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A PRELIMINARY REPORT ON THE INTERACTION OF AN ELECTROMAGNETIC FIELD WITH A MISSILE HAVING AN IONIZED TRAIL

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SUMMARY

A missile with plume (ionized trail) in flight at relatively low altitudes is treated as an isolated cylindrical receiving antenna. One portion of the antenna is considered to be perfectly conducting and represents the missile; the other section of the antenna is assumed to be imperfectly conducting and represents the plume. In the present version of the theory no charge is permitted to build up between the two portions of the antenna. The incident electromagnetic field is assumed to be a plane wave with the electric field polarized parallel to the axis of the missile and plume.

The current distributions along both the missile and plume are obtained by solving the nonhomogeneous differential equations for the currents in the two antenna sections subject to appropriate boundary conditions. Inasmuch as the missile skin contains RF leaks, the current distribution along the missile when the plume is present and when it is not is of use in predicting the vulnerability of sensitive electroexplosive devices in the missile circuitry when the missile is immersed in an intense electromagnetic field environment.

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Introduction

In this age of space exploration it has become necessary to assess the receiving characteristics of a missile with plume (ionized trail) to strong electromagnetic fields, because certain internal command functions are initiated by sensitive electroexplosive devices. The internal circuitry feeding the electroexplosives is activated by radio-frequency leaks through anodized peripheral butts between sections of the vehicle; leaks around the edges of closed access doors; leaks through improperly shielded cables that are furrowed into the outside surface of the missile, etc. If the inflight missile is unduly sensitive to the electromagnetic field environment, a degradation in performance or an abortive mission may result. The unwanted radio-frequency field environment may be generated by the operation of high-power radio stations, by local thunderstorm activity, or by a nuclear detonation in the neighborhood.

The underlying objective of this study is to determine the current distributions along a missile with ionized plume when excited by a plane wave electromagnetic field with electric field polarized parallel to the axis of the missile and plume. This permits, for example, the current at a selected point along the missile when the ionized trail is present and when it is absent to be readily compared. Experimentally, it is possible to relate the current in the electroexplosive devices to the total induced missile current at a specified point. By synthesizing the theory developed in this paper with experiment, it is possible to estimate the effect of the plume in decreasing or increasing the current in an electroexplosive device in the missile when the missile and its plume are located in a radio-frequency field environment.

The missile is represented in the present theory as a moderately thin, perfectly conducting, cylindrical receiving and scattering antenna having a smooth surface; the plume is represented in the same way, except that it is imperfectly conducting. Both the missile and plume are assumed to be of the same diameter, but not necessarily of the same length. The turbulence generated in the plume by the shock wave, the degree of inhomogeneity in the ionization of the plume in both the radial and axial directions are ignored, as well as discontinuities in the surface of the ionized trail that may give rise to reflections in the current flowing along the plume and missile, i.e., bring about partial resonances at certain frequencies. Thus, for theoretical purposes the plume is considered to be a homogeneous cylindrical rod of sufficient conductivity and small radius to permit the assumption to be made that transverse currents are small compared to the longitudinal currents, and that these currents are not coupled. The supposition is made that the hypothetical missile with ionized trail described in some detail above applies during the missile launch phase while the vehicle is at relatively low altitudes and fuel burning is in progress.

At low altitudes electromagnetic coupling exists between the missile-plume and its collinear image. Evidently, at early times the plume even makes contact with the earth. (At broadcast frequencies and longer wavelengths, when the earth acts essentially like a perfect conductor, the plume length is doubled when the image is taken into account.) These effects are ignored in the theoretical development. This is tantamount to making the assumption that immediately following launch the missile-ionized trail configuration is in free flight, and all other conductors and dielectrics in the universe are in the far zone.

On the Philosophy of Plumes

It appears that the maximum enhancement of missile current would occur when the plume is very highly conducting and of sufficient length to effect resonance at the frequency of the incident electromagnetic field. This eventually would, of course, accentuate the RF hazards to ordnance problem. If the overall length of the missile and its plume is less than one-half wavelength at the frequency of the incident field, as the length of the plume or its conductivity is decreased one would expect the amplitude of the missile current distribution to decrease. Thus the plume may be regarded as a missile tuning element. And even if the electrical properties of the plume should turn out to be those of an essentially pure dielectric at some given frequency (an unlikely circumstance), it is still possible that the missile current might be increased somewhat over the current that would exist if the plume were absent. This is true because a dielectric rod may also act as a tuning element. If the plume is considered to possess homogeneous electrical properties, have no discontinuities in its surface, and to be infinitely long, no resonances are possible. This problem has been solved at the Sandia Laboratory, but no numerical results relating to missile current for this configuration are yet available. Particular attention is invited to the fact that if a reactive junction impedance is developed between the missile and plume, resonances can obtain for missiles and plumes of almost any length. Also, for nonwavey surface (i.e., not corrugated) plumes of finite length having appropriately tapered conductivity resonances can be eliminated. This problem has also been solved theoretically at the Sandia Laboratory, but no computer program has been written to obtain numbers.

