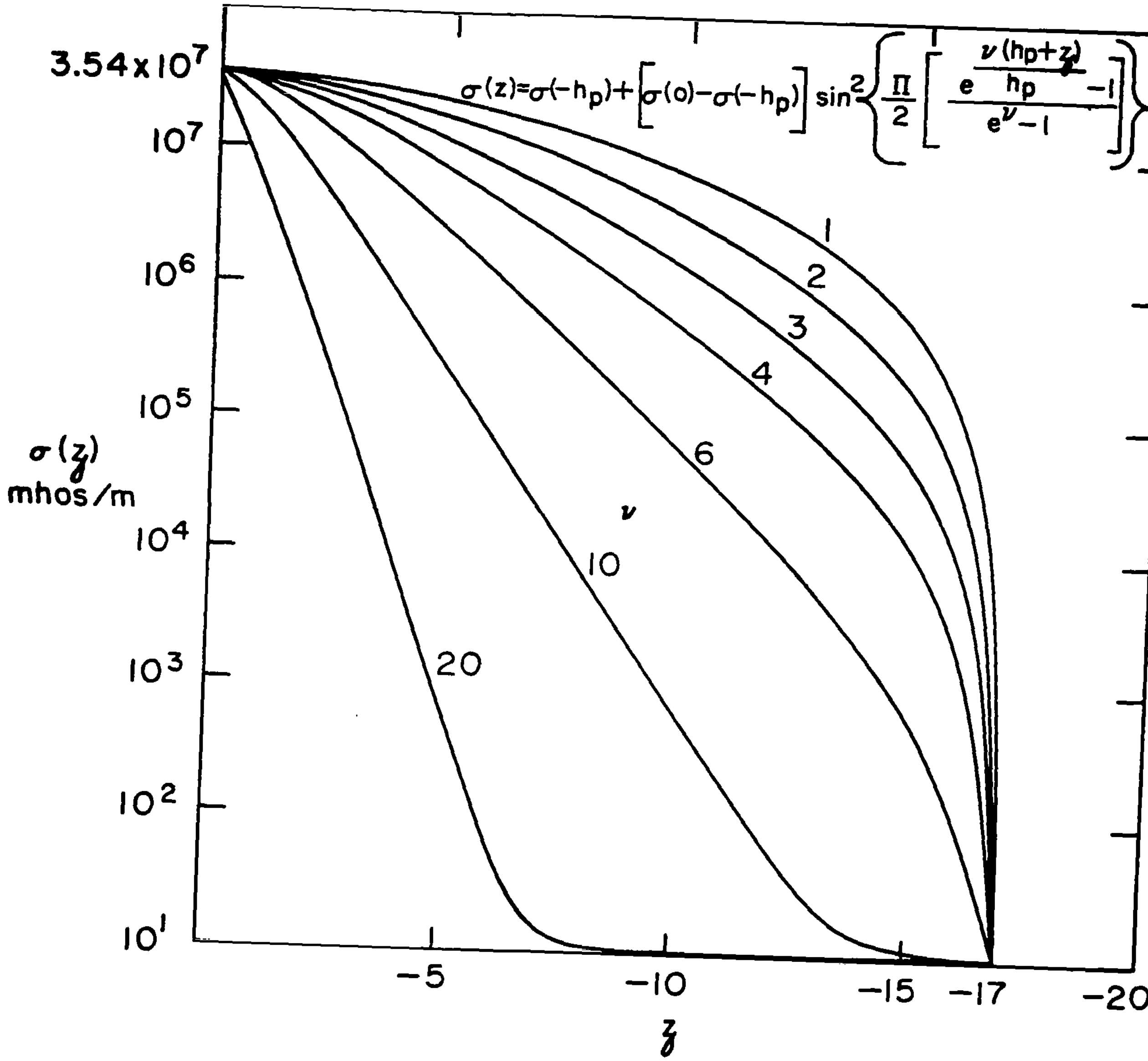



Figure 2. Ionized Trail Conductivity Profiles

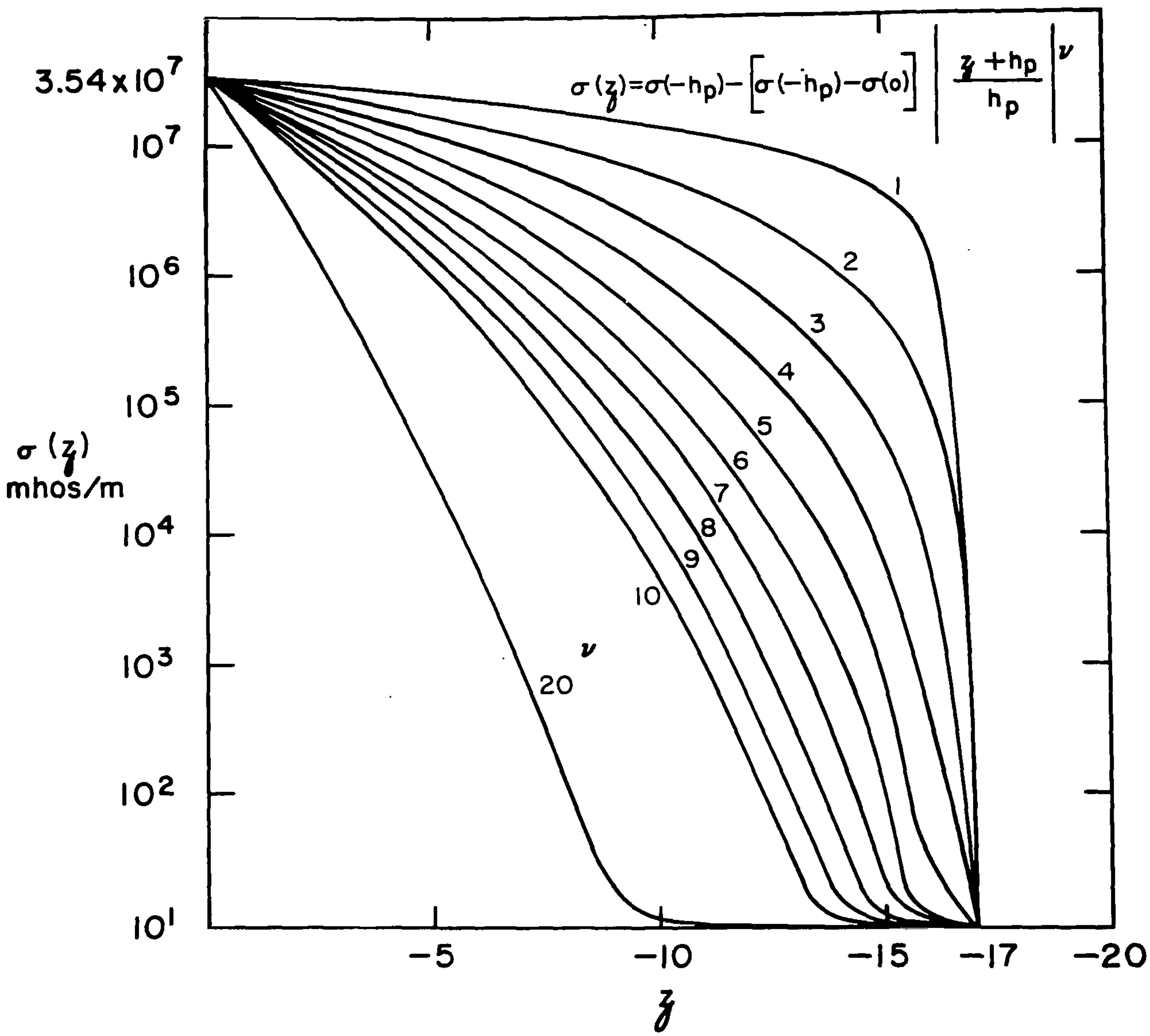



Figure 3. Ionized Trail Conductivity Profiles

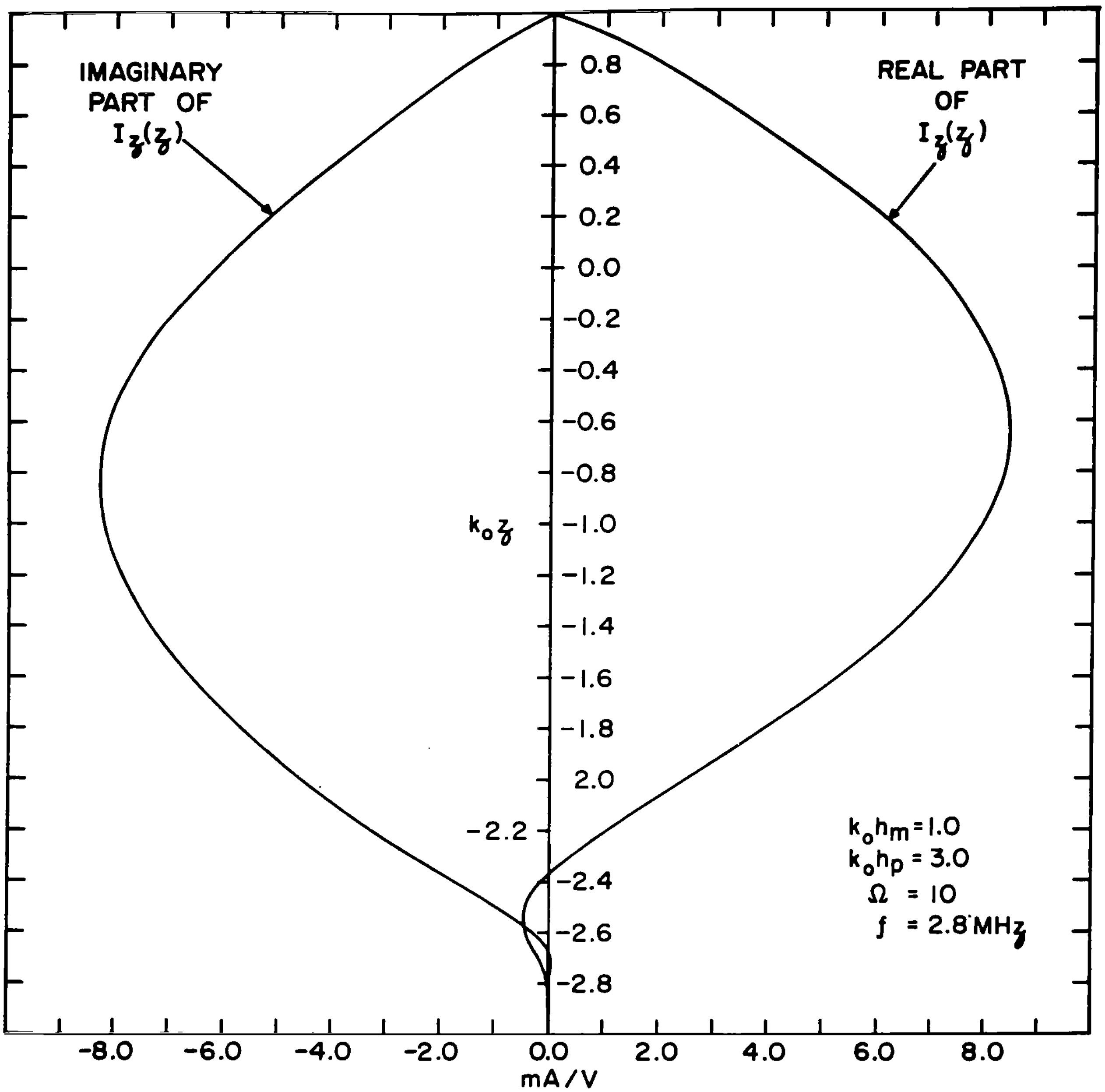
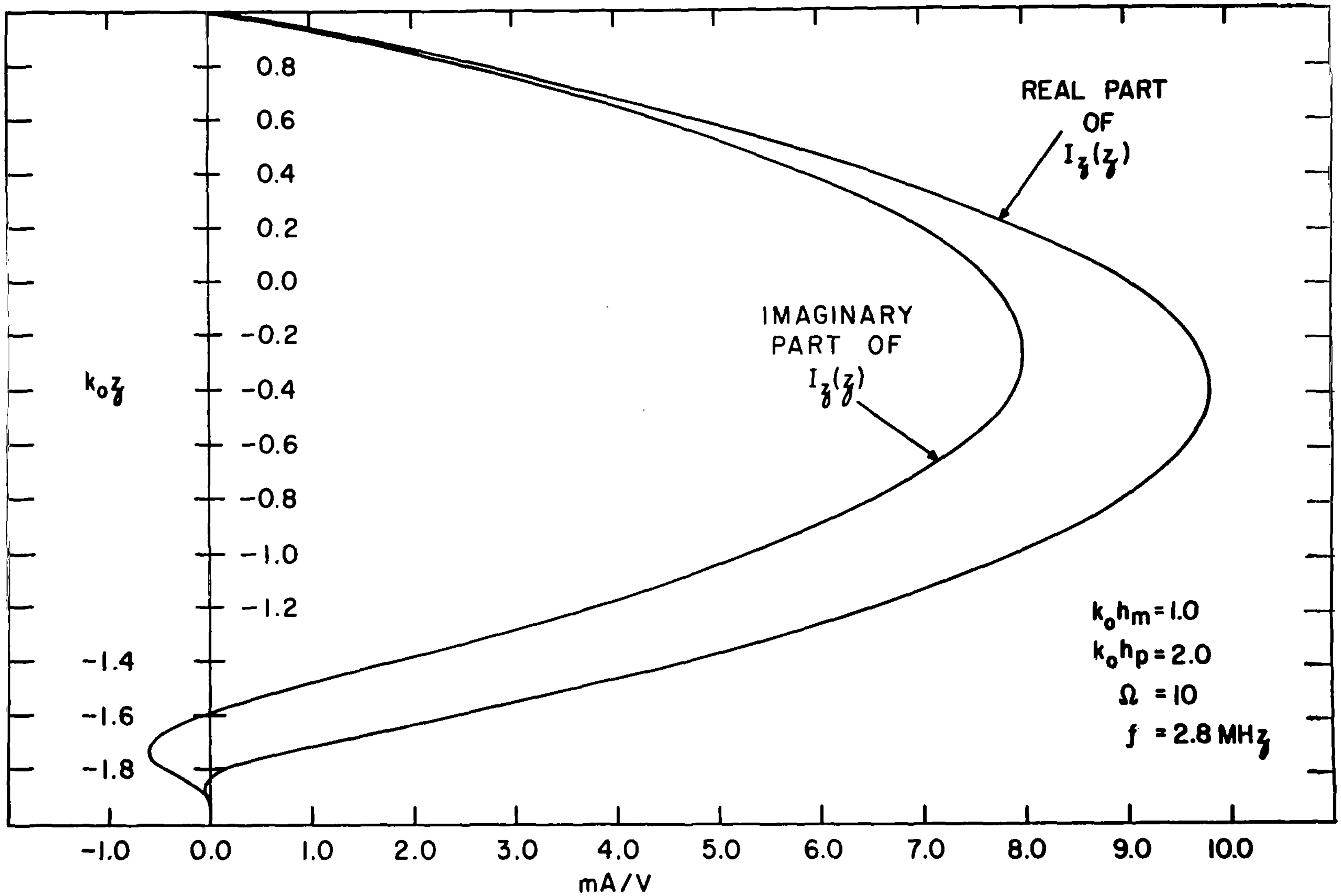



Figure 4. Curves for the Current Distributions Along Missiles with Ionized Trails of Exponential Taper

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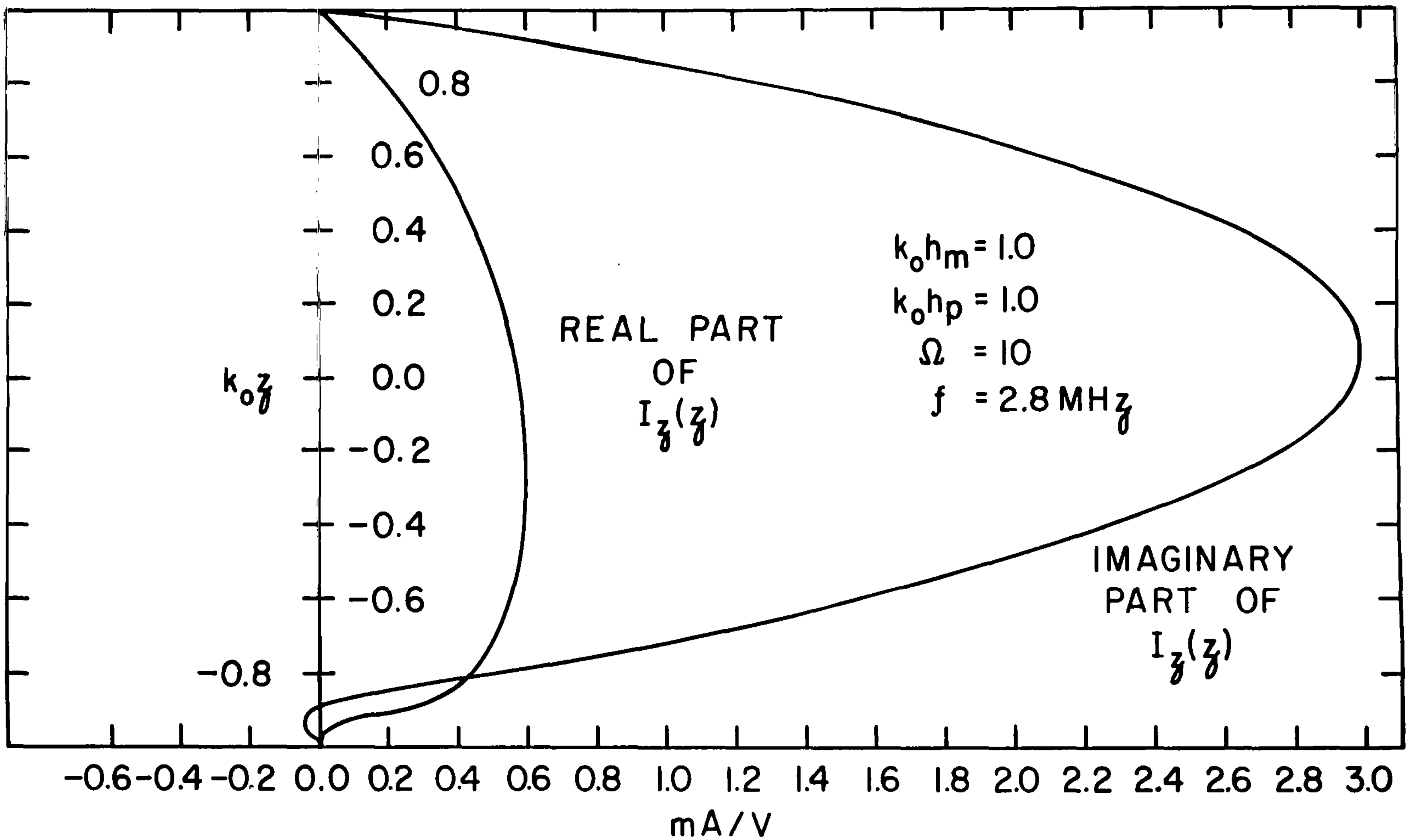


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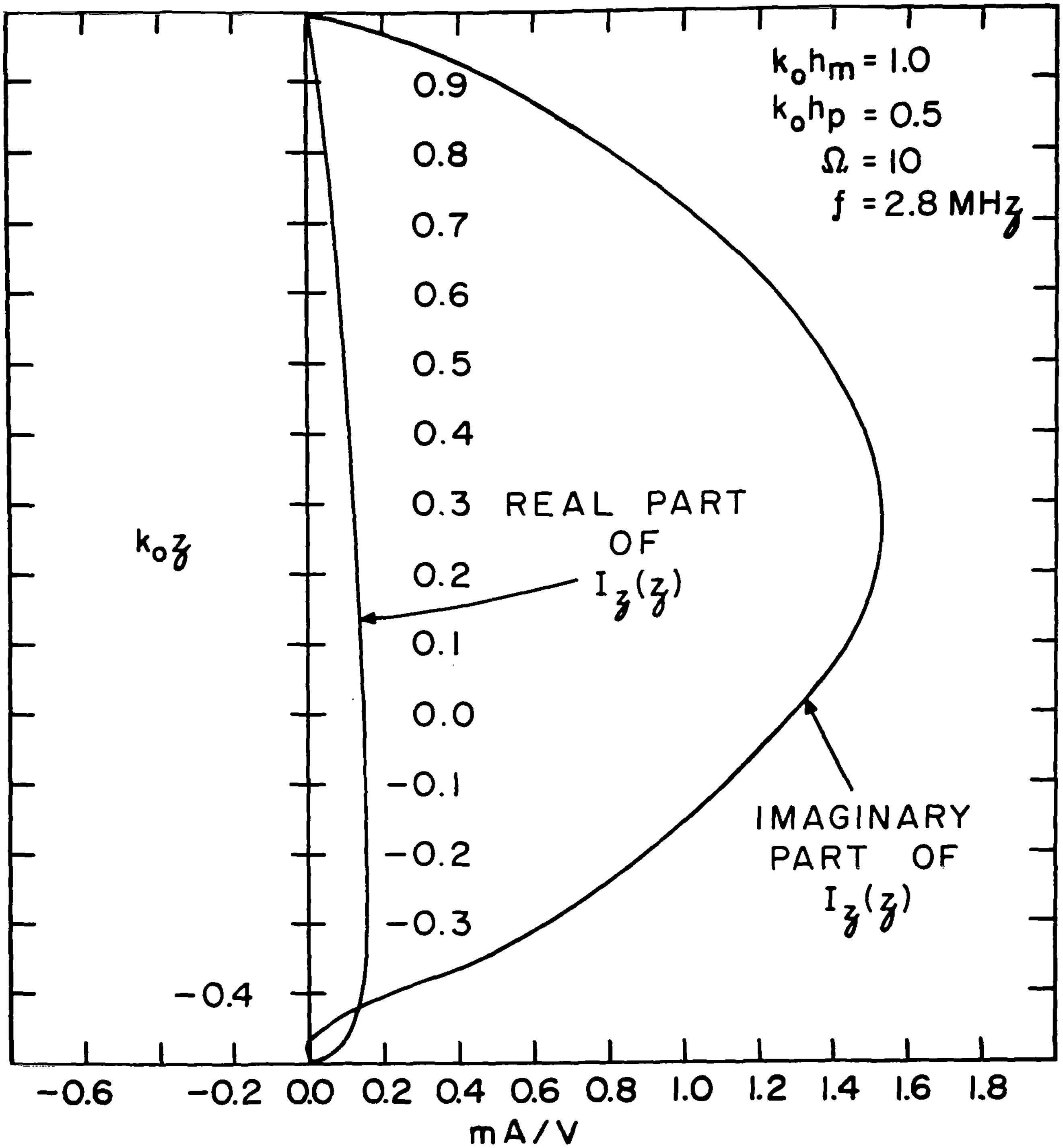