Evidently, the current at the center of the missile when the plume is considered to be perfectly conducting will be larger in magnitude than the resulting current at the same point when the plume has some uniform finite conductivity. Also, the magnitude of this current will be even smaller if the conductivity decreases along the plume axis rather than remains constant.

In concluding this section of the paper the writers would like to illustrate by a simple, but fundamentally correct, numerical example the possible effect on the current in an electrically short missile by the addition of an ionized trail of sufficient length to tune the missile to resonance. For this purpose both the missile and plume are considered to be highly conducting. In this illustration frequent reference is made to the book T.L.A., so designated for brevity.[1]

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[1] Ronold W. P. King, "Theory of Linear Antennas," Harvard University Press, Cambridge, Massachusetts (1956).

Referring to Figure 1, suppose that the missile-plume structure has the shape parameter $\Omega_{mp} = 2\ell n\left[(2h+\ell)/a\right] = 20$. The first resonance in this configuration occurs at $k_o(h + \ell/2) = 1.477$ and $Z_{mp} = 72.2 + j0.0$ ohms (Table 30.1, p 168, T.L.A.). Here k_o is the free space radian wave number, and Z_{mp} is the driving point impedance referred to the point midway between the free ends of the missile and plume. The effective length of the structure $h_{emp} \simeq 0.15\lambda$ (Figure 9.5, p 491, T.L.A.), where λ is the free space wavelength of the incident electromagnetic field. Also, let $h/a = 74.2$ with $h = 8.54$m, so that $\Omega_m = 2\ \ell n\ (2h/a) = 10$. Ω_m is the shape parameter of the missile when the plume is absent. Then

$$2h + \ell = \left(\frac{2h + \ell}{a}\right)\left(\frac{a}{h}\right)h = e^{10}\ \frac{8.54}{74.2} = 2536\text{m} .$$

The propagation constant k_o at resonance is $k_o = 1.165 \times 10^{-3}\ \text{m}^{-1}$. It follows that $\lambda = 5.390$ km and $f = 55.52$ kcs. Then $2h_{emp} = 1617$m, so that the total axial current at the center of the missile-plume configuration is $I_{zmp}\left(\frac{-\ell}{2}\right) = \frac{2h_{emp}}{Z_{mp}} E_z^{inc} = 22.40\, E_z^{inc}$.

It is most important to observe that $I_{zmp}\left(\frac{-\ell}{2}\right)$ establishes the scale factor for the current in the entire missile plume structure. The current distribution may be determined with reasonable accuracy from the leading term in the formula for the current along an unloaded receiving and scattering antenna. This formula, with origin for coordinates corresponding to Figure 1, is

$$I_{zmp}(z) = I_{zmp}\left(\frac{-\ell}{2}\right)\left[\frac{\cos k_o\left(z + \frac{\ell}{2}\right) - \cos k_o\left(h + \frac{\ell}{2}\right)}{1 - \cos k_o\left(h + \frac{\ell}{2}\right)}\right] .$$

It follows that

$$I_{zmp}(0) = 0.2459\, E_z^{inc} .$$

Now consider the missile alone. Electrically short antenna theory may be applied to such a structure when h = 8.54m and f = 55.52 kcs.

The missile impedance is

$$Z_m \simeq -j\frac{60}{k_o h}(\Omega - 3.39) \simeq -j3.99 \times 10^4 \text{ ohms}$$

[Equation 6a, p 184, T. L. A.] .

Also, the effective length of the missile is

$$2h_{em} \simeq \frac{h(\Omega - 1)}{\Omega - 0.613} \simeq 8.19\text{m}$$

[Equation 21, p 487, T.L.A.] .

It follows that the total axial current at the mid point of the missile (with plume absent) is

$$I_{zm}(0) = \frac{2h_{em}}{Z_m} E_z^{inc} = j2.05 \times 10^{-4} E_z^{inc} .$$

The ratio of these currents is

$$I_{zmp}(0)/I_{zm}(0) = 1.20 \times 10^3 .$$

Thus, if the incident electric field strength E_z^{inc} is 10^3 volts/m, the current at the center of the missile without plume is 205 milliamperes. On the other hand, the current at the center of the missile with resonating plume is 245.9 amperes.*

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*In this illustration the current amplitude was always computed at the center of the missile - whether the plume was present or not. But it would have been legitimate in this example to refer the current to the base of the missile when the plume is present, and to the center of the missile in the absence of the plume. Had this been done, the RF hazards situation would have been made much worse.

In this example the frequency is f = 55.52 kcs and the plume length ℓ = 2.52 km. These figures may appear unreasonable to the reader, but it is to be remembered that the EMP spectrum of a nuclear detonation may contain large amplitude fields several octaves lower in frequency than the frequency of the steady-state incident electromagnetic field assumed in this illustration. Until plume technology is much further advanced can one rule out highly conducting plumes in excess of 1.57 miles long?

The Differential Equation for the Current Along a Cylindrical Antenna with Perfectly Conducting and Imperfectly Conducting Sections in Contact

Consider a cylindrical antenna of radius a and length $2h + \ell$, as illustrated by Figure 1. The section of length 2h is perfectly conducting and represents the missile. The section of length ℓ is imperfectly conducting and represents the plume. The origin of a cylindrical coordinate system is taken on axis midway in the perfectly conducting section. The axis of the cylinder lies parallel to the z-coordinate.

The scalar potential ϕ and vector potential $\vec{A}$ are defined in terms of the electric field $\vec{E}$ and magnetic field $\vec{B}$ by the relations

$$-\nabla\phi = \vec{E} + j\omega\vec{A} , \tag{1}$$

$$\nabla \times \vec{A} = \vec{B} , \tag{2}$$

$$\nabla \cdot \vec{A} = -j \frac{k_o^2}{\omega} \phi . \tag{3}$$

Here k_o is the free space propagation constant $k_o = 2\pi/\lambda$ and $\omega = 2\pi f$ is the radian frequency. The assumed but suppressed time dependence used in this work is $\exp(j\omega t)$.

The superscripts t, s, and i appearing in the following analytical development have the meaning total, scattered, and incident, respectively. It is necessary to carefully delineate the potentials and fields being discussed.

It follows from (1) and (3) that

$$\nabla\nabla \cdot \vec{A}^t + k_o^2 \vec{A}^t = j \frac{k_o^2}{\omega} \vec{E}^t \; . \tag{4}$$

An important result due to Stratton[2] is that

$$\vec{A}^i = j \frac{\vec{E}^i}{\omega} \; . \tag{5}$$

Now in a charge free region

$$\nabla \cdot \vec{E}^i = 0 \; . \tag{6}$$

Using (5) in (6) it follows that

$$\nabla \cdot \vec{A}^i = 0 \; . \tag{7}$$

Hence

$$\nabla\nabla \cdot \vec{A}^t = \nabla\nabla \cdot \vec{A}^s \; . \tag{8}$$

Provided $A_r^s << A_z^s$ and $A_\theta^s = 0$, (8) may be written

$$\nabla\nabla \cdot \vec{A}^s \simeq \frac{\partial^2}{\partial z^2} A_z^s \; . \tag{9}$$

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[2] Julius A. Stratton, "Electromagnetic Theory," McGraw-Hill Book Co., Inc, p 26, Eq. (24), (1941).

Substituting (9) into (4) and using the fact that

$$\vec{A}_z^t = \vec{A}_z^s + \vec{A}_z^i , \tag{10}$$

one obtains

$$\left(\frac{\partial^2}{\partial z^2} + k_o^2\right) A_z^s(a, z) = j \frac{k_o^2}{\omega} E_z^t(a, z) - k_o^2 A_z^i(a, z) . \tag{11}$$

If the ionized trail of the missile is sufficiently conducting so that an internal impedance per unit length z^i can be defined it is correct to write

$$E_z^t(a, z) = z^i I_z(z) , \tag{12}$$

where $I_z(z)$ is the total cross-sectional current in the plume flowing in the z-direction. Note that z^i is given by the ratio $E_z^t(a, z)/I_z(z) = \left[E_z^i(a, z) + E_z^s(a, z)\right]/ I_z(z)$ and not by the ratio $E_z^s(a, z)/I_z(z)$, for example, as is true for transmission when the incident field is absent. Thus Equation (12) emphasizes the importance of maintaining the proper superscripts on the potentials and fields, especially on those appearing in Equation (11). Equation (12) applies also to the missile if it is considered to be imperfectly conducting. But normally, $z_m^i \neq z_p^i$, where the subscripts m and p denote the missile and plume, respectively. If the missile is assumed to be perfectly conducting then from (12) it is clear that $z_m^i = 0$.

Assume that the incident electric field is directed parallel to the axis of the missile and plume. It follows that

$$\vec{E}^i = \hat{z} E_z^i . \tag{13}$$

Using (5),

$$A_z^i(a, z) = j\frac{E_z^i(a, z)}{\omega} . \tag{14}$$

If it is further assumed that $k_o a \ll 1$, $a \ll (2h + \ell)$, the scattered vector potential evaluated on the surface of the missile or plume due to the current flow in the structure is

$$A_z^s(a, z) = \frac{\mu_o}{4\pi} \int_{-(h+\ell)}^{h} dz' I_z(z') K(z, z') . \tag{15}$$

Here $\mu_o = 4\pi \times 10^{-7}$ henry/m; $K(z, z') = \exp(-jk_o R)/R$ and $R = \sqrt{(z - z')^2 + a^2}$.