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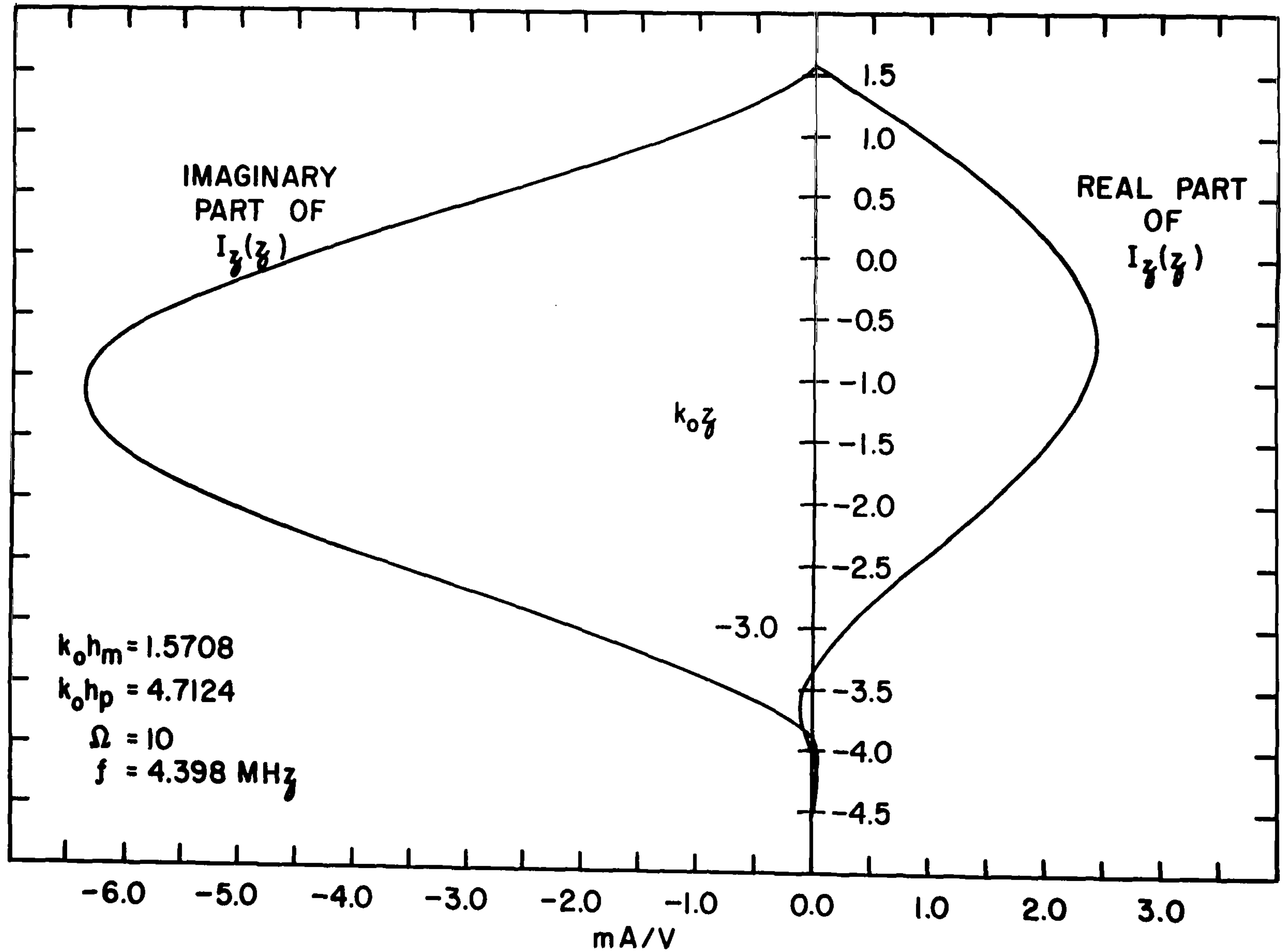


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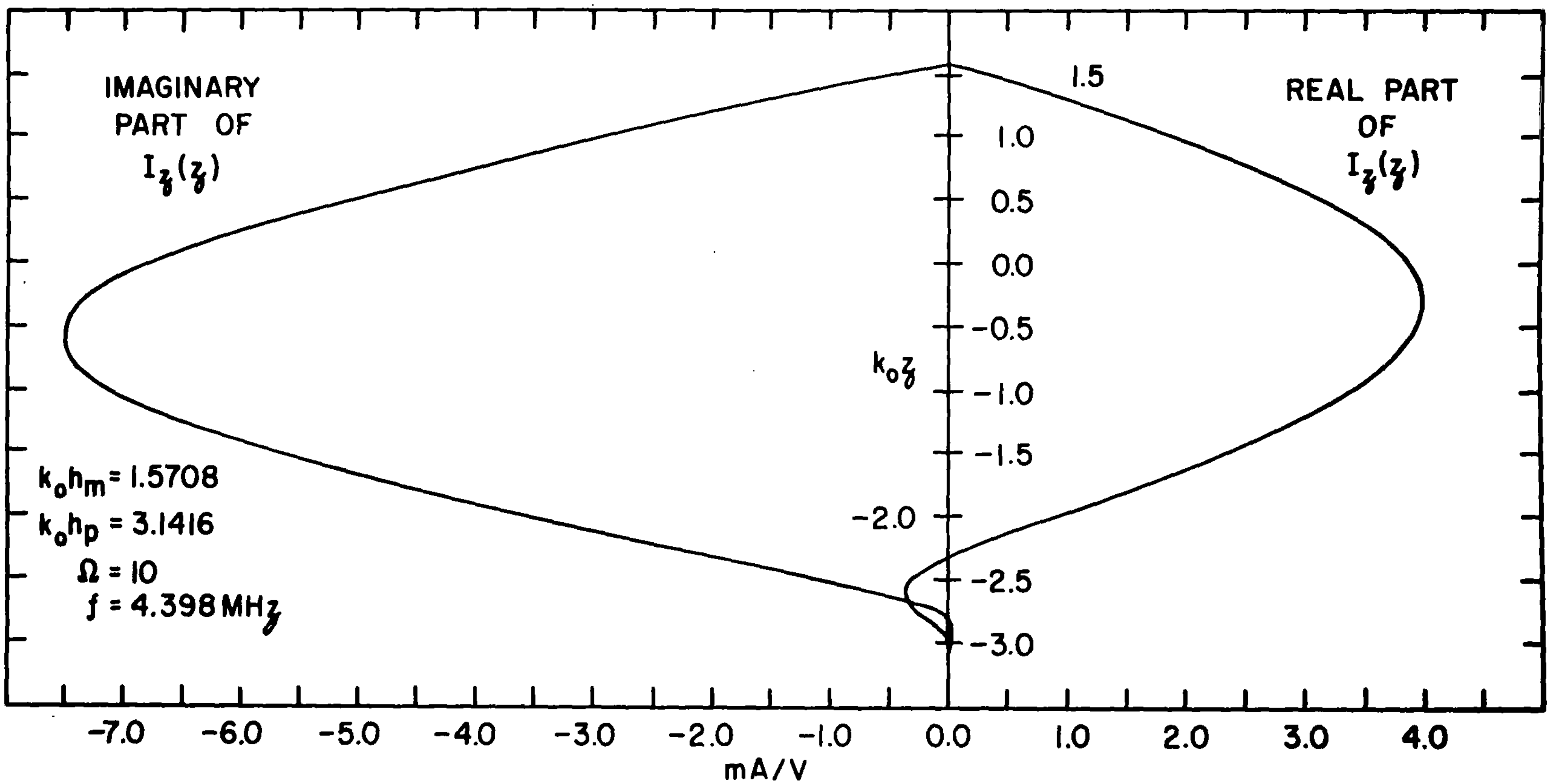


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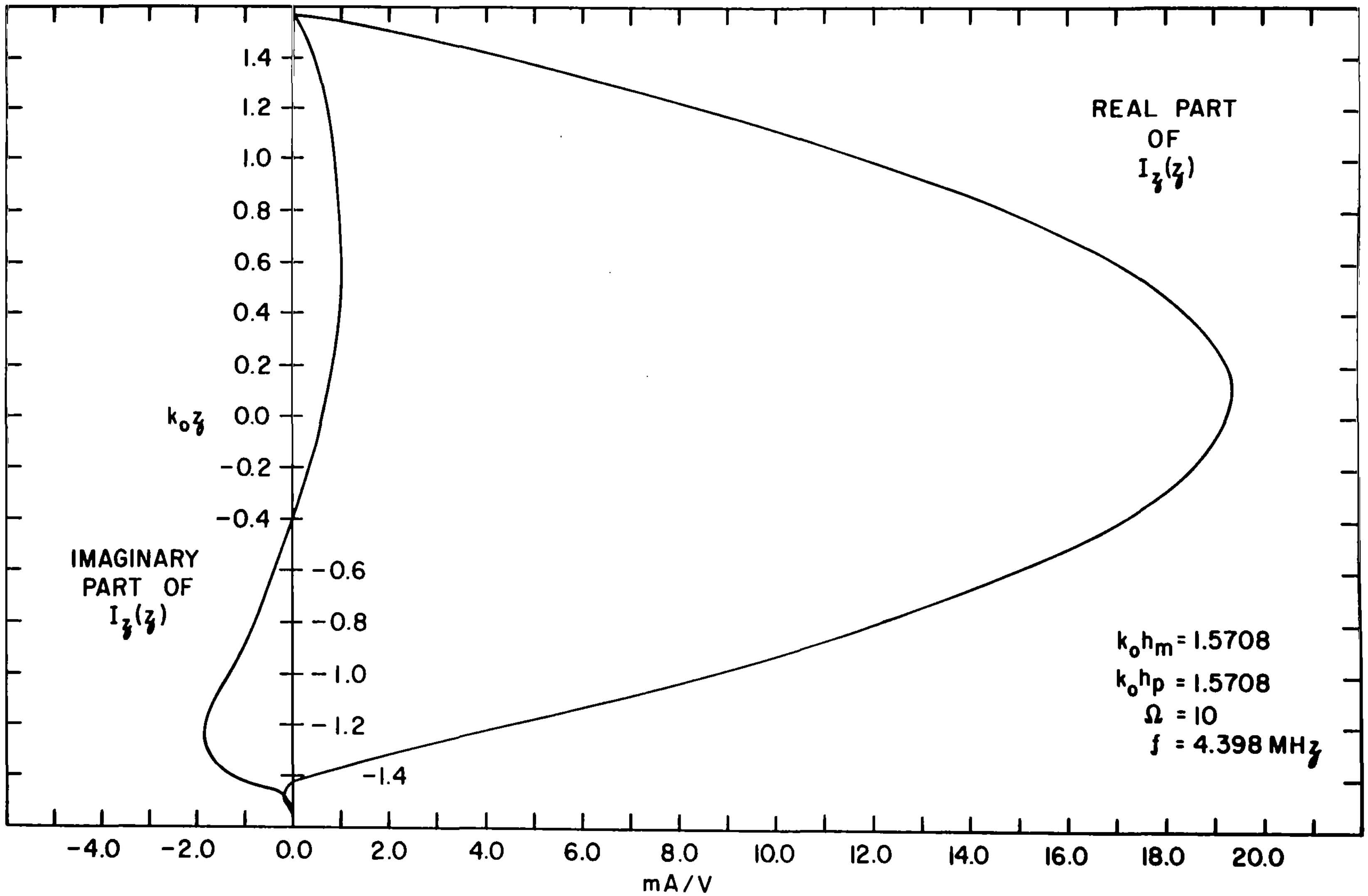


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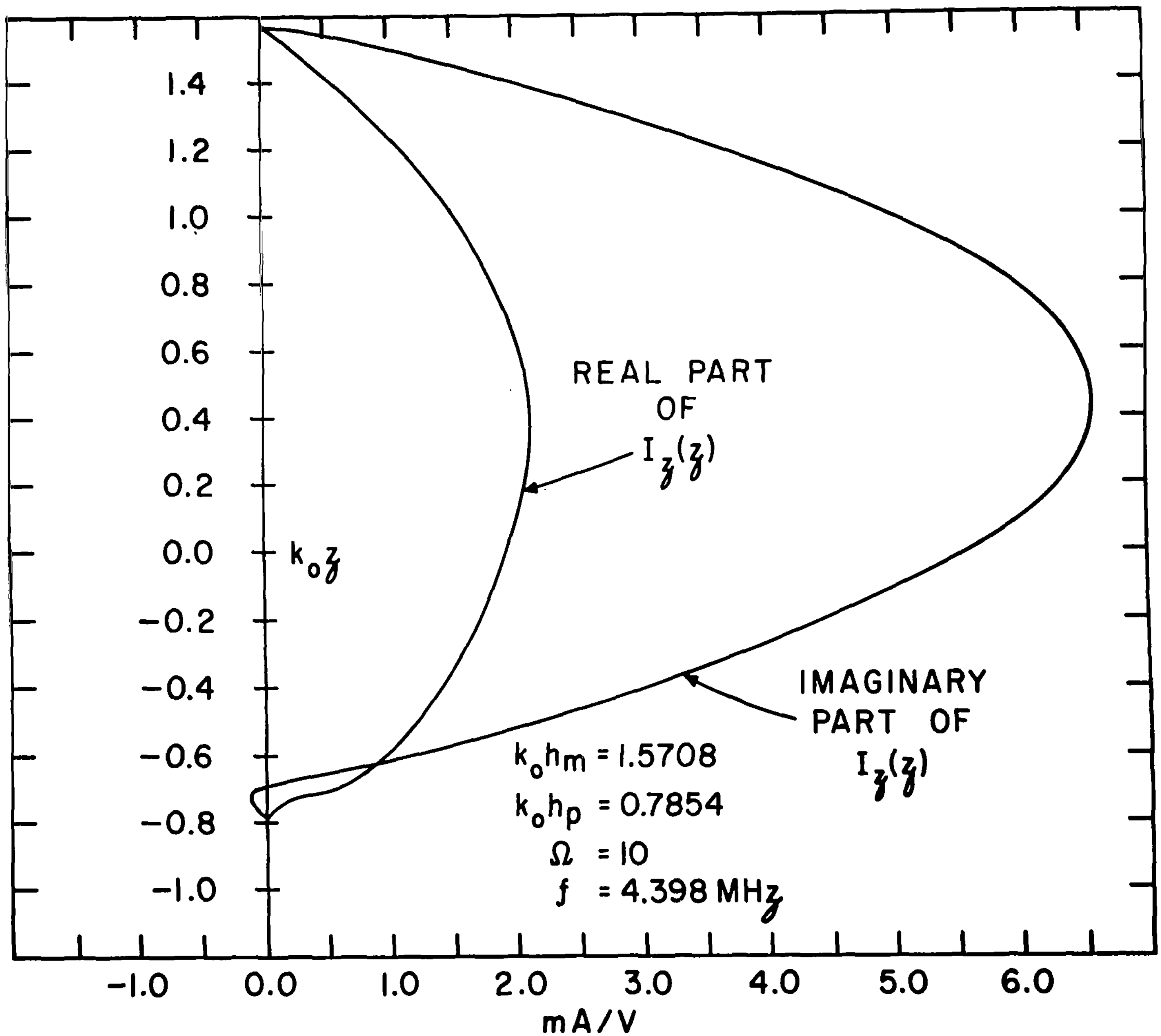


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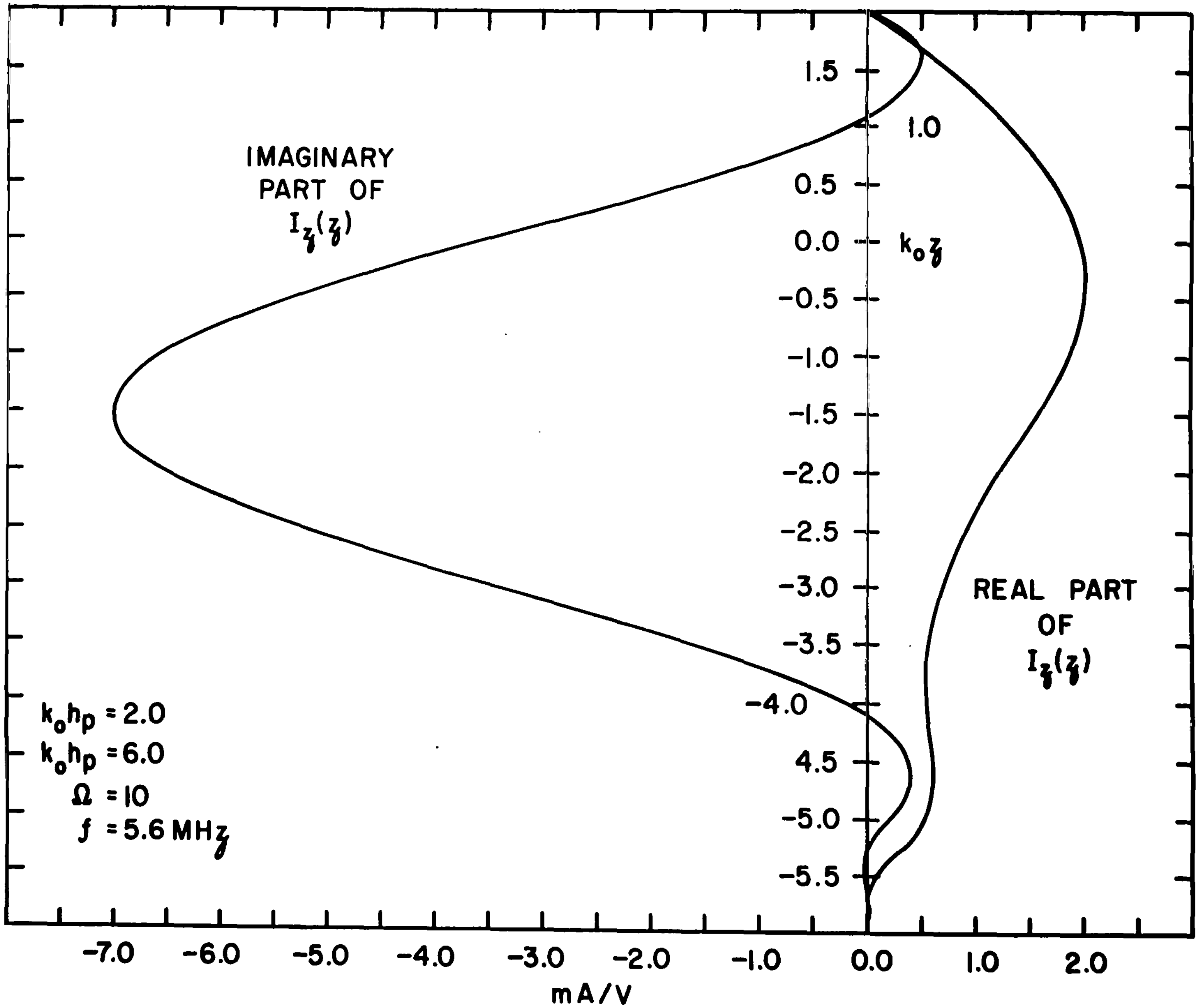