Substituting (13) through (15), inclusive, into (11) leads to the integral equation for the current distribution along the missile and ionized trail. It is

$$\frac{\mu_o}{4\pi}\left(\frac{\partial^2}{\partial z^2} + k_o^2\right)\int_{-(h+\ell)}^{h} dz' I_z(z') K(z, z') - j\frac{k_o^2}{\omega} z^i I_z(z) = -j\frac{k_o^2}{\omega} E_z^i . \tag{16}$$

Rather than attempt to solve Equation (16) for the current $I_z(z)$ directly it is more convenient to obtain the differential equation governing the distribution of current along the missile and plume and to solve it approximately.

An equivalent form of (15) is

$$A_z^s(a, z) = \frac{\mu_o}{4\pi} \psi(z) I_z(z) , \tag{17}$$

where

$$\psi(z) = \frac{1}{I_z(z)} \int_{-(h+\ell)}^{h} dz' I_z(z') K(z, z') . \tag{18}$$

It turns out that whether the dipole antenna is used either as a wave launcher or collector the function $\psi(z)$--which is proportional to the ratio of vector potential to current--is sensibly constant along perfectly conducting antennas except at and near very small or zero values of the current. Since the vector potential is determined largely by the current at or near the point where the vector potential is computed, with the exceptions noted above, it should not make much difference what distribution $I_z(z)$ is used in evaluating (18). In other words, a current distribution may be assumed for establishing the scale factor for the ratio of vector potential to current. However, this point is being considered and will be discussed in a subsequent report. The authors wish to point out emphatically that this is not tantamount to assuming a distribution of current along the missile and plume. Admittedly if $\ell > 2h$ (especially if $\ell \gg 2h$) it is likely that the least error would be made in evaluating $\psi(z)$ if $I_z(z)$ along an unloaded resistive receiving and scattering antenna were used in (18). But if $2h > \ell$ (especially if $2h \gg \ell$), it is probable that a better value of $\psi(z)$ would be obtained by evaluating (18), using $I_z(z)$ developed from the theory of the unloaded perfectly conducting receiving and scattering antenna.[3] An intensive investigation is being conducted by the authors to settle this matter. This is an important point because the present theory is of zeroth-order accuracy, and cannot be used to predict with any precision missile-plume-current distributions except when $2h + \ell < \lambda/2$, i.e., the length of the missile and plume is shorter than one-half wavelength.

In the present study the writers have chosen to use the ψ function[1]

$$\psi(u) = \int_{-\left(\frac{2h+\ell}{2}\right)}^{\left(\frac{2h+\ell}{2}\right)} \frac{\sin k_o\left[\left(\frac{2h+\ell}{2}\right) - |u'|\right]}{\sin k_o\left[\left(\frac{2h+\ell}{2}\right) - |u|\right]} K(u, u')du' . \qquad (19)$$

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[3] Charles W. Harrison, Jr., "Response of Transmission Lines Excited by the Nonuniform Resultant Field in Proximity to a Cylindrical Scatterer of Finite Length," SC-R-65-978, Sandia Corporation, Eq. (22), August 1965.

$$\left.\begin{aligned} \psi &= |\psi(0)| , & k_o\left(h + \frac{\ell}{2}\right) &\le \frac{\pi}{2} \\ \psi &= \left|\psi\left(h + \frac{\ell}{2} - \frac{\lambda}{4}\right)\right| , & k_o\left(h + \frac{\ell}{2}\right) &> \frac{\pi}{2} . \end{aligned}\right\} \tag{20}$$

In (18), $K(u, u') = \exp(jk_o R_u)/R_u$, $R_u = \sqrt{(u - u')^2 + a^2}$.

It is important to note that when ψ is not considered a function of z, (17) requires the ratio of vector potential to current to be a constant. Accordingly, if the vector potential is sinusoidally distributed along the missile and plume, the current distribution is also sinusoidal. (It is worth mentioning that even though the vector potential on a perfectly-conducting center-driven dipole is always sinusoidally distributed the current can never be so distributed.)

Now the differential equation for the current distribution along either the missile or the plume may be obtained by substituting (12), (13), (14), and (17) [with $\psi(z) \simeq \psi$ where ψ is given by (20)] into (11). It is

$$\left(\frac{\partial^2}{\partial z^2} + k^2\right) I_z(z) = -j \frac{4\pi k_o^2 U^i}{\zeta_o \psi} . \tag{21}$$

The notation

$$k^2 = k_o^2 - j \frac{4\pi k_o z^i}{\zeta_o \psi} , \tag{22}$$

is used. k is the wave propagation constant along the plume. ζ_o is the characteristic resistance of free space; $\zeta_o = 120\pi$ ohms. In the present situation, U^i is defined by the relation[1]

$$U^i = -\frac{E^i_{(\theta=\pi/2)}}{k_o} = \frac{E^i_z}{k_o} , \tag{23}$$

because the incident electric field is assumed to be directed parallel to the axis of the missile and plume. Note that $E^i_{(\pi/2)}$ is antiparallel with respect to E^i_z. Observe also, that as written, (21) applies only to the plume. But by writing k_o for k the equation then applies to the missile if the missile is considered to be perfectly conducting, i.e., $z^i = 0$.

The Solution of the Differential Equation for the Currents Along the Missile and Plume

The general solution of (21) applying to the missile is

$$I_{zm}(z) = A\, e^{-jk_o z} + B\, e^{jk_o z} + R_1 \,, \tag{24}$$

where

$$R_1 = -j \frac{4\pi}{\zeta_o \psi} U^i \,, \tag{25}$$

and along the plume the solution of (21) is

$$I_{zp}(z) = C\, e^{-jkz} + D\, e^{jkz} + R_2 \,, \tag{26}$$

where

$$R_2 = \frac{k_o^2}{k^2} R_1 \,. \tag{27}$$

That the expressions for $I_{zm}(z)$ and $I_{zp}(z)$ are valid may be verified by direct substitution into the differential equation.

It is assumed that the boundary conditions to be satisfied by the solutions for the current are

$$I_{zm}(h) = I_{zp}(-h - \ell) = 0 \,, \tag{28}$$

$$I_{zm}(-h) = I_{zp}(-h) \,, \tag{29}$$

$$\frac{\partial}{\partial z} I_{zm}(z)\Big|_{z=-h} = \frac{\partial}{\partial z} I_{zp}(z)\Big|_{z=-h} . \qquad (30)$$

The normally small junction impedance between the plume and missile has been neglected in writing (30).

Applying (28) to (24), and eliminating the constant B, leads to an expression for the current distribution along the missile in terms of the constant A. It is

$$I_{zm}(z) = j2A\, e^{-jk_o h} \sin k_o(h - z) - R_1\, e^{-jk_o(h-z)} + R_1 . \qquad (31)$$

Similarly, applying (28) to (26), and eliminating the constant D leads to an equation for the current distribution along the plume in terms of the constant C. It is

$$I_{zp}(z) = -j2C\, e^{jk(h+\ell)} \sin k(z + h + \ell) - R_2\, e^{jk(z + h + \ell)} + R_2 . \qquad (32)$$

Equating (31) to (32) at $z = -h$, as required by (29), leads to the first of the simultaneous equations for the constants A and C. The expression is

$$j2A\, e^{-jk_o h} \sin 2k_o h - R_1\, e^{-j2k_o h} + R_1 = -j2C e^{jk(h+\ell)} \sin k\ell - R_2\, e^{jk\ell} + R_2 . \qquad (33)$$

The second of the simultaneous equations for the constants A and C is obtained by differentiating (31) and (32) as required by (30), and equating the results at $z = -h$. It follows that

$$2k_o A\, e^{-jk_o h} \cos 2k_o h + k_o R_1\, e^{-j2k_o h} = 2kC\, e^{jk(h+\ell)} \cos k\ell + kR_2\, e^{jk\ell} . \qquad (34)$$

Solving (33) and (34) for the constants A and C yields

$$A = j\, \frac{1}{2Q}\, R_2\, e^{jk_o h}\left[1 - \frac{R_1}{R_2}\left(\cos k\ell - j\,\frac{k_o}{k} \sin k\ell\right) e^{-j2k_o h} + \left(\frac{R_1 - R_2}{R_2}\right)\cos k\ell\right], \qquad (35)$$

and

$$C = -j \frac{1}{2Q} R_1 e^{-jk(h+\ell)} \frac{k_o}{k} \left[1 - \frac{R_2}{R_1}\left(\cos 2k_o h + j \frac{k}{k_o} \sin 2k_o h\right) e^{jk\ell} \right.$$

and

$$\left. - \left(\frac{R_1 - R_2}{R_1}\right) \cos 2k_o h \right] , \tag{36}$$

where

$$Q = \frac{k_o}{k} \cos 2k_o h \sin k\ell + \sin 2k_o h \cos k\ell . \tag{37}$$

For convenience the following notation is introduced:

$$T_m = j \frac{2A}{R_2} e^{-jk_o h} , \tag{38}$$

and

$$T_p = -j \frac{2C}{R_1} e^{jk(h+\ )} . \tag{39}$$

Then

$$T_m = -\frac{1}{Q}\left[1 - \frac{k^2}{k_o^2}\left(\cos k\ell - j \frac{k_o}{k} \sin k\ell\right) e^{-j2k_o h} - \left(1 - \frac{k^2}{k_o^2}\right) \cos k\ell \right] , \tag{40}$$

and

$$T_p = -\frac{k_o}{kQ}\left[1 - \frac{k_o^2}{k^2}\left(\cos 2k_o h + j \frac{k}{k_o} \sin 2k_o h\right) e^{jk\ell} - \left(1 - \frac{k_o^2}{k^2}\right) \cos 2k_o h \right] . \tag{41}$$