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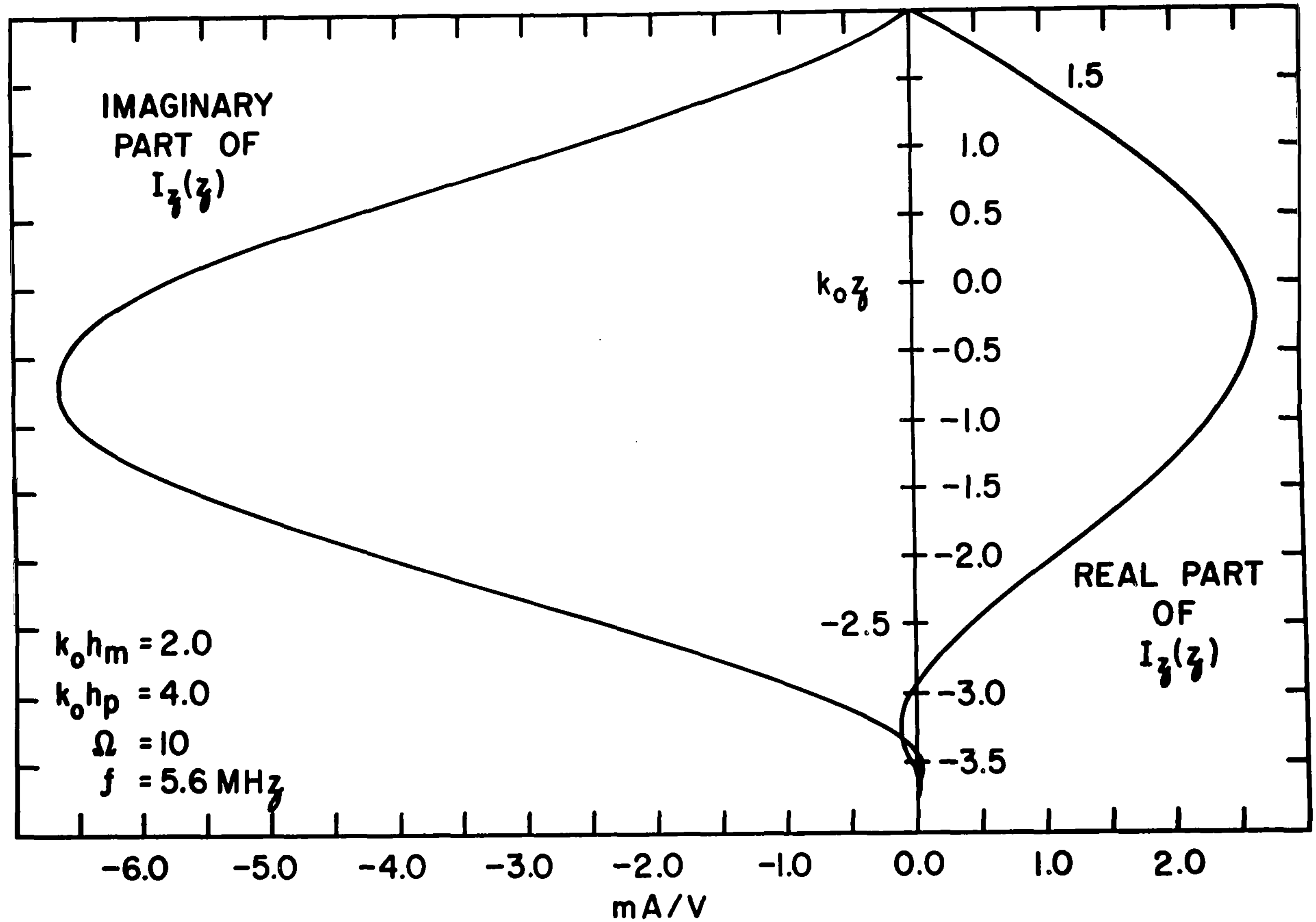


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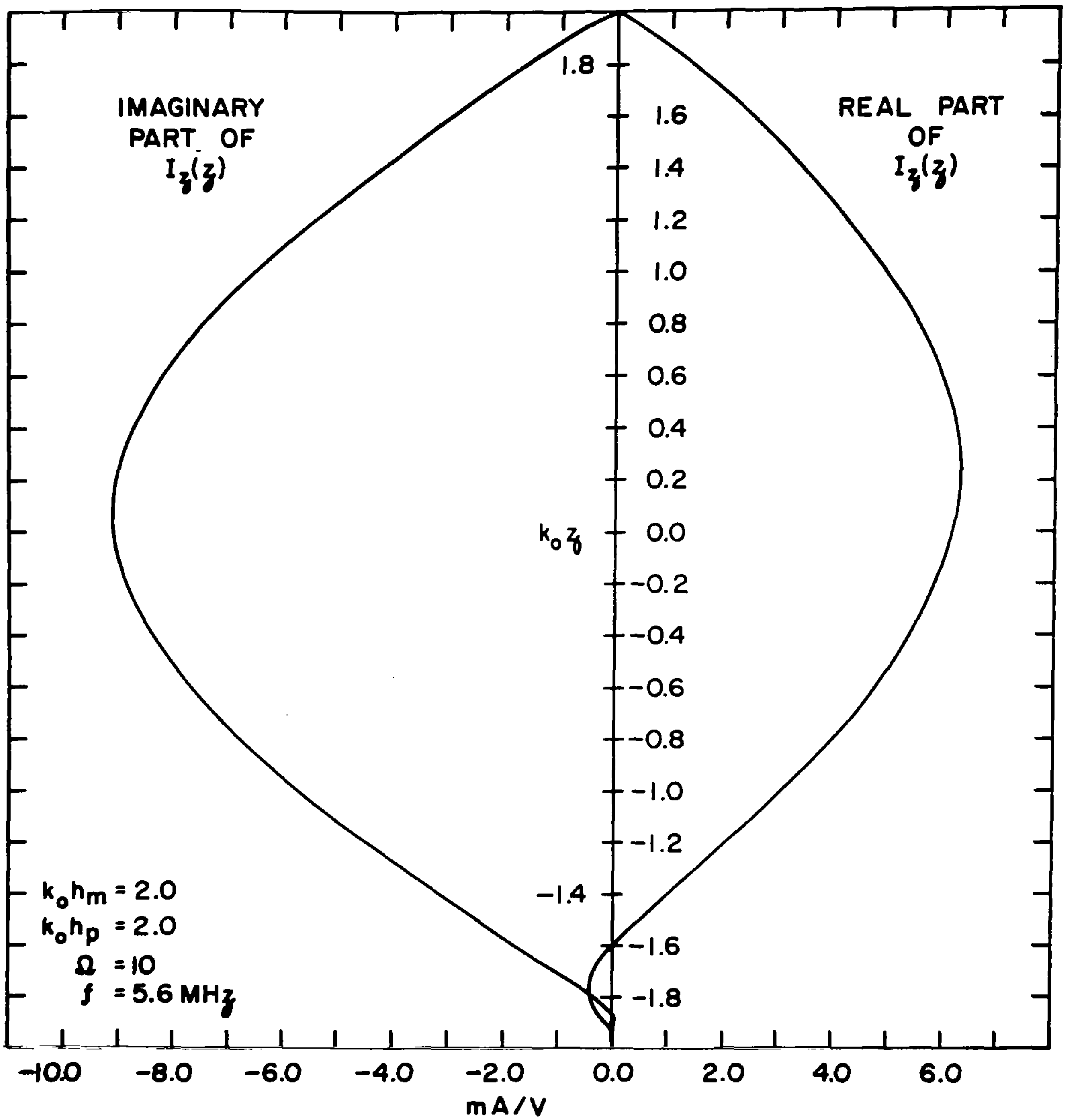


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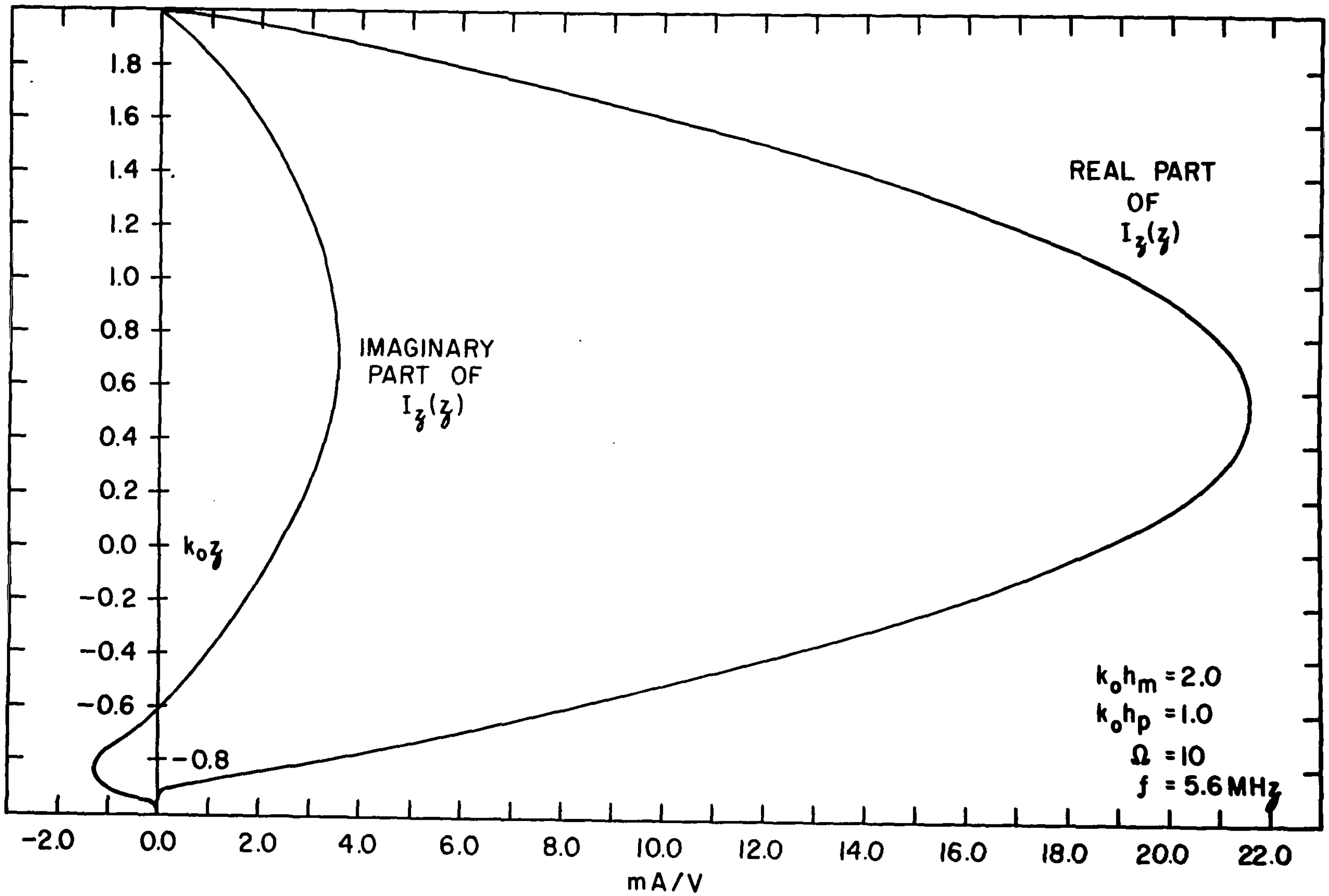


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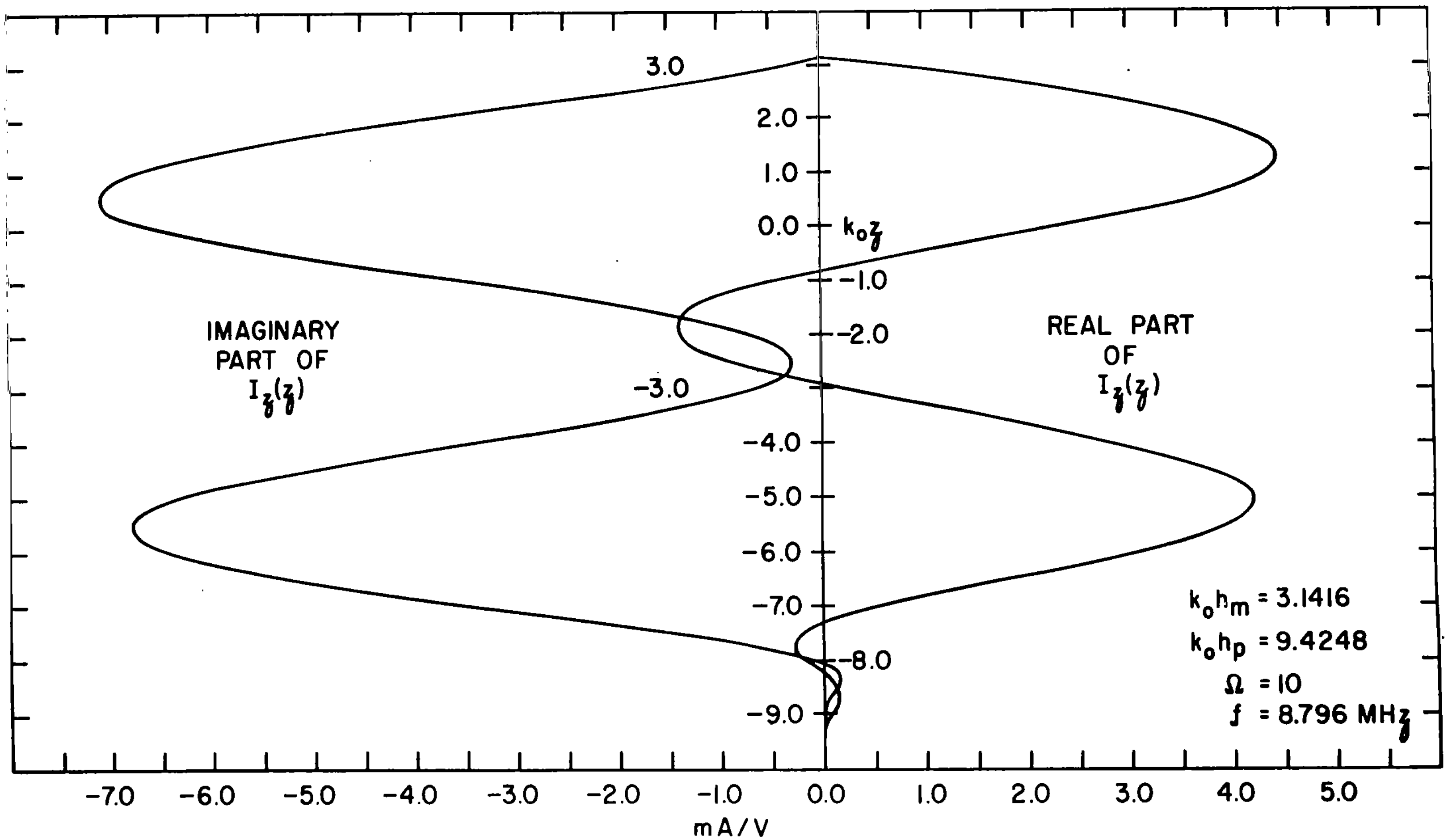


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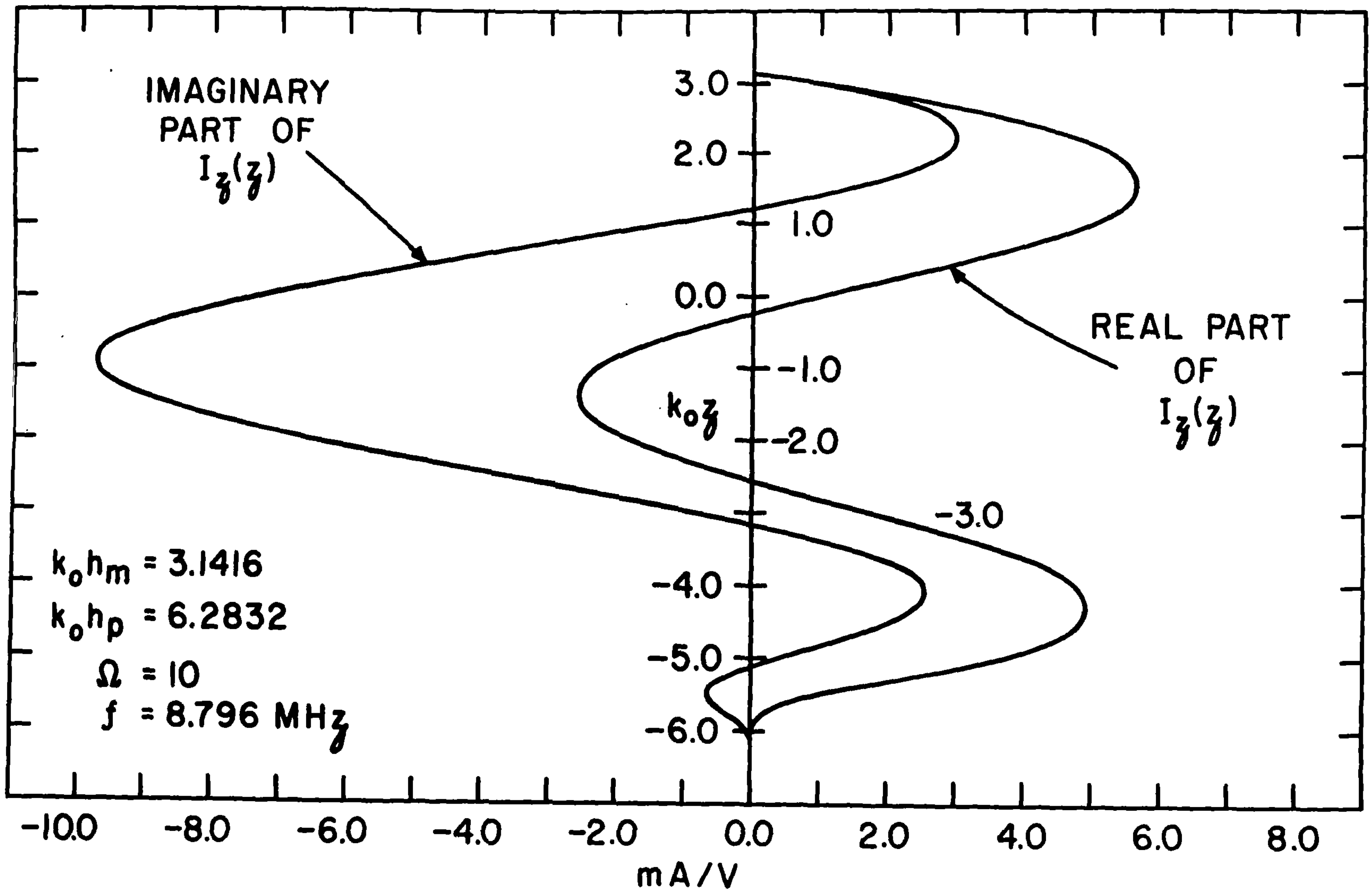


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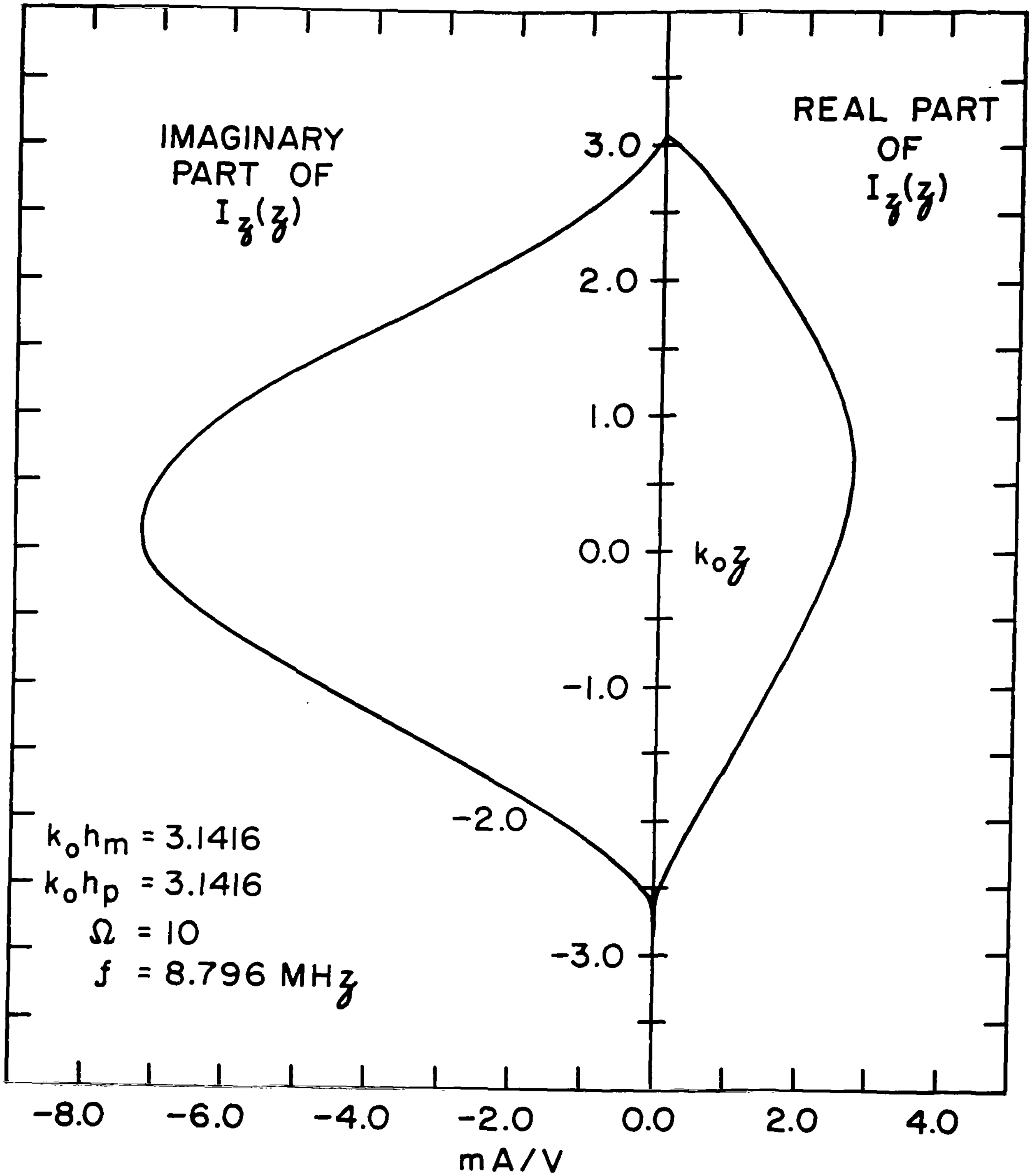


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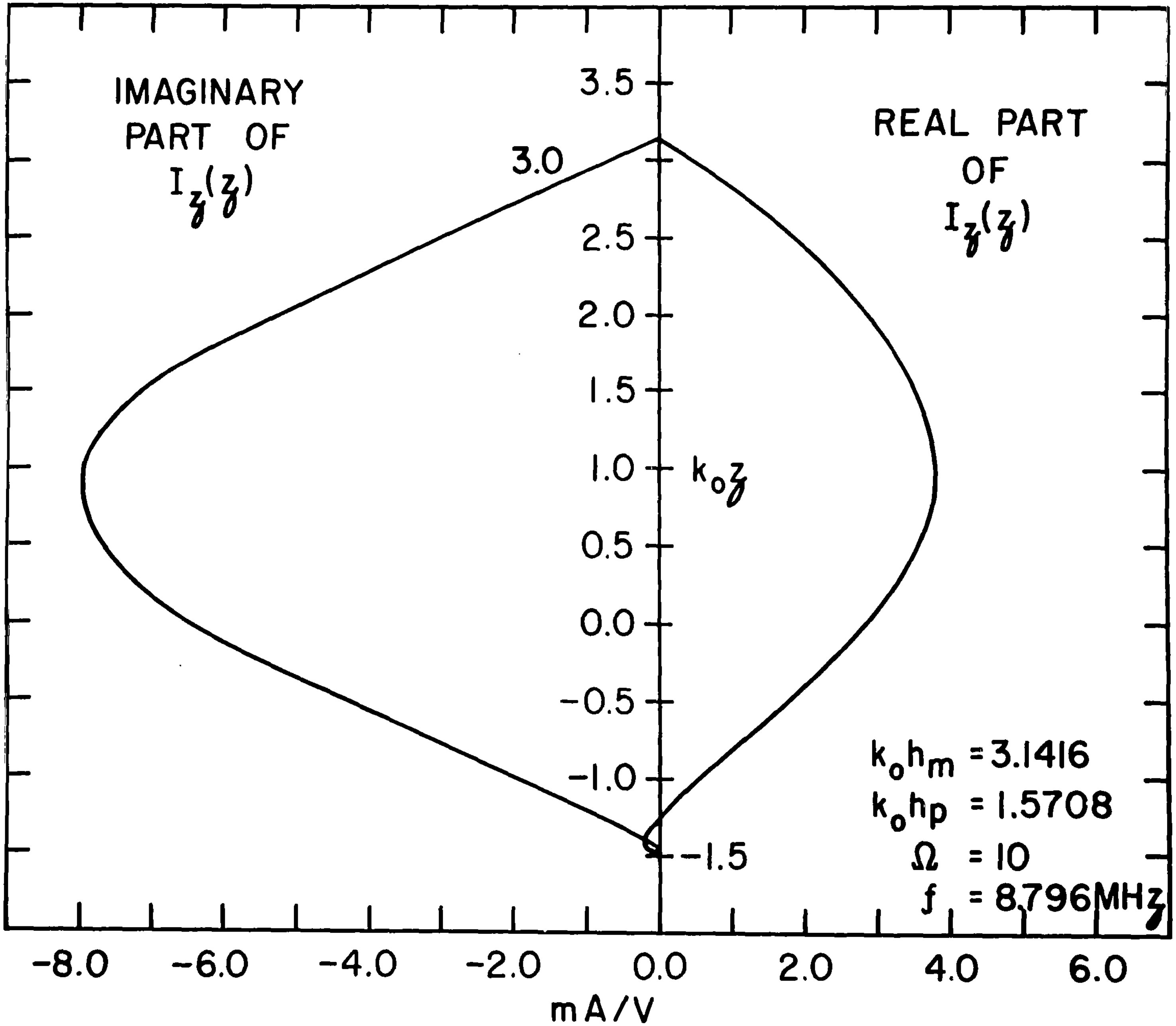


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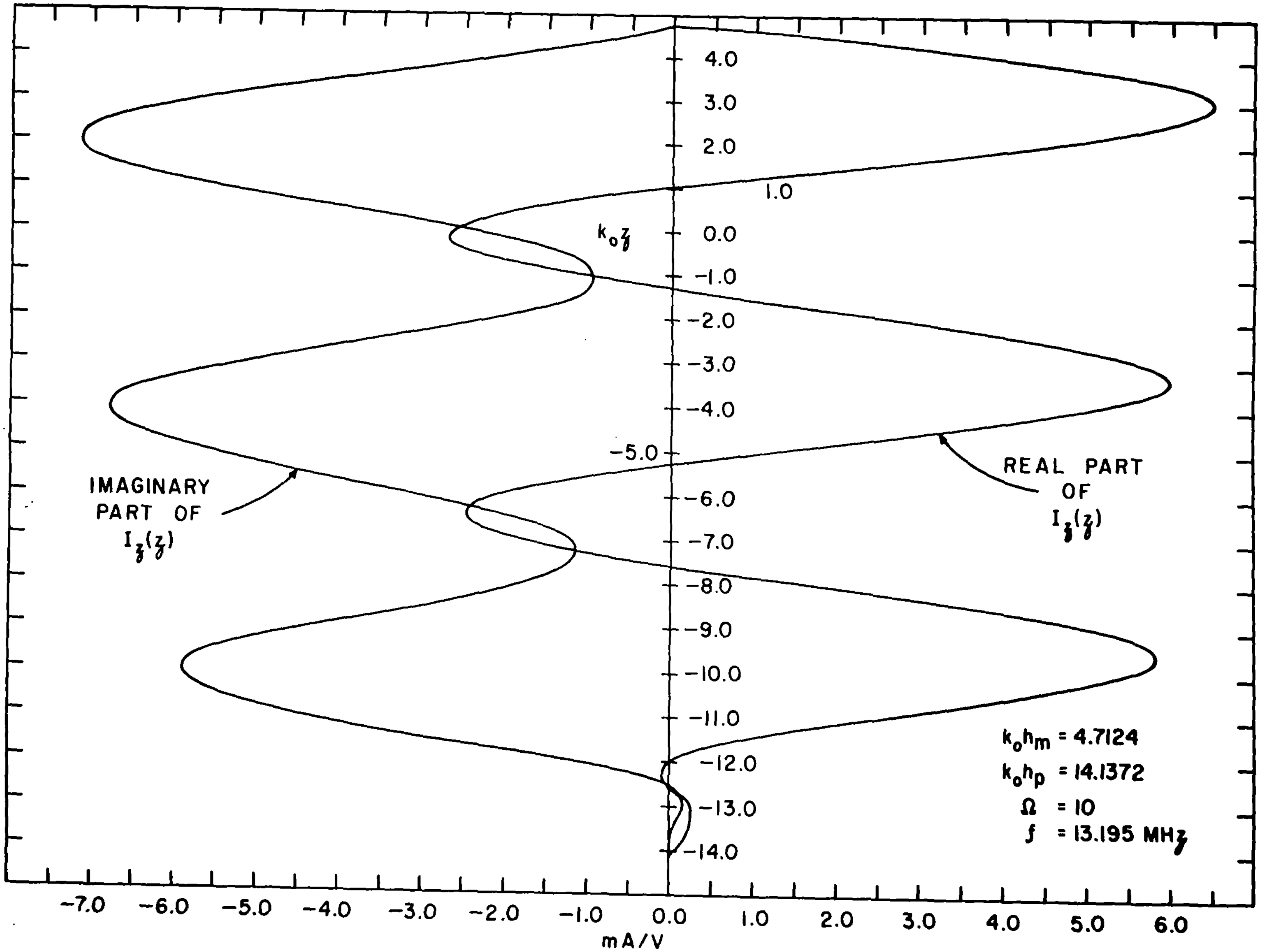


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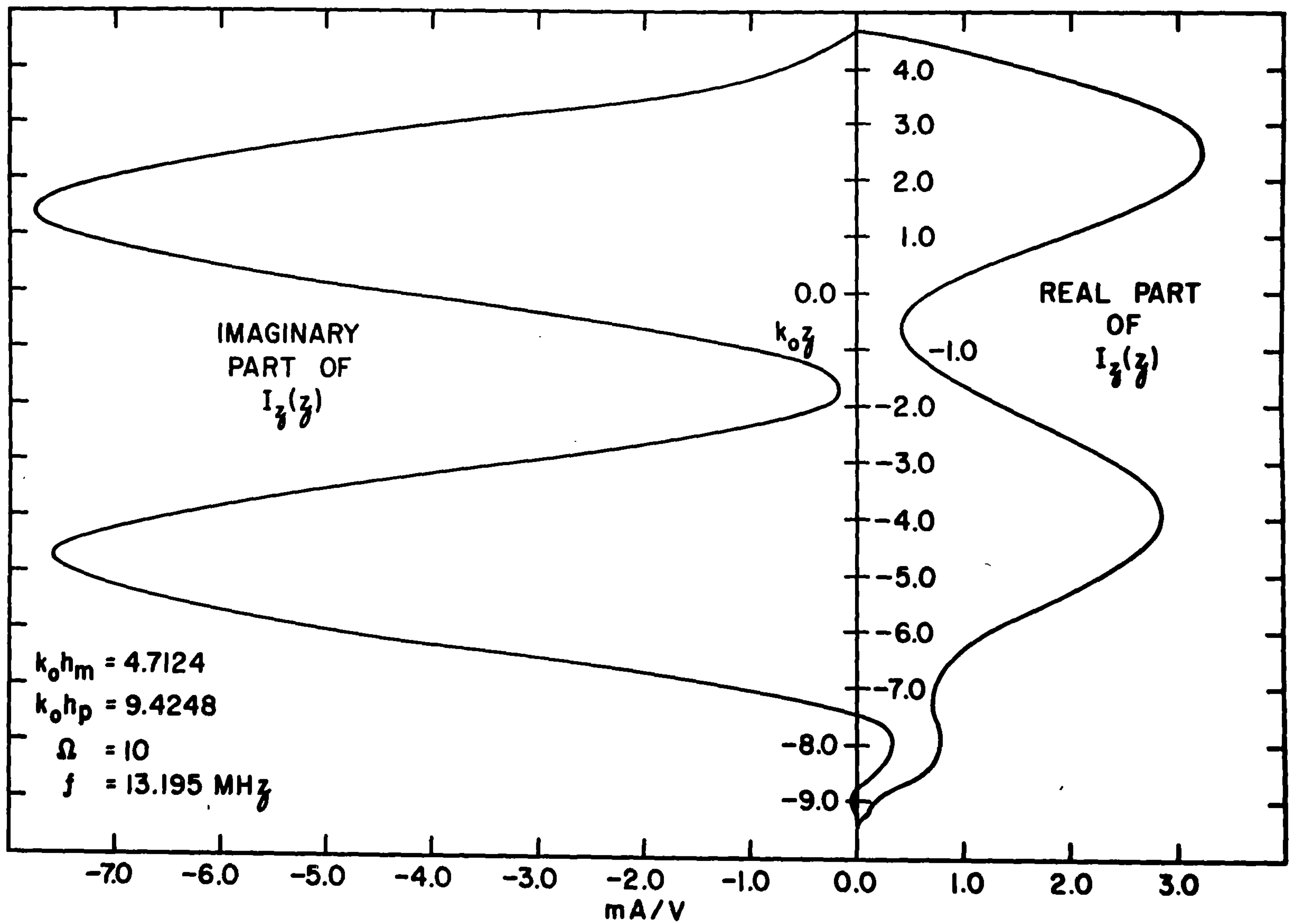


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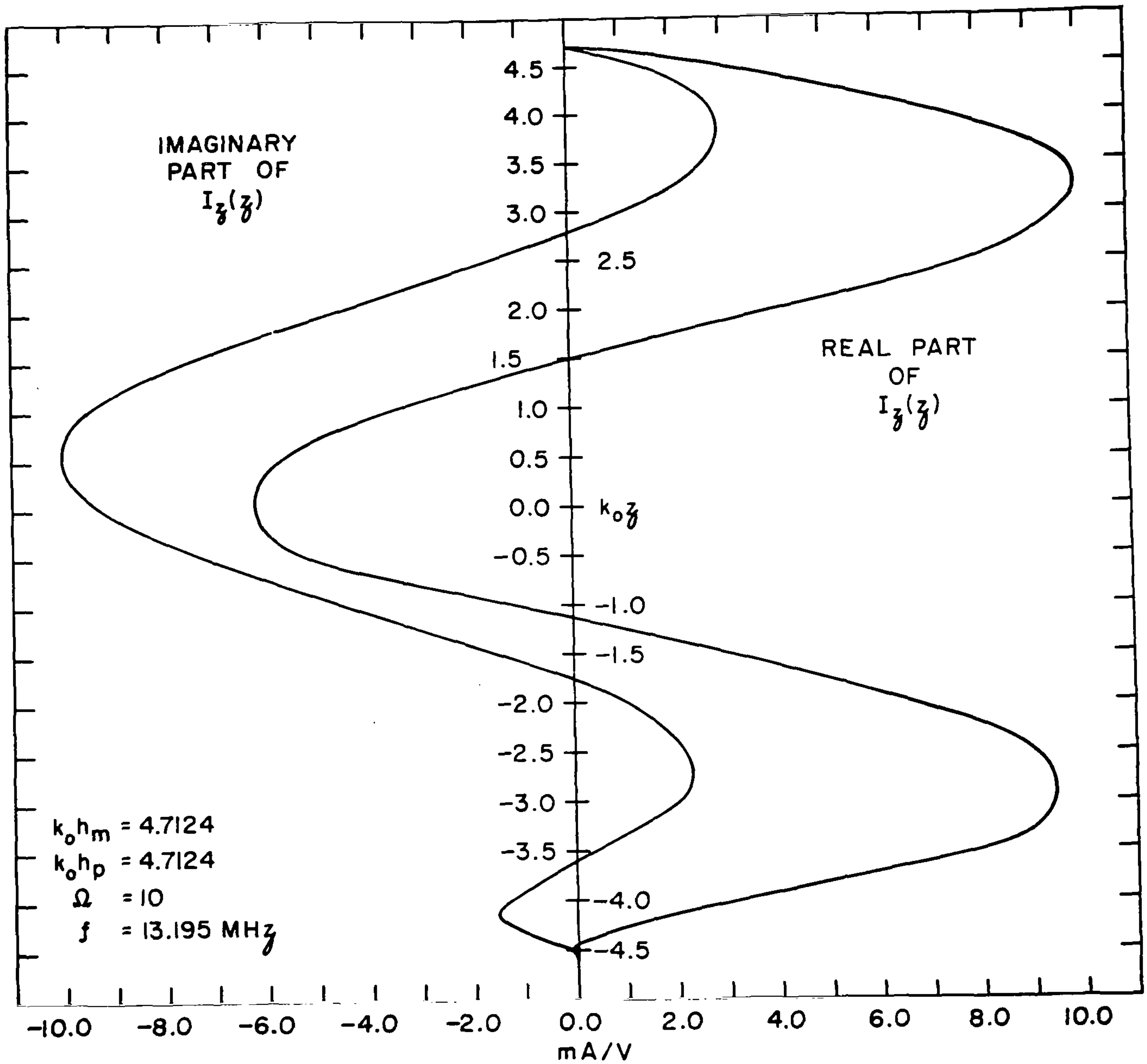


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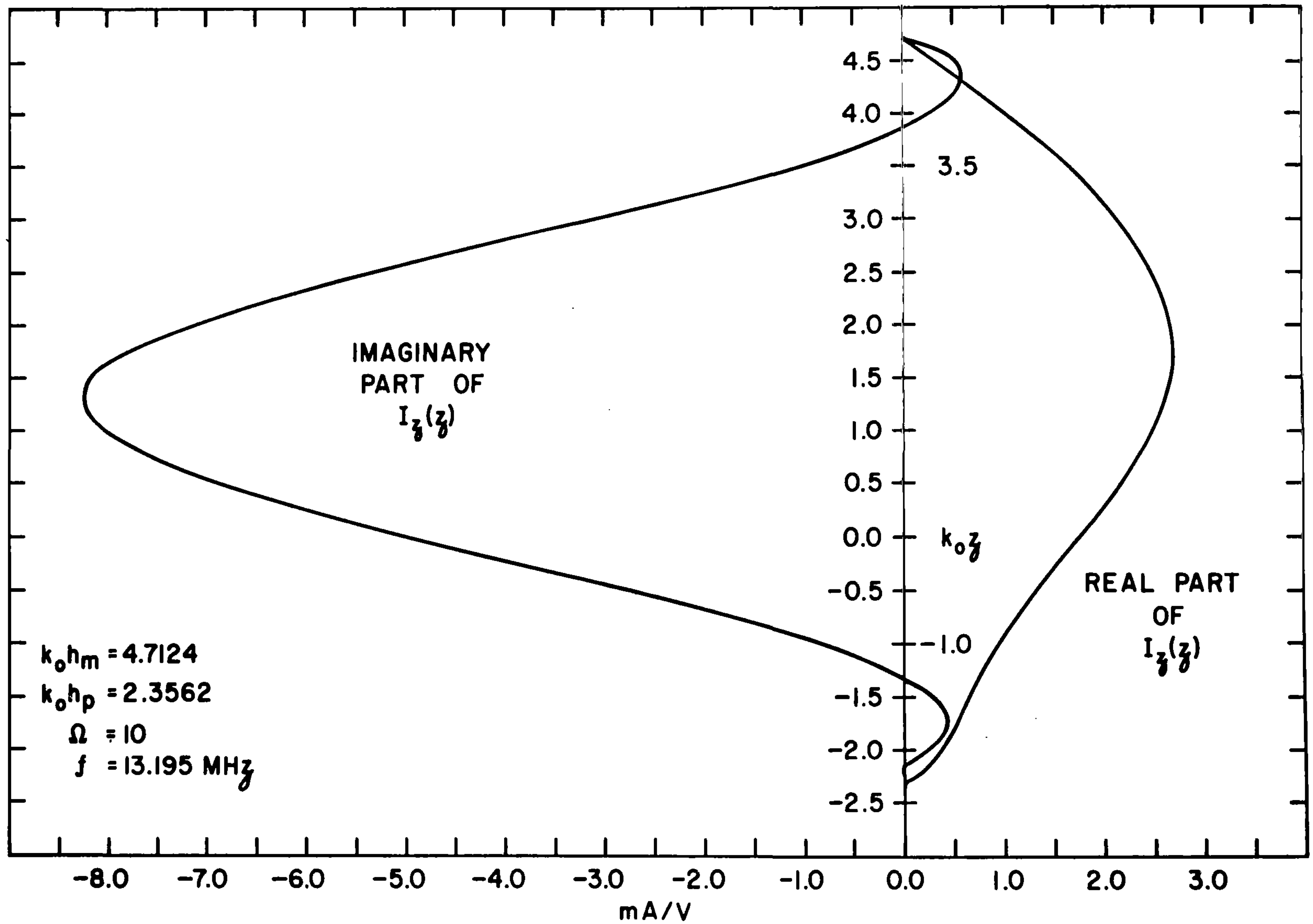


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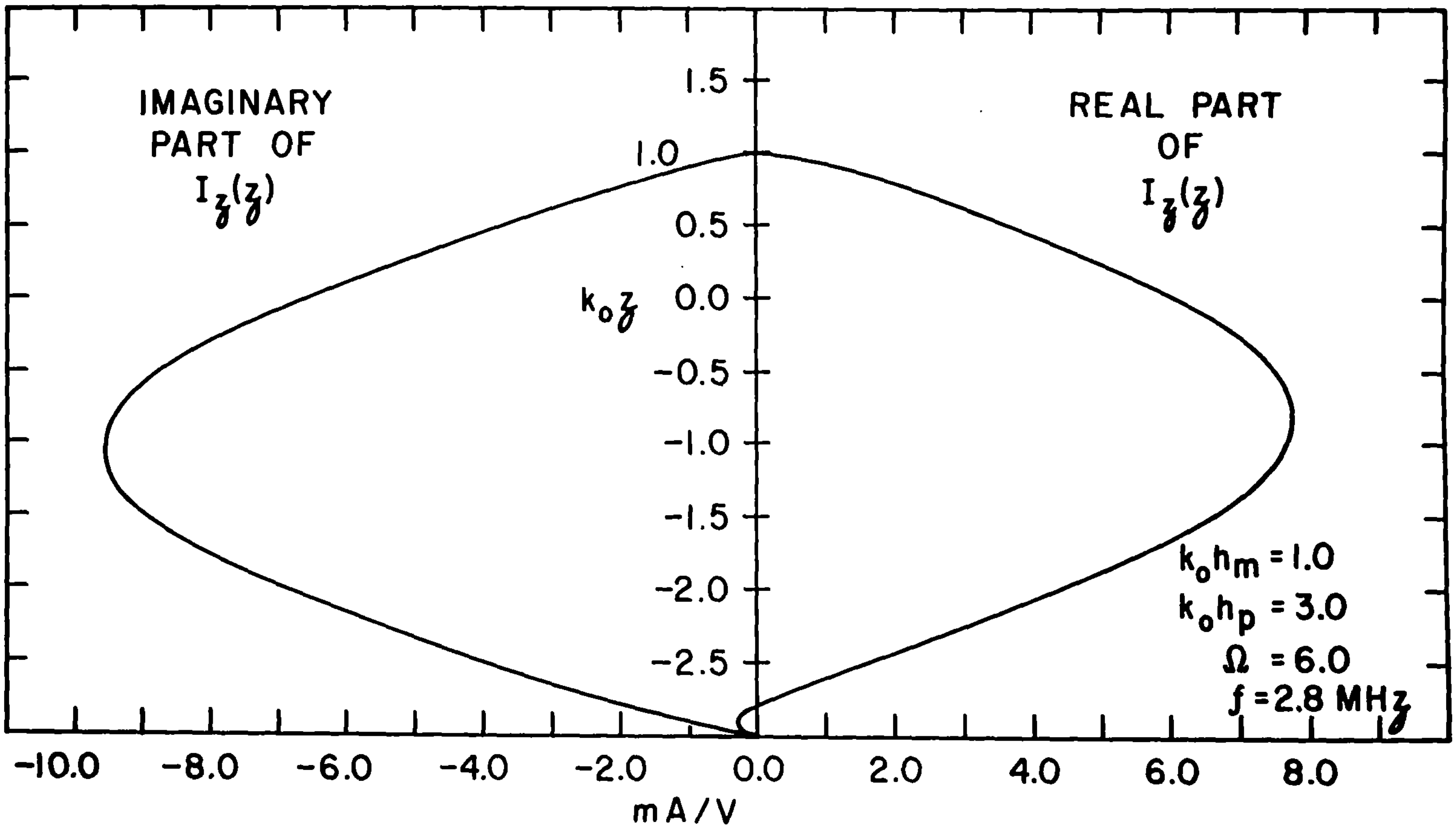