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* It should be observed that $T_m \rightarrow \infty$ and $T_p \rightarrow \infty$ as $Q \rightarrow 0$. Evidently, if $k = k_o$, $Q = 0$ when $k_o(2h + \ell) = n\pi$. It follows that the theory in its present form is not valid under the circumstances set forth here when $(2h + \ell) = n(\lambda/2)$, $n = 1, 2, 3, \ldots$ Even if $k \neq k_o$, erratic results should be anticipated when the overall length of the missile and plume is in the neighborhood of a multiple of a half-wavelength of the incident radiation. Also, since reradiation is not considered best results are obtained when $(2h + \ell) < \lambda/2$.

Solving (38) for A and (39) for C, and substituting the values of these constants in (31) and (32), respectively, leads to the final expressions for the current distributions along the missile and plume. These are

$$I_{zm}(z) = -j\frac{4\pi U^{i}}{\zeta_{o}\psi}\frac{k_{o}^{2}}{k^{2}}\left[T_{m}\sin k_{o}(h-z) + \frac{k^{2}}{k_{o}^{2}}\left(1 - e^{-jk_{o}(h-z)}\right)\right], \tag{42}$$

and

$$I_{zp}(z) = -j\frac{4\pi U^{i}}{\zeta_{o}\psi}\left[T_{p}\sin k(z+h+\ell) + \frac{k_{o}^{2}}{k^{2}}\left(1 - e^{jk(z+h+\ell)}\right)\right]. \tag{43}$$

The assumptions have been made that the missile and plume have the same diameter; that there is no accumulation of charge at the juncture of the missile and plume, and that the plume is of uniform (as opposed to tapered) conductivity. Most of the above restrictions applying to the present treatment of the problem can be removed in a more advanced theoretical analysis of a missile with ionized trail.

Internal Impedance per Unit Length of an Imperfectly Conducting Cylinder

In Equation (12) the internal impedance per unit length of a cylindrical conductor is defined. To the best knowledge of the writers all derivations leading to a formula for z^{i} are based on the assumption that the conductor is infinitely long. This leads to an expression for z^{i} that is independent of z. The procedure followed to develop a formula for z^{i} in the literature is to write, in cylindrical coordinates, the one-dimensional wave equation in the variable r for the current density flowing in the z-direction. Solution of the equation leads to an expression for the current density at any point r in terms of the current density at the surface of the conductor. The total current is then obtained by integrating the current density across the transverse plane of the conductor. Since the current density at the surface of the conductor equals the conductivity multiplied by the electric field on the surface, the internal impedance per unit length of a conductor may be found from its definition.

All skin effect formulas are based on the assumption that ordinary circuit theory applies. This requires that the conductor be sufficiently good so that the surface field at a given point is controlled essentially by currents and charges in the neighborhood of that point. It is to be remembered that at a given frequency the wavelength is extremely short in a good conductor compared to the wavelength in free space. By definition, a near-zone circuit problem is one in which the distance between interacting currents and charges is small in terms of the wavelength. It is worth mentioning that in addition to the assumption that the cylinder is infinitely long and highly conducting it is also implicit in the derivation for z^i that the axial distribution of current is not coupled to the transverse distribution.

It is possible to use with precision a constant value of z^i, determined in the manner outlined above, in dissipative antenna theory provided $\sigma \gg \omega\epsilon$, $h \gg a$, and $k_o a \ll 1$. This is interesting when it is remembered that $E_z(a, z)$ and $I_z(z)$ are far from uniform on this structure of finite length! Of course, fields will penetrate into a good conductor when the radius of the cylinder is small compared to the skin depth.

In a relatively poorly conducting plasma the situation is completely changed. The problem of determining z^i is no longer one of ordinary circuit theory because the volume element in which significant interaction of currents and charges take place is enlarged in terms of the wavelength in the plasma medium. The fields penetrate into the plasma column giving rise to coupled axial and transverse currents. If the plasma cylinder does not satisfy the inequality $k_o a \ll 1$, rotational symmetry of the incident field does not obtain, and this gives rise to axial and transverse currents. Furthermore, in a "transmitting antenna" consisting of a plasma column, an initially axial current must give rise to substantial transverse currents in the column.

In view of the foregoing, the authors must assume that the plume properties satisfy the inequality $\sigma \gg \omega\epsilon$. It then follows that the displacement current in the plasma column is small and cannot build up a significant charge at the junction between the missile and its ionized trail. This means that the junction impedance is small. As mentioned earlier in this paper, this impedance was not considered.