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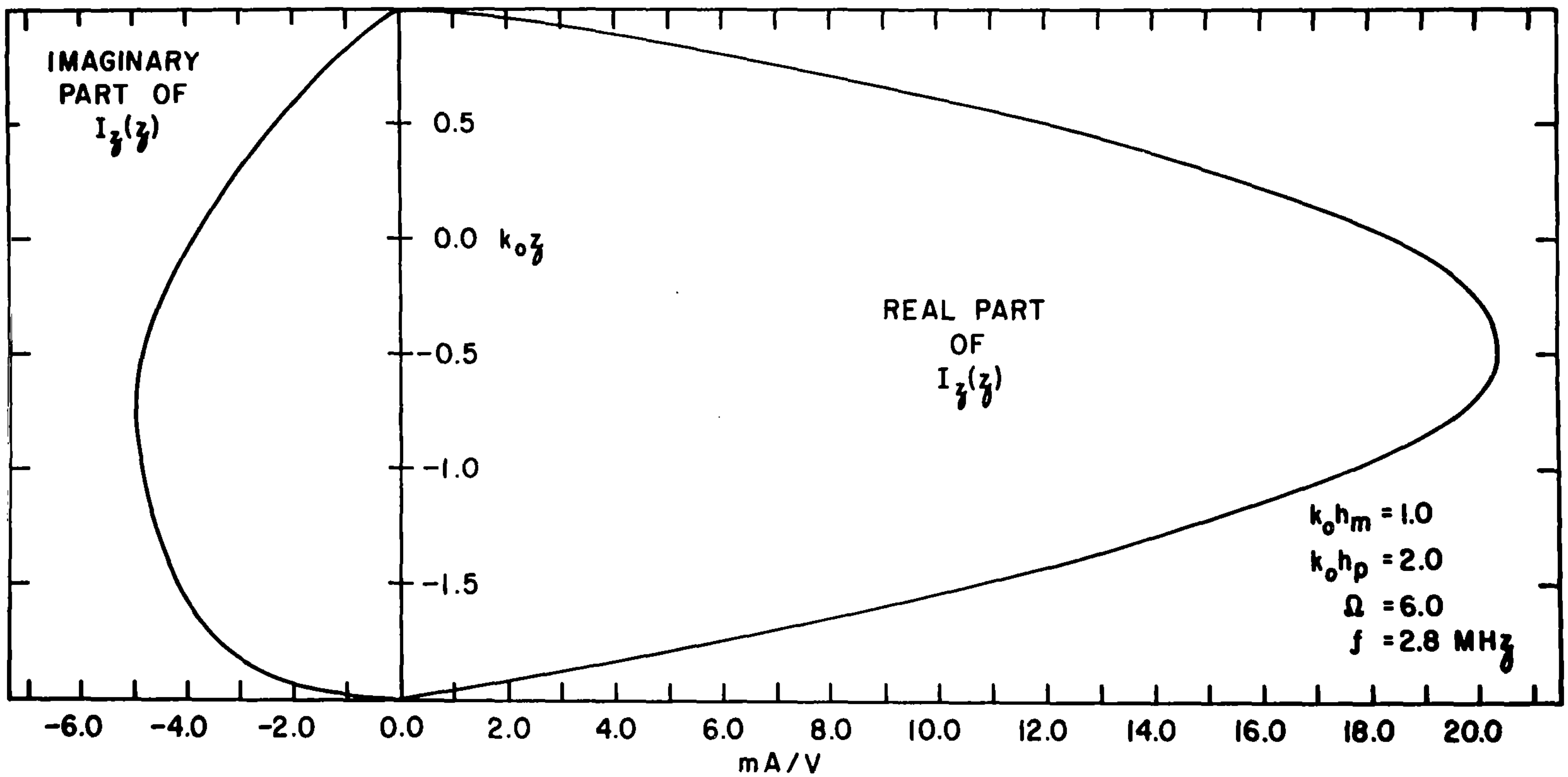


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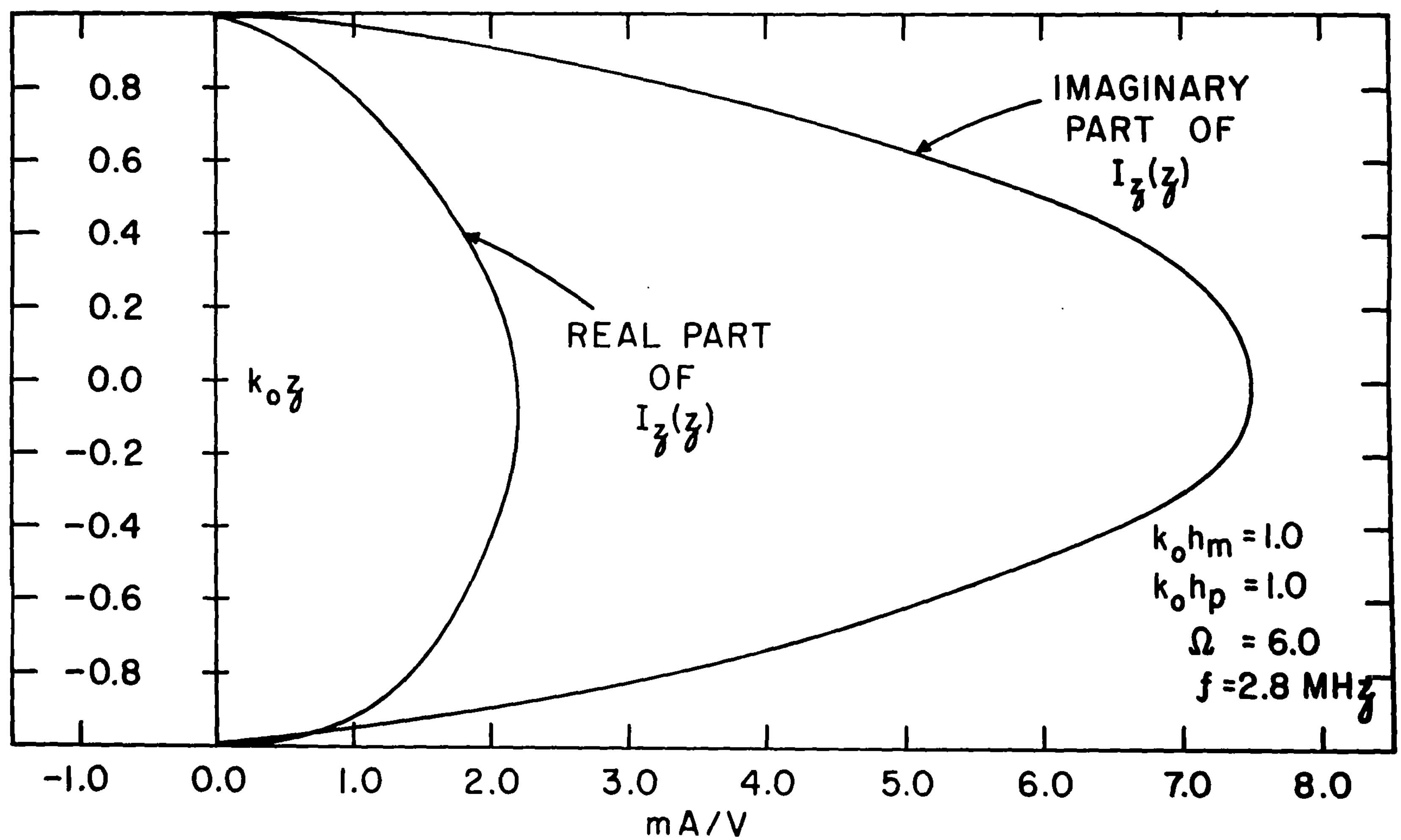


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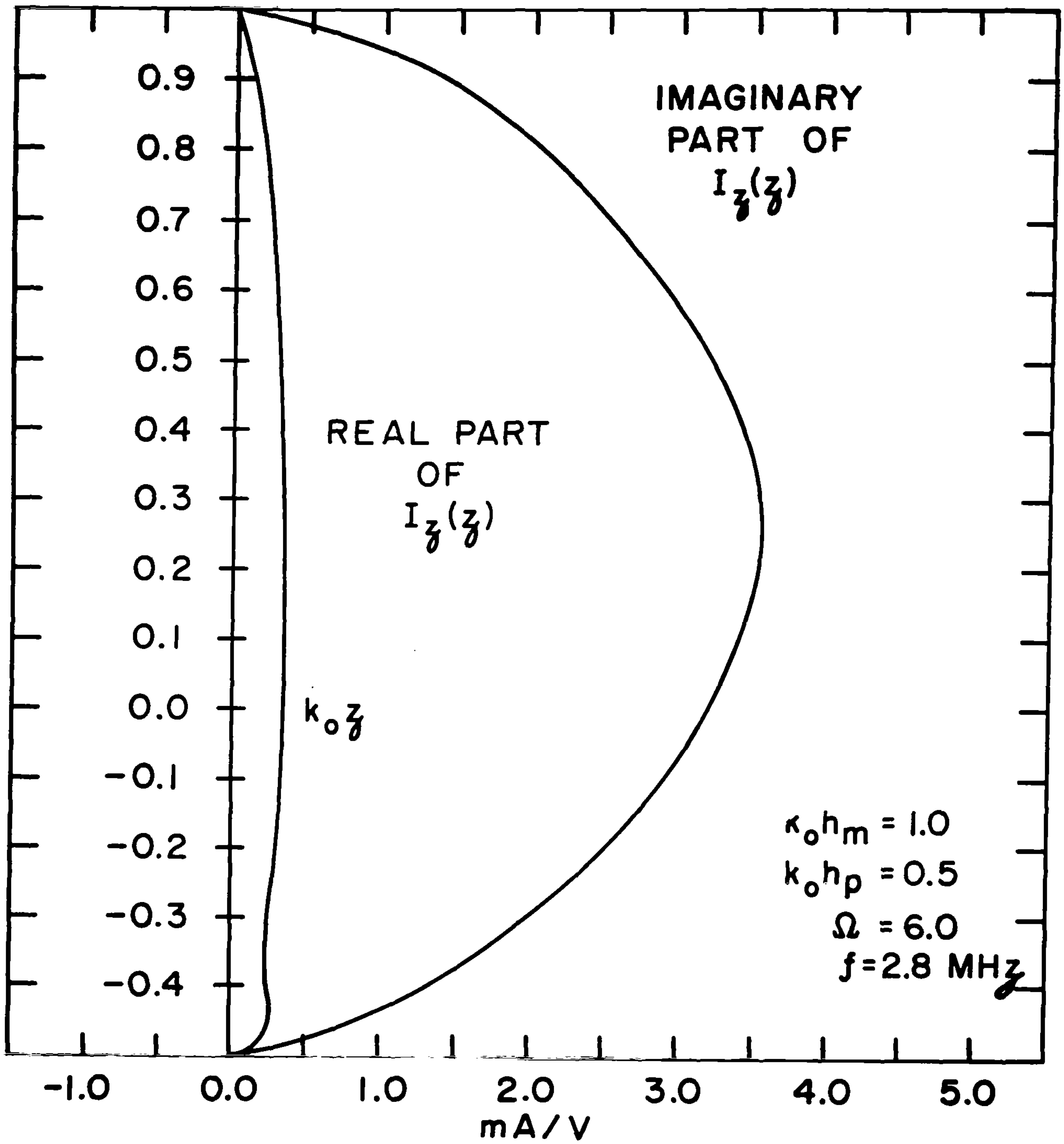


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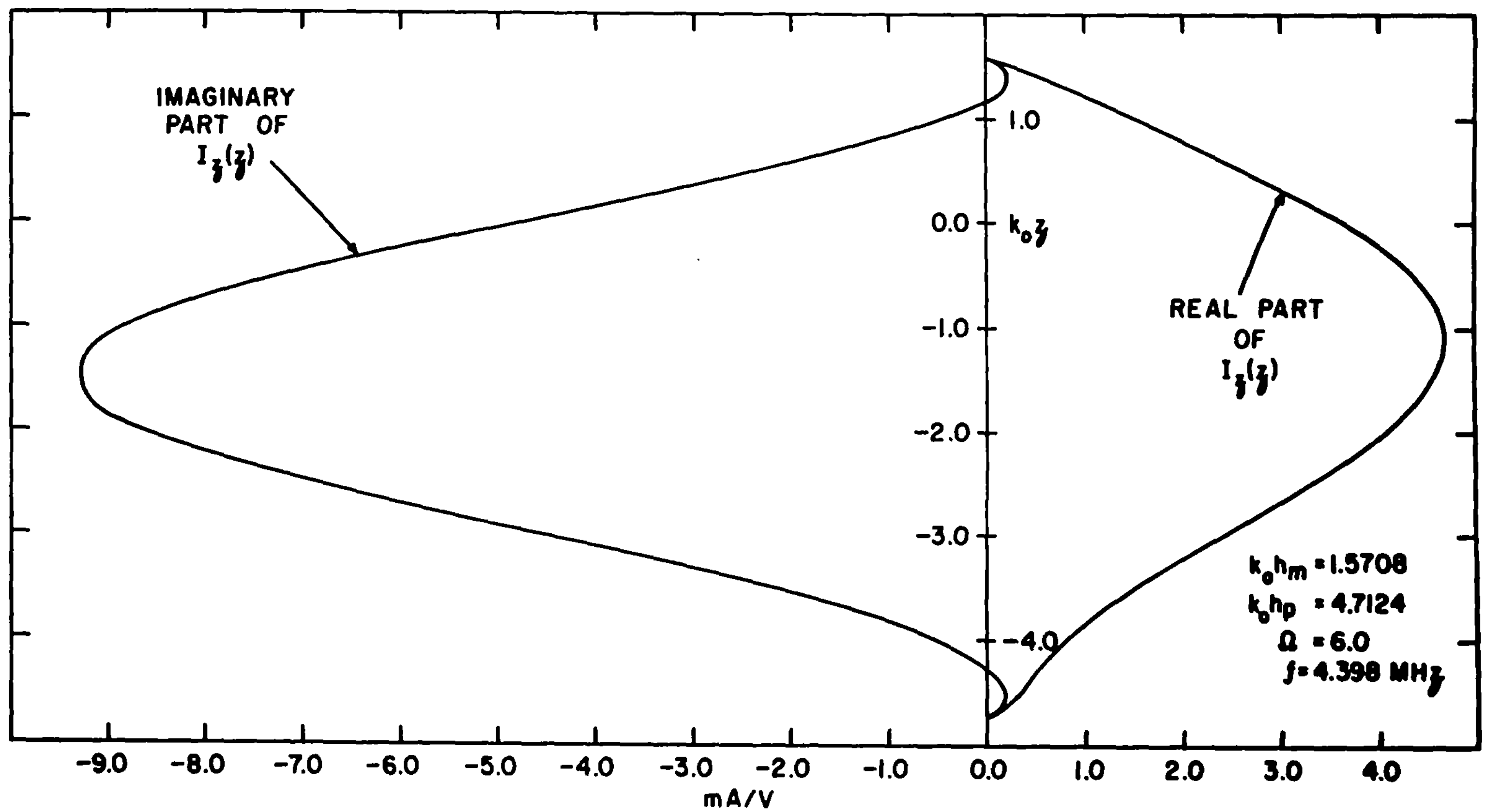


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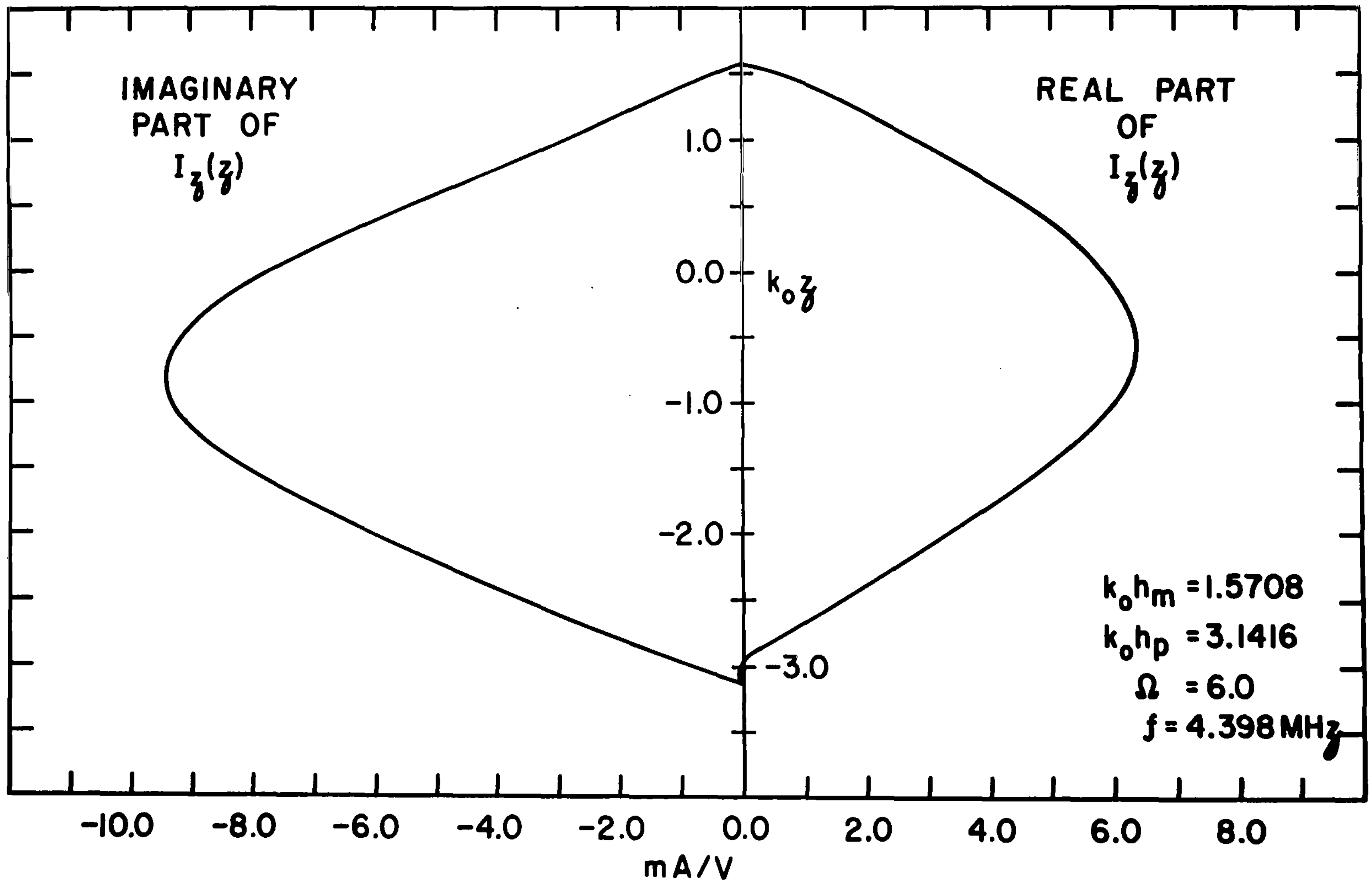


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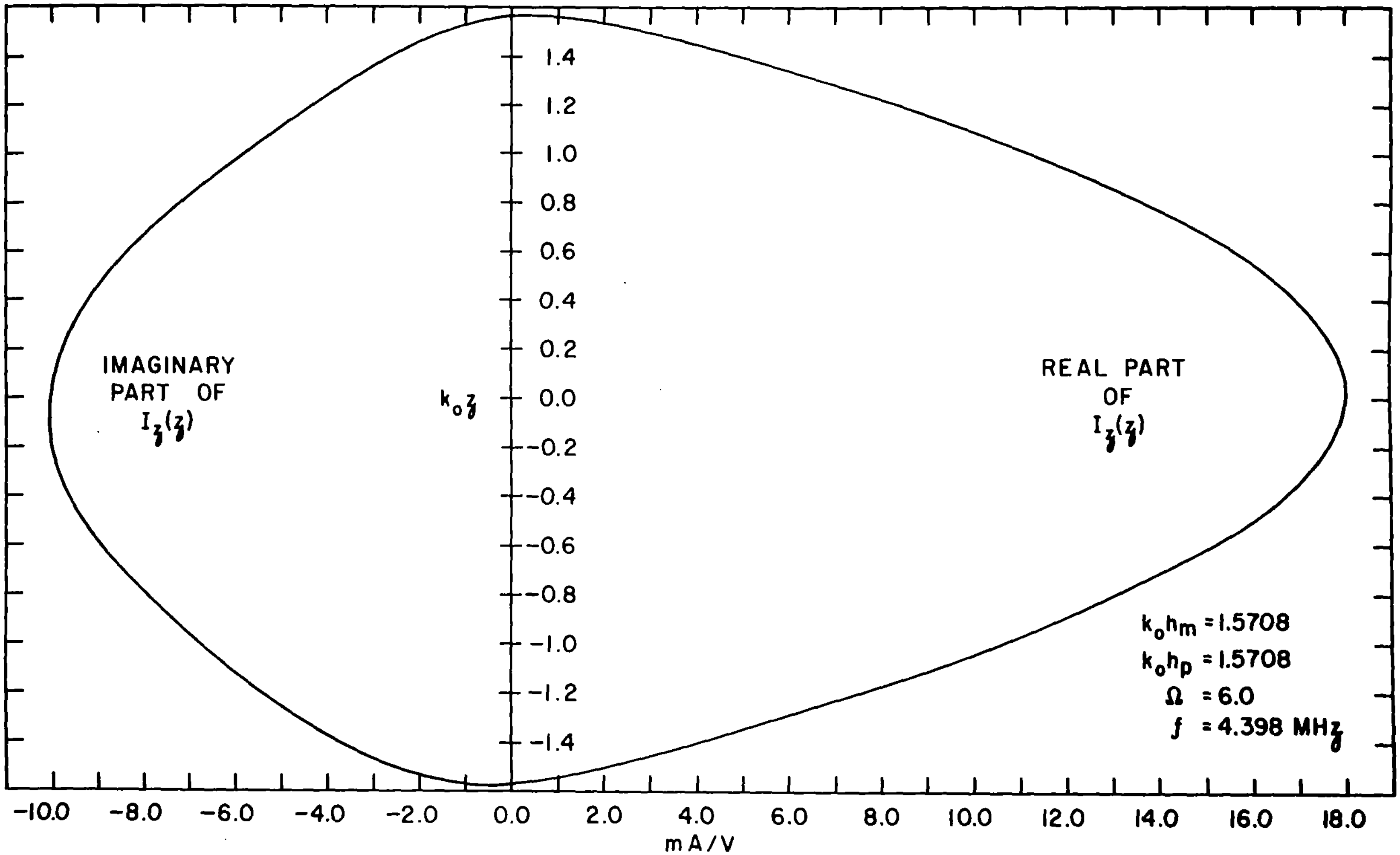


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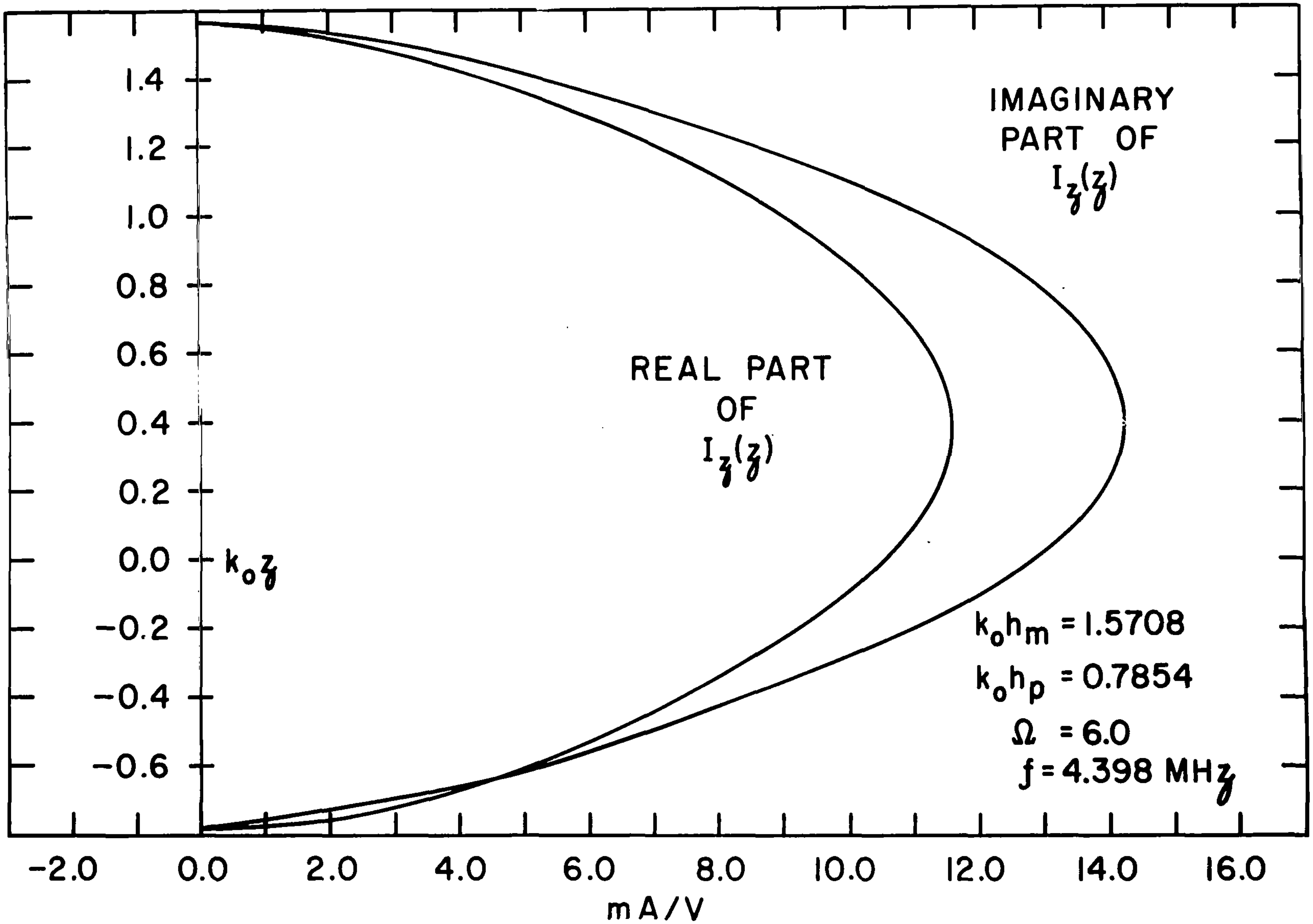


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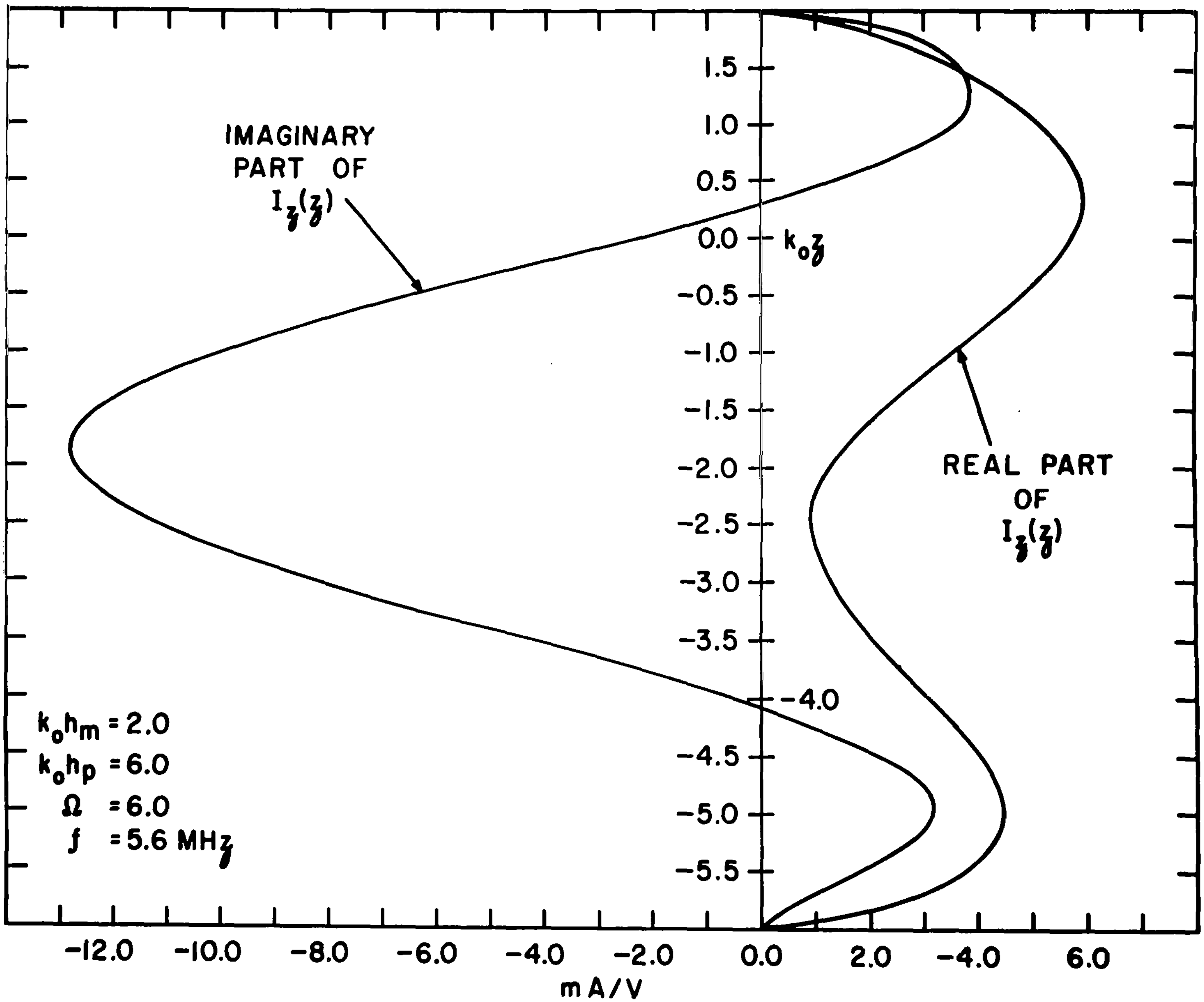


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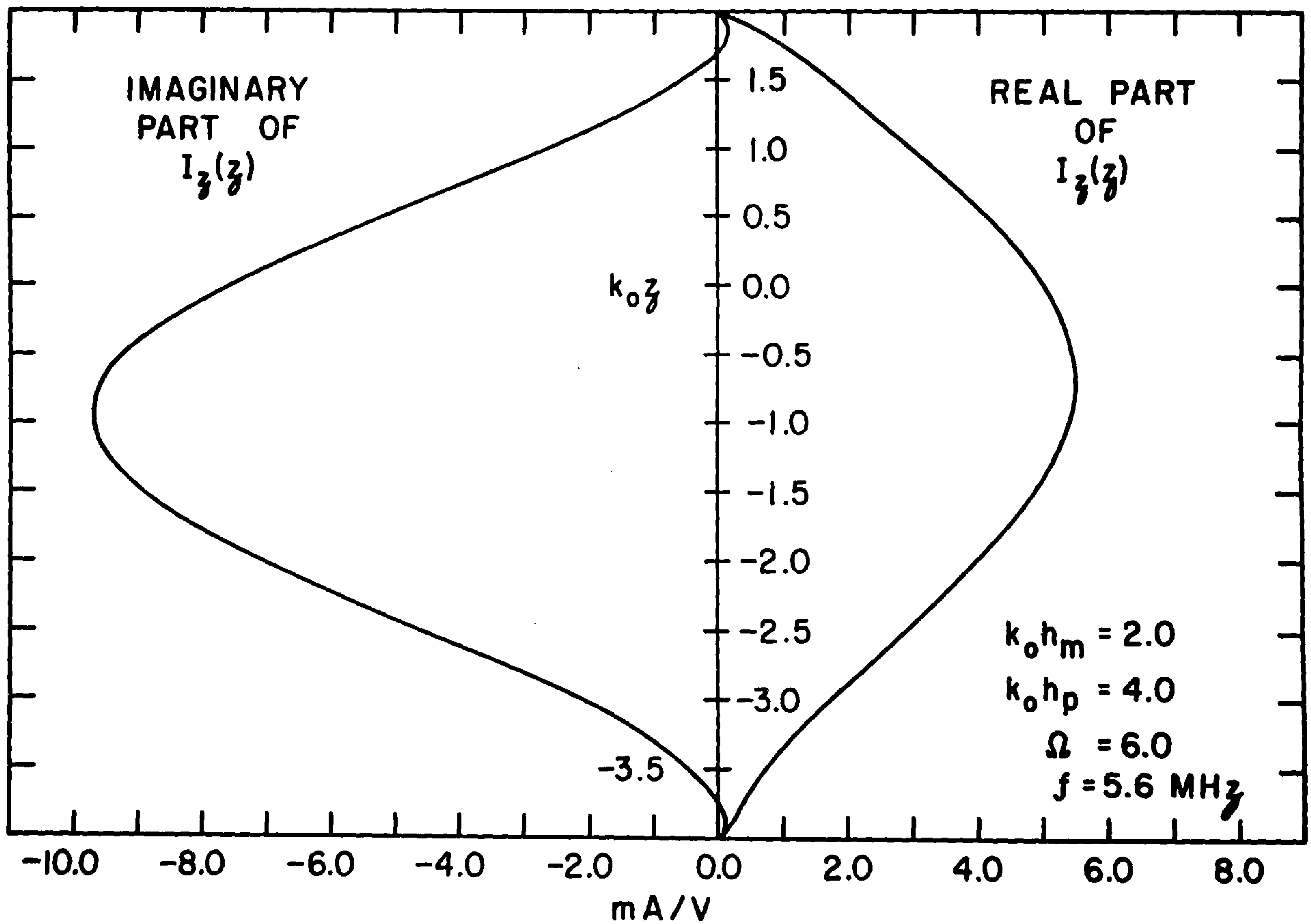


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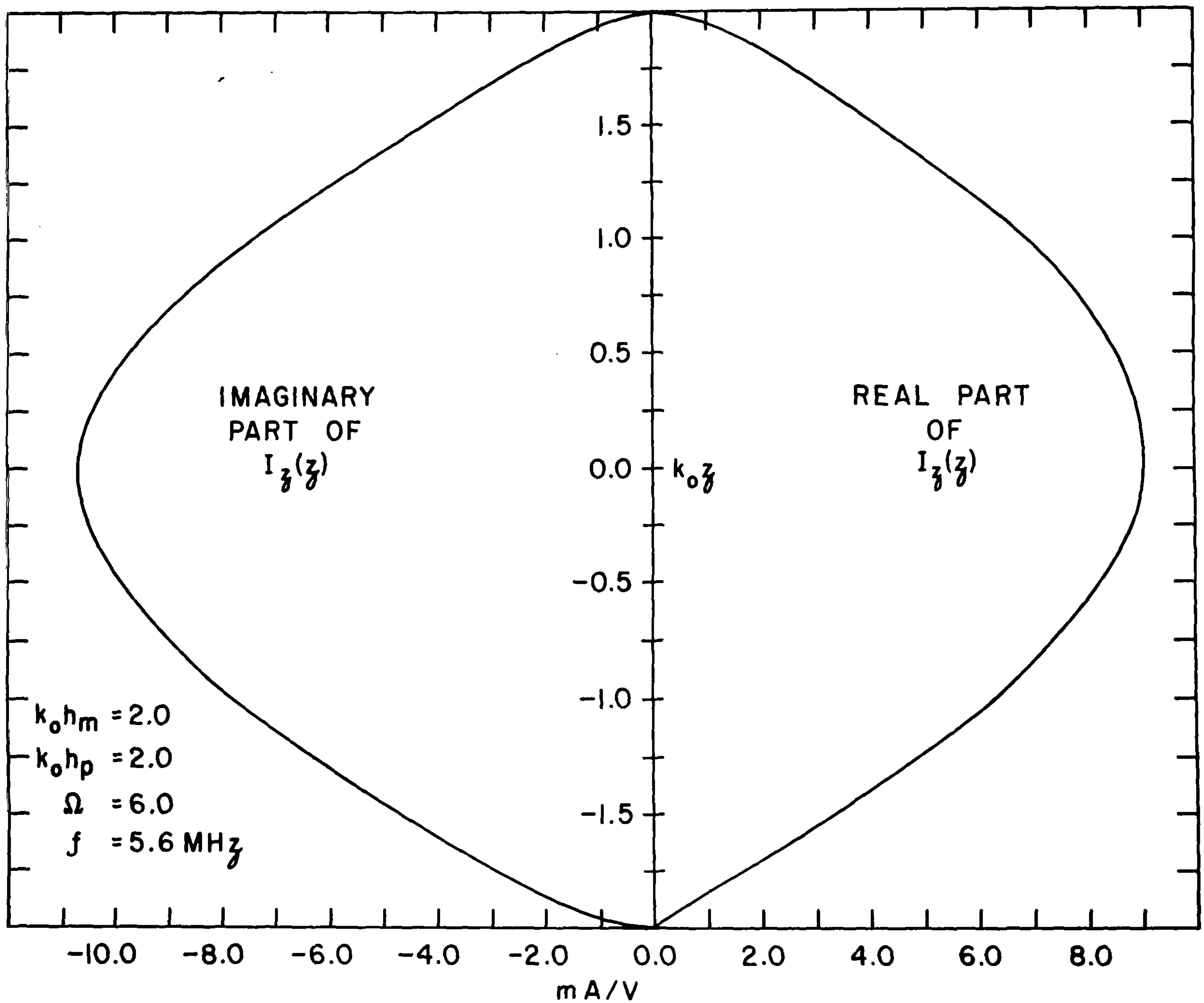


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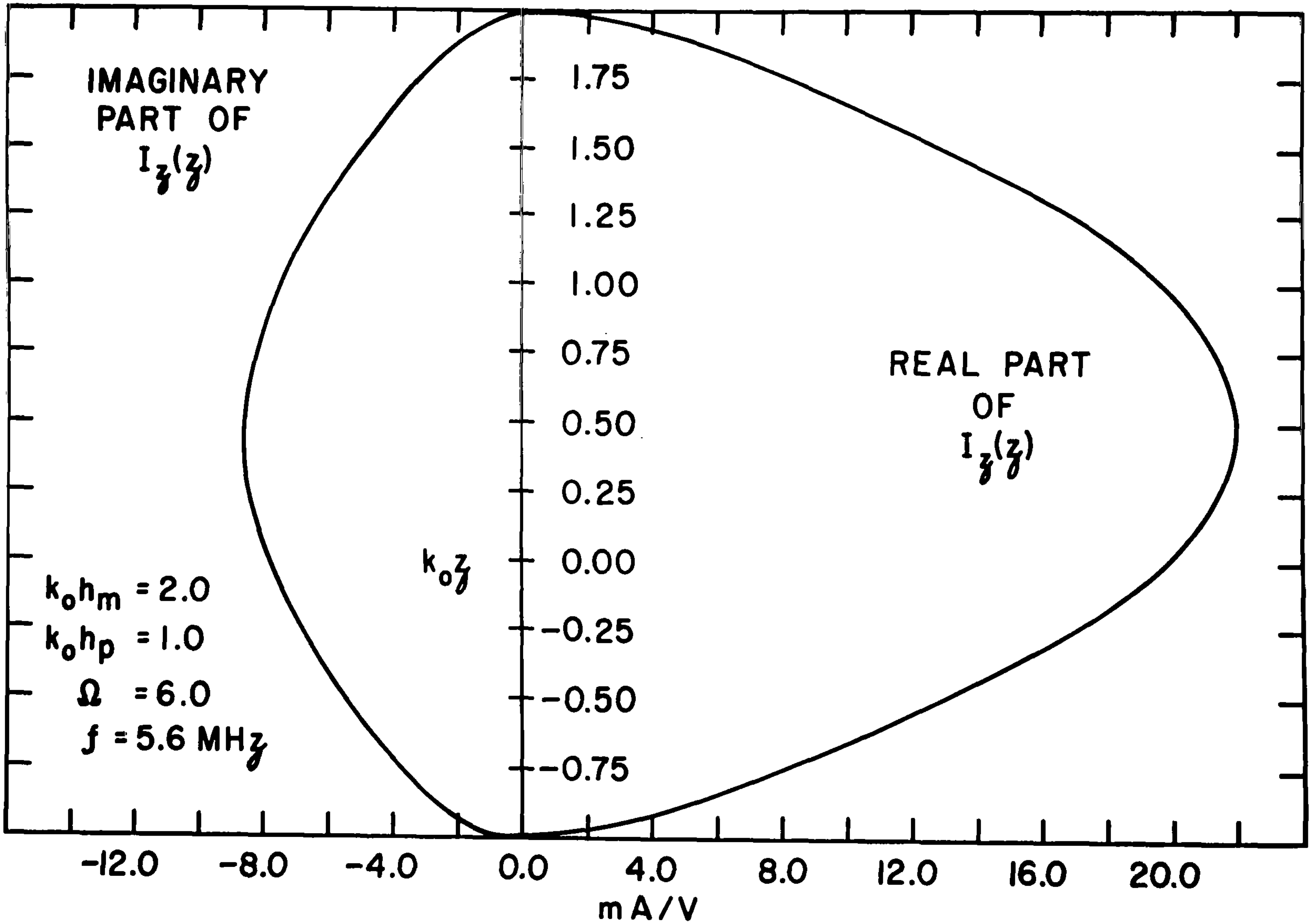