By solving a simple one-dimensional boundary value problem in cylindrical coordinates, it is readily shown that[4]

$$E_z(r) = E_z(a) \frac{J_o(\gamma r)}{J_o(\gamma a)} . \tag{44}$$

Here $E_z(r)$ is the electric field in the z-direction at radial distance r from the center of an infinite imperfectly conducting cylinder of outer radius a. Here

$$\gamma^2 = \omega^2 \mu \xi \ , \tag{45}$$

where

$$\xi = \epsilon \left(1 - j \frac{\sigma}{\omega\epsilon}\right) \ , \tag{46}$$

and μ, ϵ, and σ are the constitutive parameters of the cylinder (ionized trail or plume in this case).

Now,

$$I_z(z) = \oint H_\theta r d\theta = \int_S \nabla \times \vec{H} \cdot \hat{z} ds \ , \tag{47}$$

and

$$\nabla \times \vec{H} = (\sigma + j\omega\epsilon)\vec{E} \ . \tag{48}$$

It follows that on substituting (48) into (47), and using (44), that

$$I_z = 2\pi(\sigma + j\omega\epsilon) \frac{E_z(a)}{J_o(\gamma a)} \int_0^a J_o(\gamma r) r dr \ , \tag{49}$$

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[4] Ronold W. P. King, "Fundamental Electromagnetic Theory," Dover Publications, Inc., New York, Chapter V, pp 321-401, (1963).

or

$$I_z = \frac{2\pi a(\sigma + j\omega\epsilon)}{\gamma} \frac{E_z(a)J_1(\gamma a)}{J_o(\gamma a)} . \qquad (50)$$

If the approximation

$$\sigma \gg \omega\epsilon , \qquad (51)$$

is made,

$$\gamma = (1 - j)\sqrt{\frac{\omega\mu\sigma}{2}} . \qquad (52)$$

Then,

$$z^i = \frac{E_z(a)}{I_z} = \frac{(1 - j)\sqrt{\frac{\omega\mu\sigma}{2}}\, J_o\left[a(1 - j)\sqrt{\frac{\omega\mu\sigma}{2}}\right]}{2\pi a\sigma J_1\left[a(1 - j)\sqrt{\frac{\omega\mu\sigma}{2}}\right]} \text{ ohms/m} . \qquad (53)$$

This is the final formula for z^i used in the computations reported in this paper.

On the Radar Cross Section of a Missile Having an Ionized Trail

The radar cross section of a missile-plume configuration, assuming the incident electric field is directed parallel to the axis of the structure, is

$$\sigma_{11} = 4\pi r^2 \left|\frac{E_z^s}{E_z^i}\right|^2 . \qquad (54)$$

It is easily shown that the magnitude of the scattered field E_z^s in the far zone from the missile and its exhaust is[5]

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[5]C. W. Harrison, Jr., and R. O. Heinz, "On the Radar Cross Section of Rods, Tubes, and Strips of Finite Conductivity," IEEE Transactions on Antennas and Propagation, Vol. AP-11, No. 4, pp. 459-468, July 1963. See also R. W. P. King and C. W. Harrison, Jr., "Current Distribution and Impedance per Unit Length of a Thin Strip," submitted for publication as a Communication in the IEEE Transactions on Antennas and Propagation.

$$\left|E_z^s\right| = \left|\frac{\zeta_o k_o}{4\pi r} \int_{-(h+\ell)}^{h} I_z(z')dz'\right| . \tag{55}$$

Thus,

$$\left|E_z^s\right| = \left|\frac{\zeta_o k_o}{4\pi r}\left(\int_{-h}^{h} I_{zm}(z')dz' + \int_{-(h+\ell)}^{-h} I_{zp}(z')dz'\right)\right| , \tag{56}$$

where $I_{zm}(z)$ and $I_{zp}(z)$ are given by (42) and (43), respectively. On substituting these expressions for the current along the missile and its plume into (56), integrating, and then using the definition (54) for the radar cross section σ_{11} results in the expression

$$\sigma_{11} = \frac{4\pi}{|\psi|^2}\left|\frac{k_o^2}{k^2} T_m\left(\frac{1 - \cos 2k_o h}{k_o}\right) + T_p\left(\frac{1 - \cos k\ell}{k}\right)\right.$$

$$\left. + 2h + \frac{k_o^2 \ell}{k^2} + \frac{j}{k_o}\left(1 - e^{-j2k_o h}\right) - j\frac{k_o^2}{k^2}\left(\frac{1 - e^{jk\ell}}{k}\right)\right|^2 . \tag{57}$$

This is the final formula for the radar cross section of the missile and its plume, when the incident electric field is polarized parallel to the axis of the missile.

On the Merit of Comparing Numerically the Missile-Plume Propagation Constants

There are circumstances when considerable insight into the effect of the plume on the current in the missile may be obtained by simply computing k, the ionized trail propagation constant, and comparing it with k_o, the missile propagation constant. Evidently, if $k \simeq k_o$ the plume is acting essentially like a perfectly conducting extension of the missile. The current distribution along both the missile and plume is then obtained easily from an application of classical linear antenna theory.[1, 3] This avoids programming of equations (42) and (43) in this paper to obtain missile and plume currents. A simple numerical example will illustrate the writers' point.