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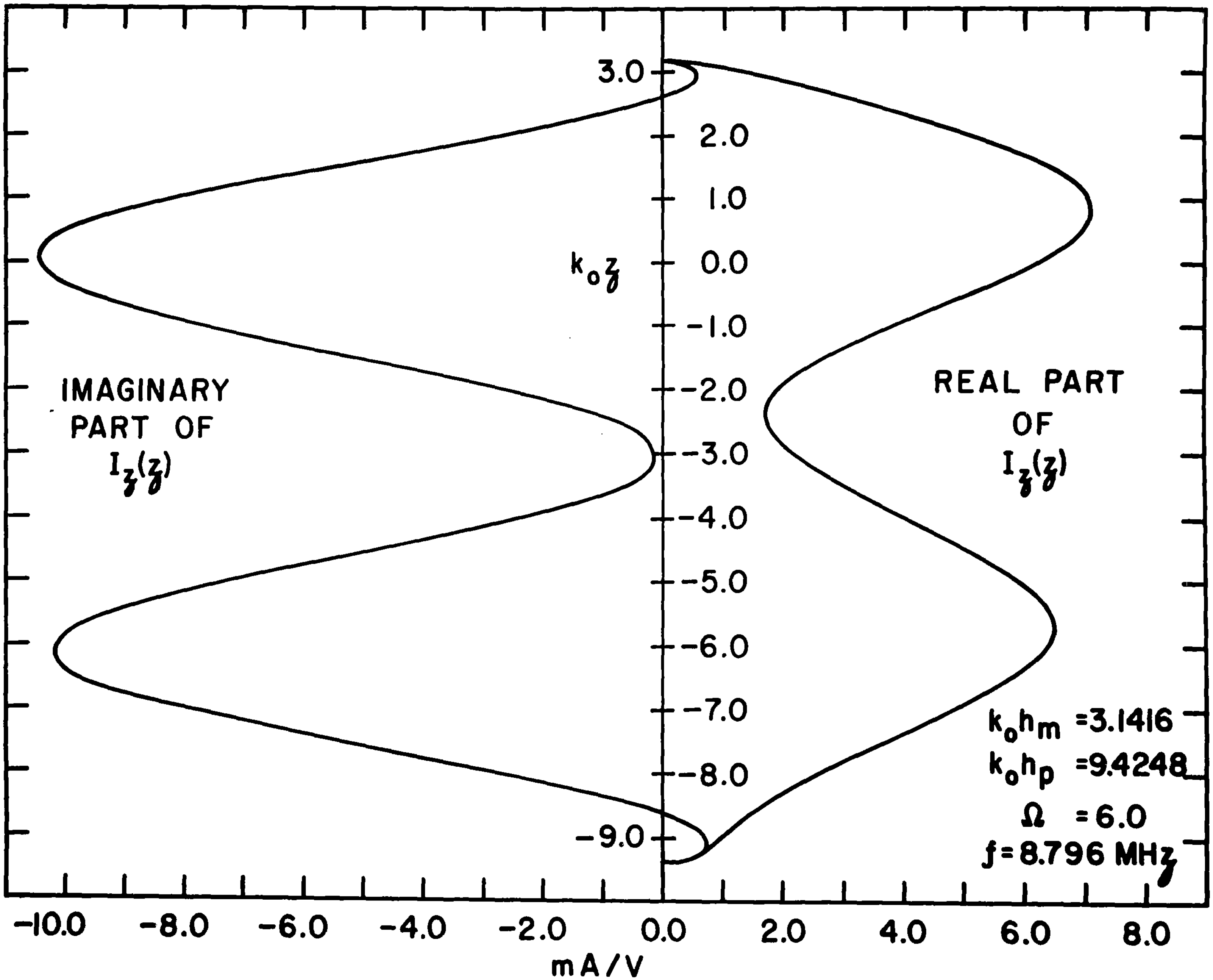


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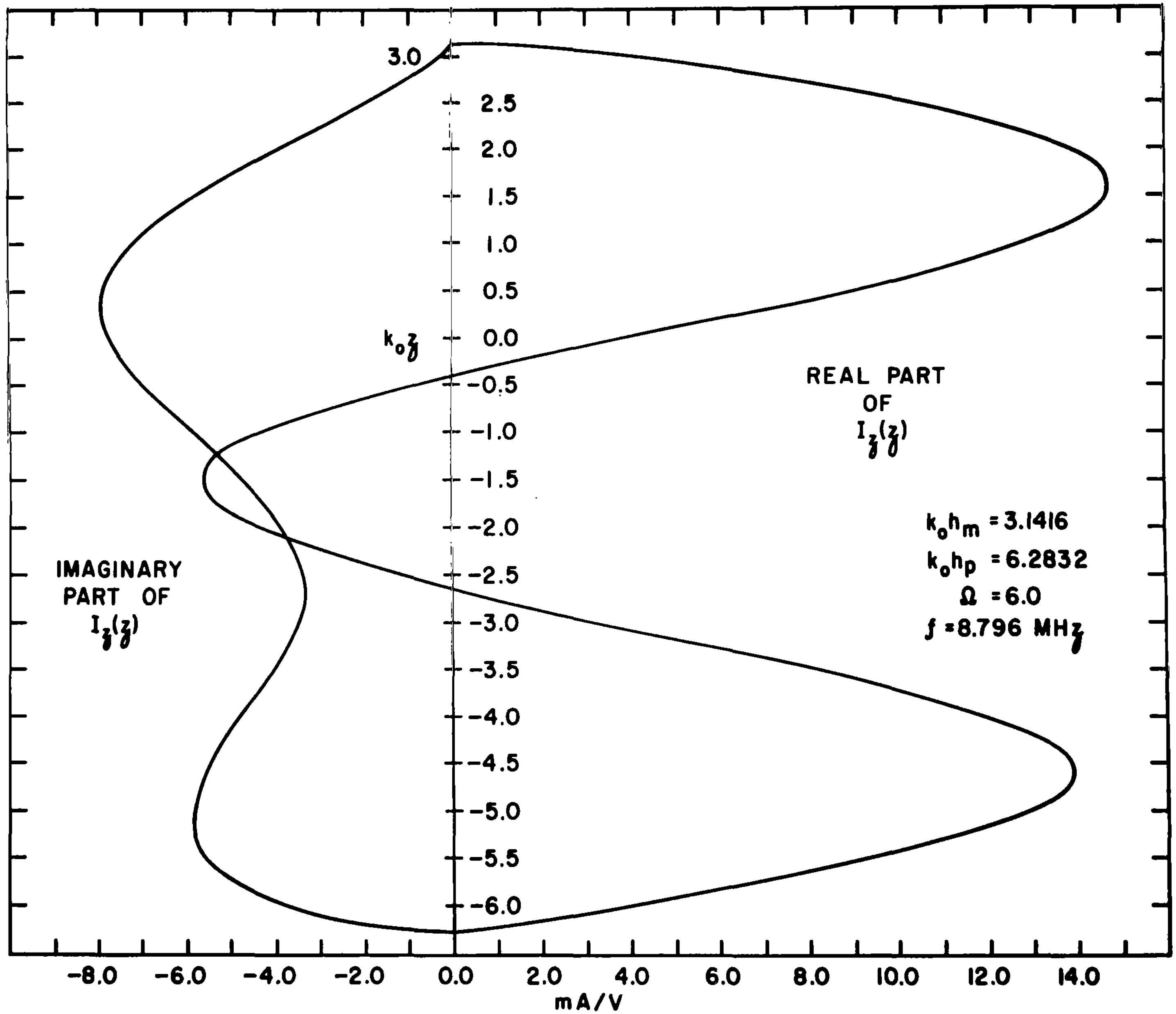


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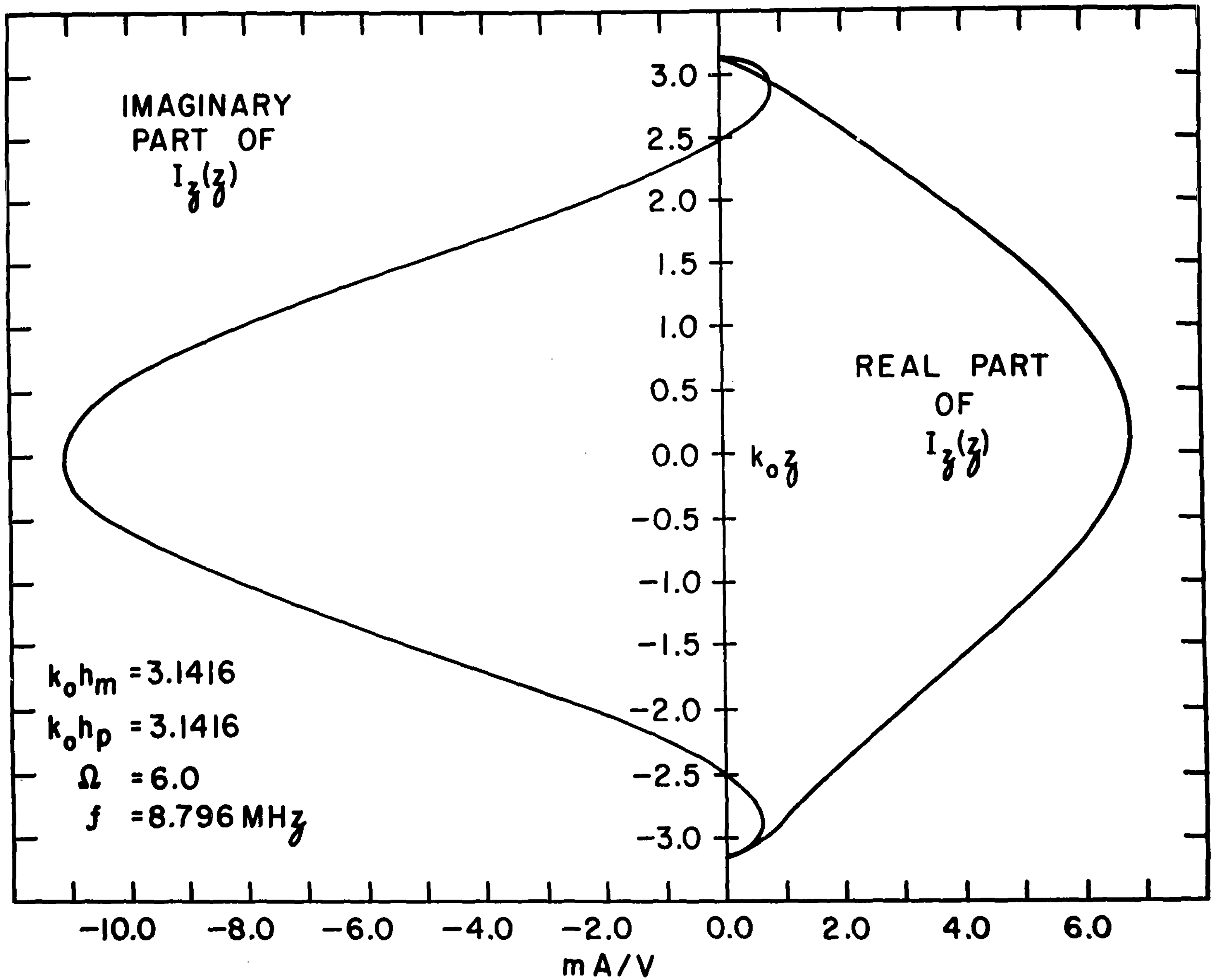


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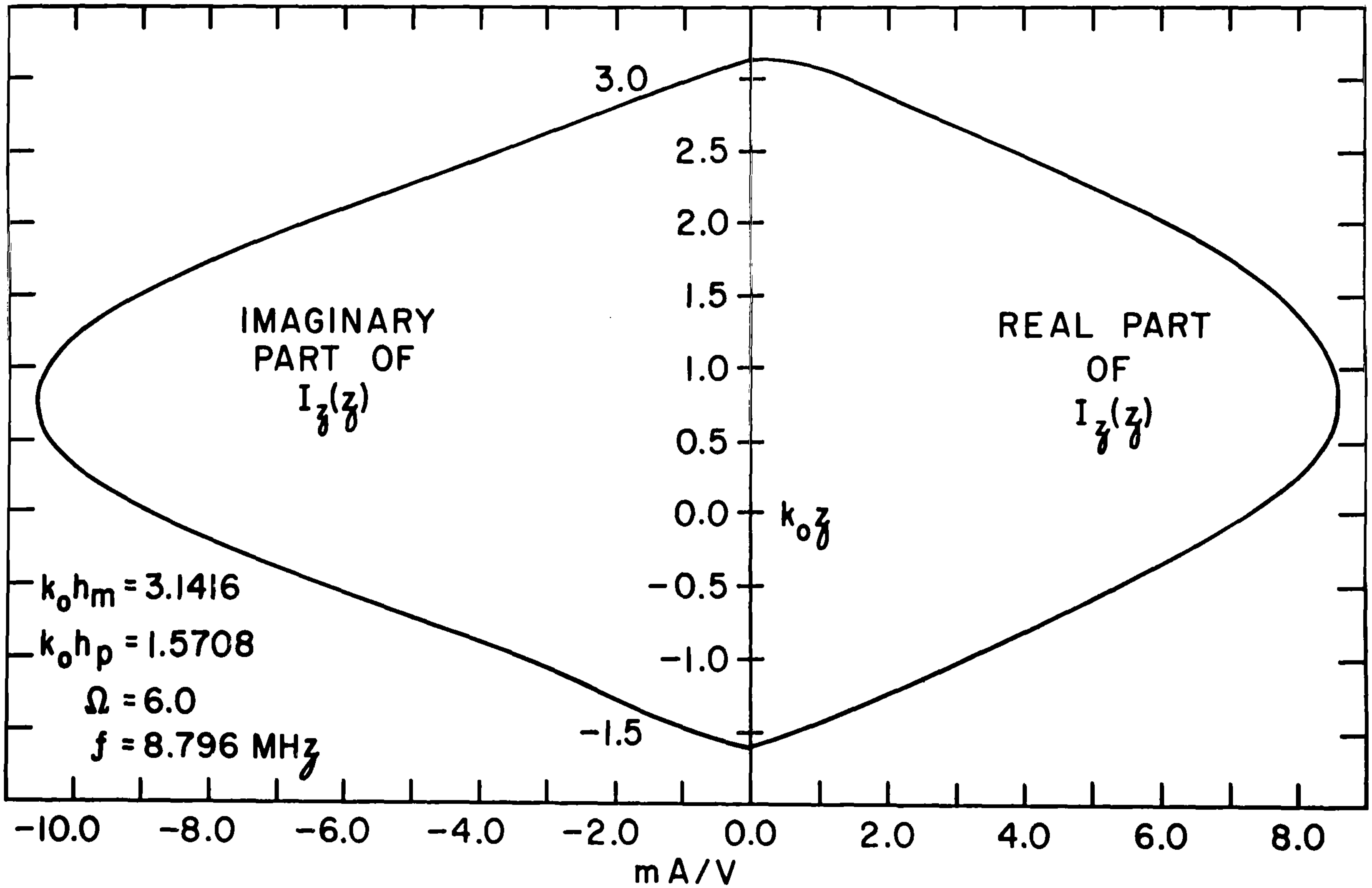


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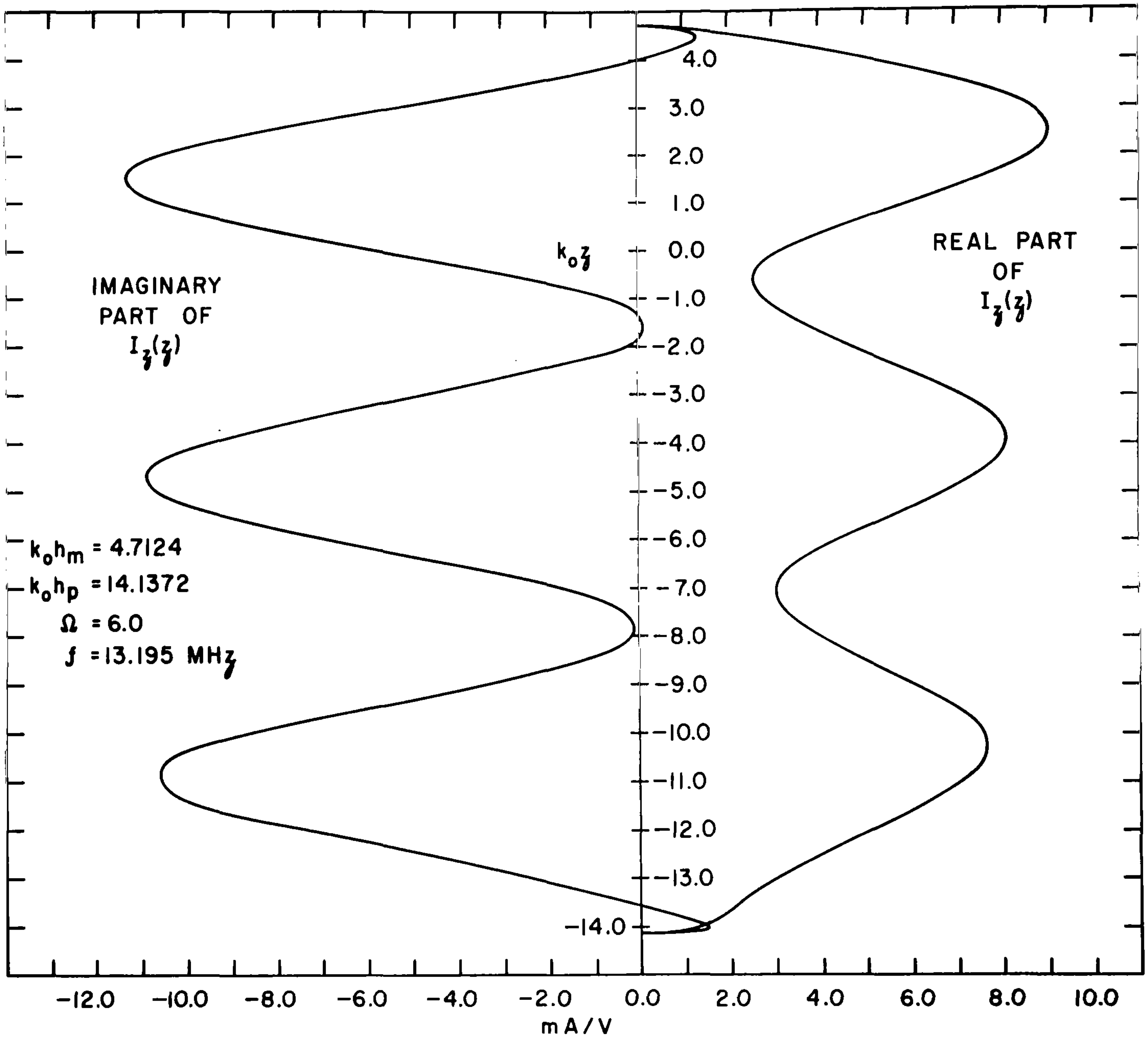


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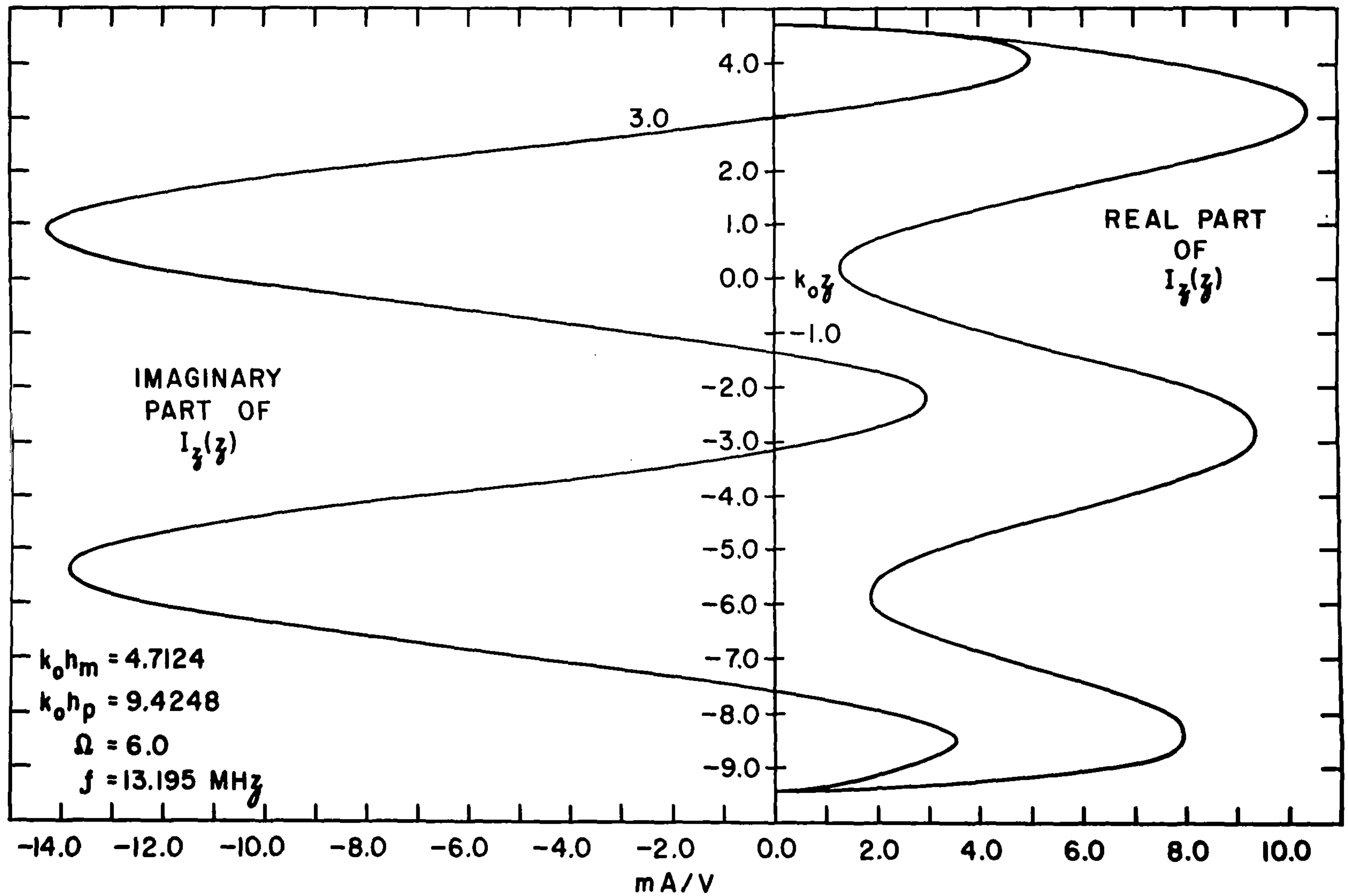


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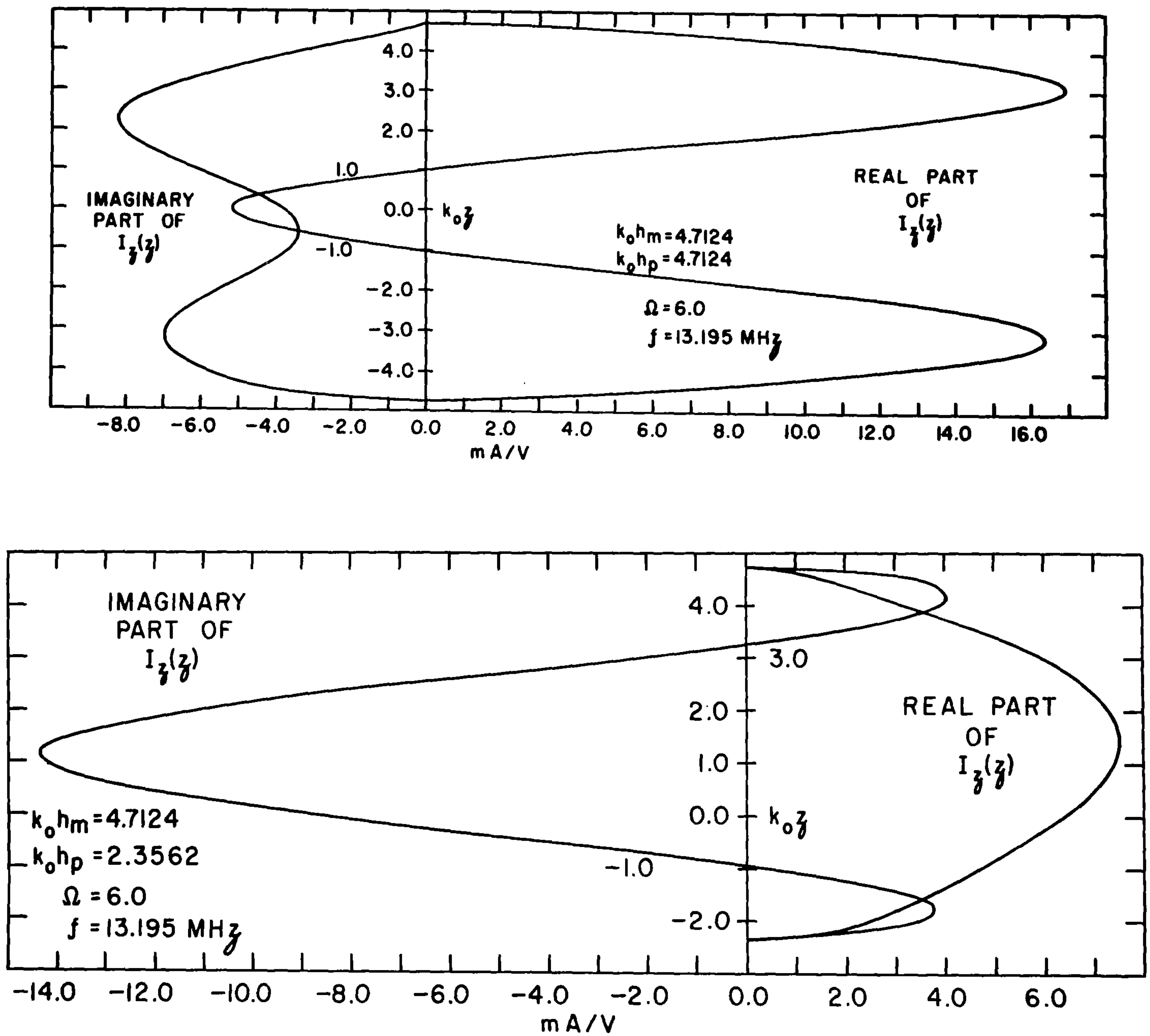


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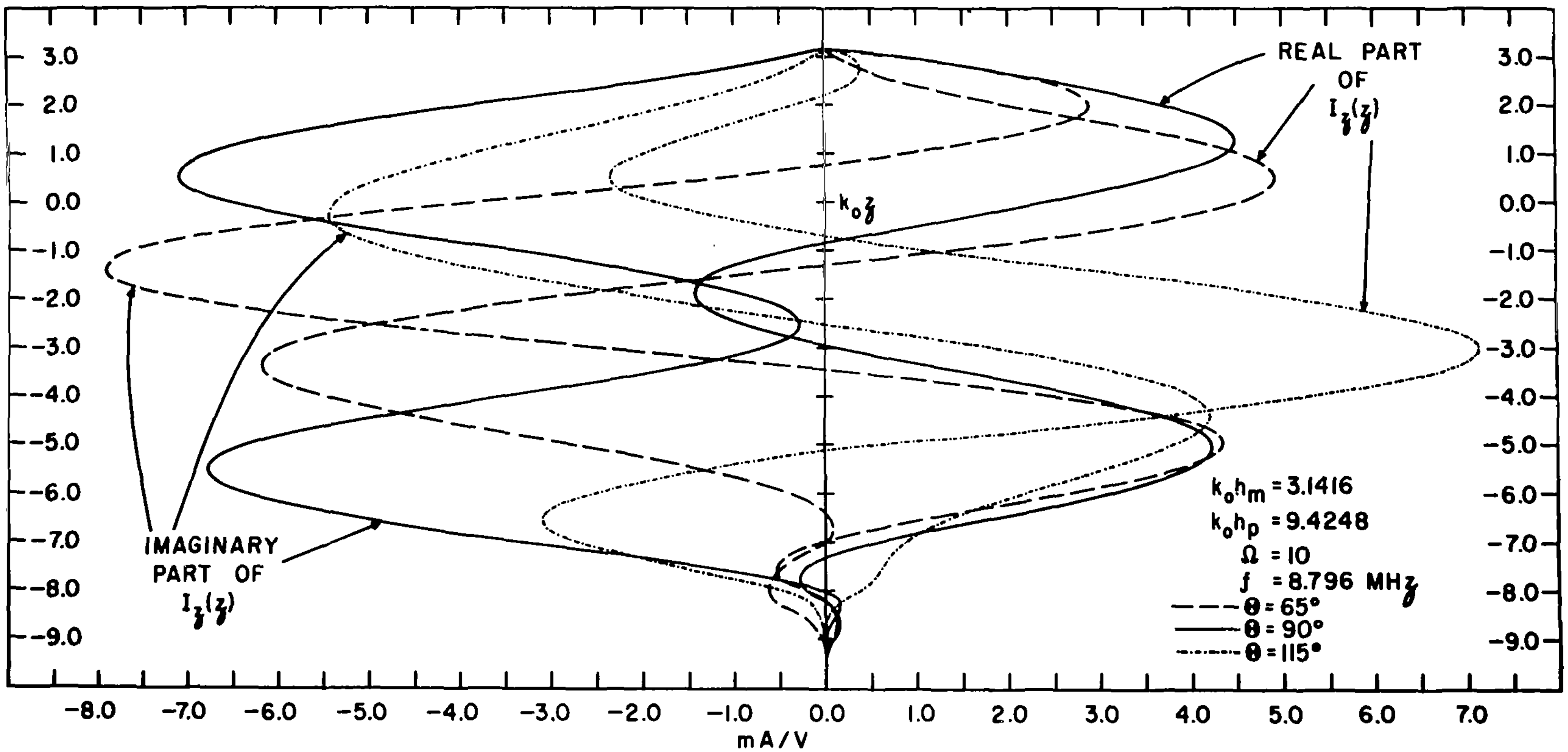


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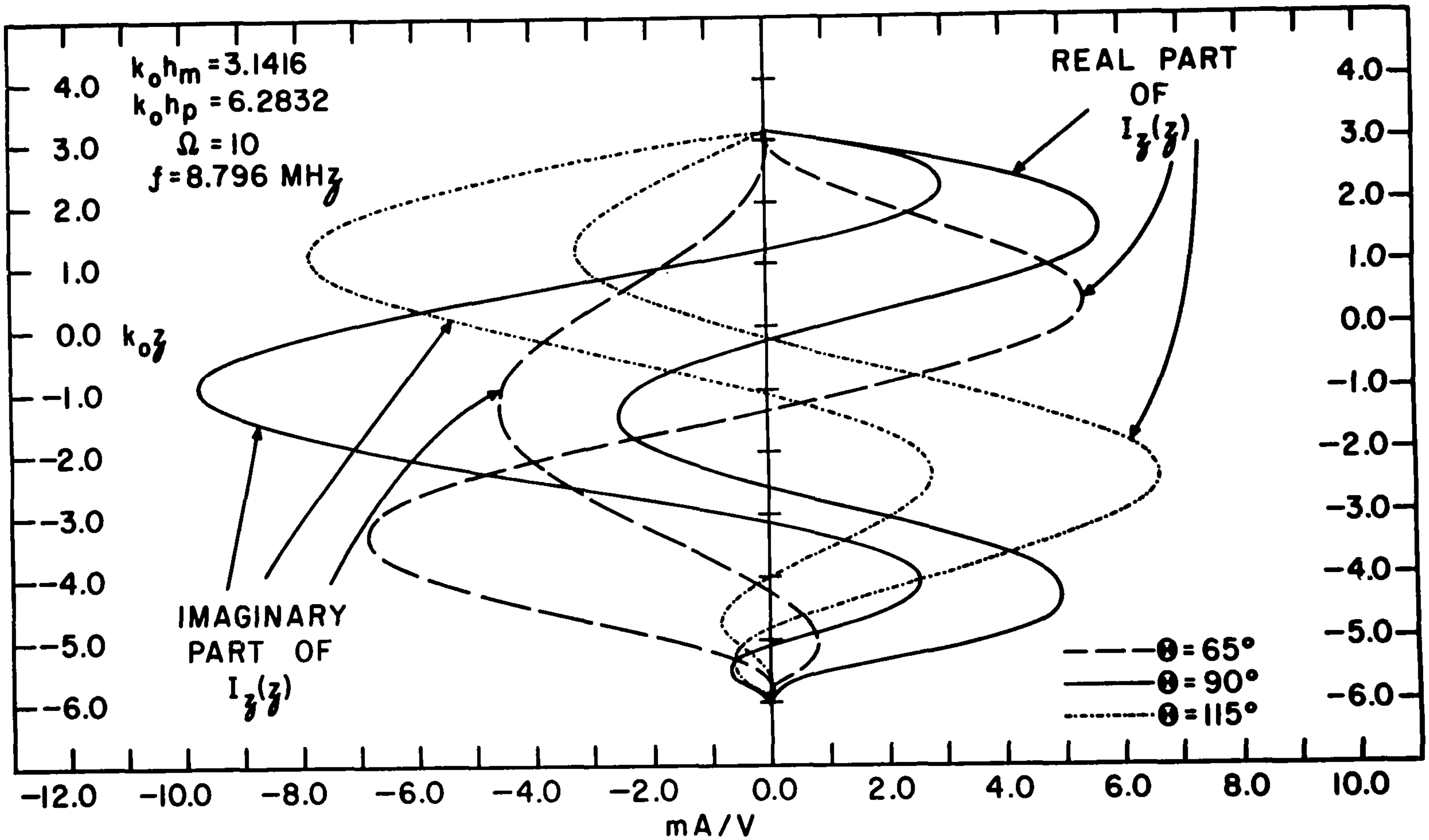


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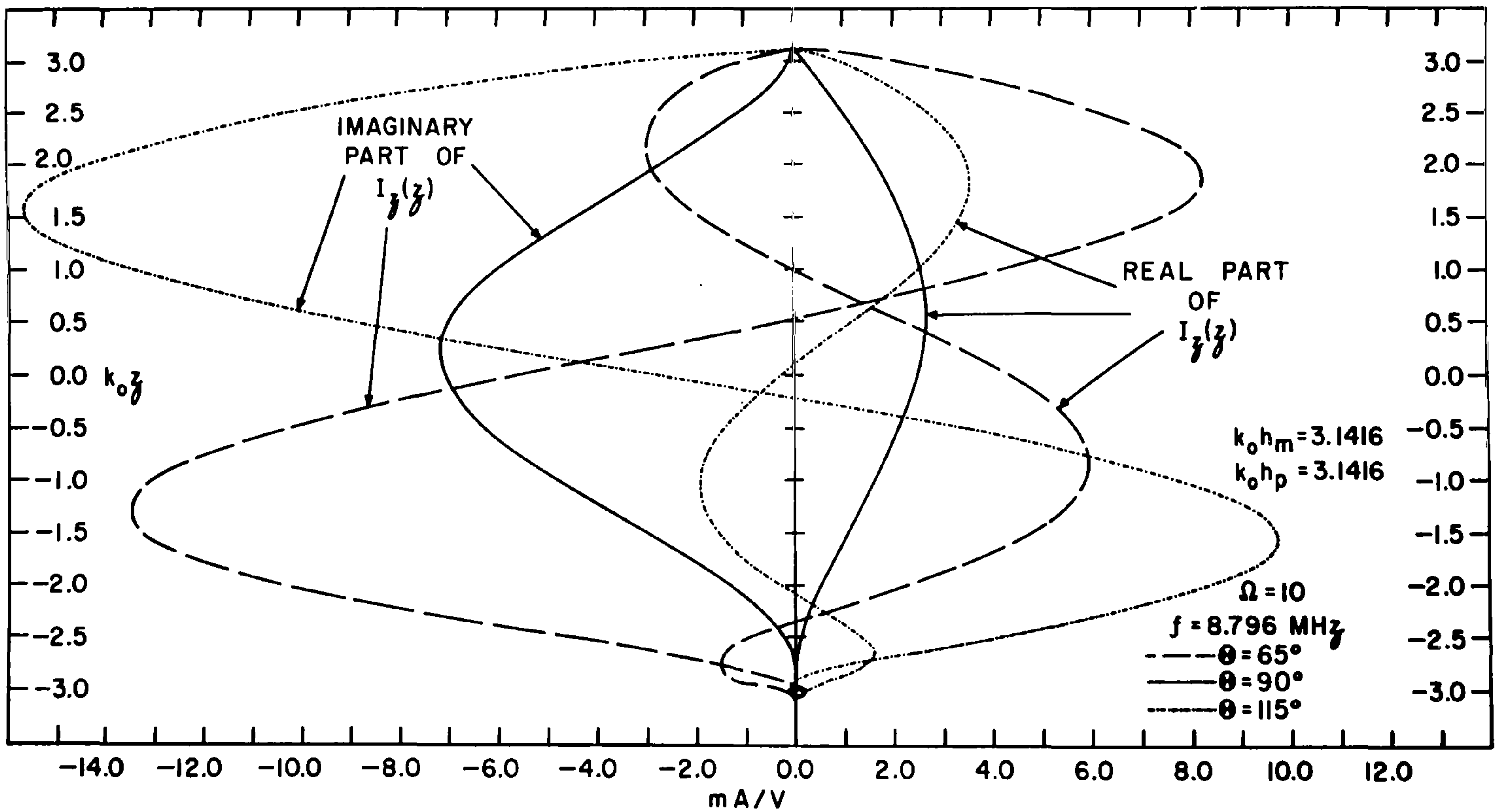


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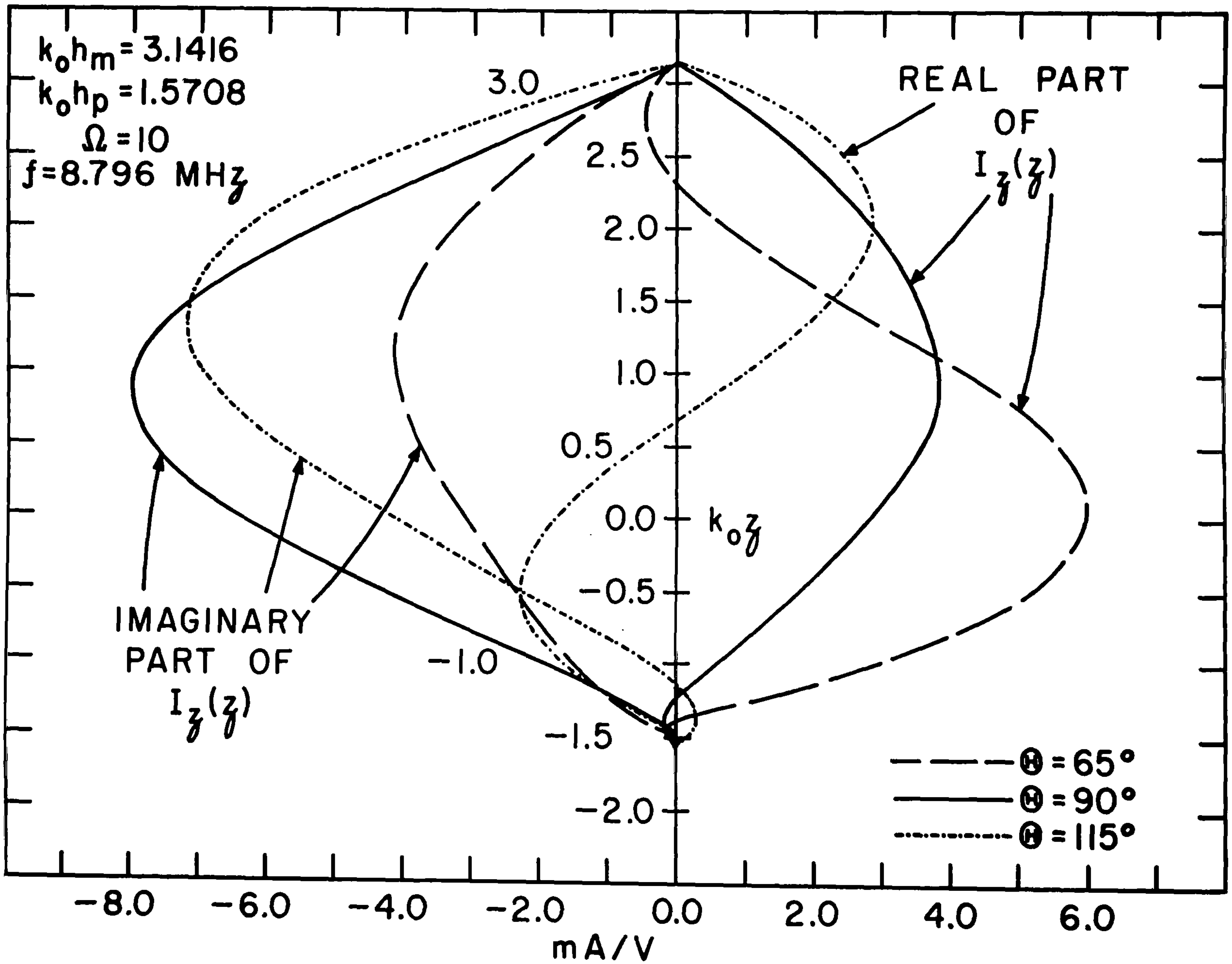


Figure 4. (cont)

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2. E. A. Aronson and C. D. Taylor, "Matrix Methods for Solving Antenna Problems," IEEE Transactions on Antennas and Propagation, Vol. AP-15, No. 5, pp. 696-697, September 1967.

3. E. A. Aronson and C. W. Harrison, Jr., "A Note on Scattering from Linear Antennas of Uniform Conductivity," Sandia Laboratories SC-R-68-1850, July 1968. A copy of this report may be obtained from the Technical Information Department.

4. C. W. Harrison, Jr. and R. O. Heinz, "On the Radar Cross Section of Rods, Tubes, and Strips of Finite Conductivity," IEEE Transactions on Antennas and Propagation, Vol. AP-11, No. 4, pp. 459-468, Equation 51, July 1963.