The plume propagation constant k is given by (22); that is

$$k^2 = k_o^2 - j\frac{4\pi k_o z^i}{\zeta_o \psi} .$$

For convenience let the missile length 2h = 60' = 18.29 m, and the plume length ℓ = 18.29 m, so that $(2h + \ell)$ = 36.58 m. Also, let $\Omega = 2\ \ell n\left(\frac{2h + \ell}{a}\right) = 7$. It follows that $(2h + \ell)/a = 16.56$, hence a = 1.104 m. Suppose that at the frequency of the incident electromagnetic field $k_o(2h + \ell) = \pi/2$ radians. Then $f = 2.05 \times 10^6$ cps, $\omega = 2\pi f = 12.88 \times 10^6$ radians/sec, λ = 146.34 m, and $k_o = 2\pi/\lambda = 4.3 \times 10^{-2}$ radians/m. Assume that σ = 40 mhos/m (10 times the conductivity of sea water). Then from (53), $z^i \simeq 6.49(1 + j)10^{-2}$ ohms/m. One now computes ψ from (20), assuming for this purpose that the missile and plume are perfectly conducting. Then $\psi = 5.66$.* Remember also that $\zeta_o = 120\pi$ ohms.

Substituting the above numbers in the formula for k^2, one obtains $k_o^2 = 1.85 \times 10^{-3}$ radians/m^2, and $-j\left(4\pi k_o z^i/\zeta_o\psi\right) = 1.64(1 - j)10^{-5}$ radians/m^2. It therefore follows that $k \simeq k_o$. Thus the plume may be considered perfectly conducting in this instance and the distribution of current along the missile (and plume) determined using the standard formulas for the short-circuit current in a perfectly conducting receiving and scattering antenna of length $(2h + \ell)$ and radius a.[1,3] If it turns out that the value of $\left|4\pi k_o z^i/\zeta_o\psi\right|$ is significant compared to k_o^2 it will be necessary to program the theory presented in this paper for a digital computer in order to obtain numerical information revealing exactly what effect the ionized trail of an inflight missile has on the amplitude of the current in the

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*Some readers may be concerned over the fact that ψ, as computed from (20), is based on the over-all length of the missile and plume. It can be shown[1] that $\psi \simeq \Omega - 2$. If $\Omega \geq 7$ and $2h \sim \ell$ it obviously does not make a great deal of difference in ψ if Ω is taken to be $2\ \ell n(2h/a)$, $2\ \ell n(\ /a)$, or $2\ \ell n[(2h + \ell)/a]$.

missile, when the missile with plume is illuminated by a plane wave electromagnetic field with the electric vector directed parallel to the axis of the missile.

Numerical Results

Two tables of numerical results are presented that are essentially self-explanatory. In both tables the missile half-length is h = 8.54 meters. In the first table h/a = 10.7, and in the second h/a = 85.4. In all cases the incident electric field is taken as 1 volt/m rms, i.e., $E_z^{inc} = 1$ volt/m. The numbers appearing to the right in each column indicate the power of 10 to which the number in the same column to the left is to be raised. For example, 6.9840 + 003 means 6.9840×10^3. Similarly, 1.2500 - 003 means 1.2500×10^{-3}.

Conclusions

A preliminary zeroth-order theory has been presented leading to formulas for the distribution of current along a perfectly conducting missile in free flight with collinear ionized trail of finite length that is considered to be imperfectly conducting. Reradiation from the missile-plume configuration is ignored in the present theory. The structure so formed is excited by a plane wave electromagnetic field with electric vector directed parallel to the missile axis. Although in the present theory the internal impedance per unit length of the plume z^i is taken to be constant, the problem has been solved at the Sandia Laboratory for a tapered z^i, i.e., $z^i(z)$. The analytical form of the axial taper would be determined by experiment. If $\sigma(r)$ is known from experiment, rotational symmetry obtains and the cylinder is long compared to its radius, z^i can be calculated from the definition (12). But the evaluation of z^i may require the solution of a differential equation using machine methods. In all cases it is necessary that $\omega\epsilon/\sigma \ll 1$ because it appears that in light of the present technology z^i cannot be defined for a very poor conductor--a plasma for instance.

The writers are of the opinion that radar cross-section measurements of missile-plume configurations must be carried out with caution, especially if the plume is a poorly conducting plasma in a state of turbulence. The plume may be partially transparent to the incident field at the frequency in use, and one must expect the back-scattered signal to be of random polarization. Thus an incorrect result is obtained if the radar antenna is sensitive to vector fields of fixed orientation.

It has been shown that once the current $I_z(z)$ along both the missile and plume are known in terms of E_z^i, the radar cross-section of the missile-plume configuration may be determined easily for parallel incidence of the electric field.

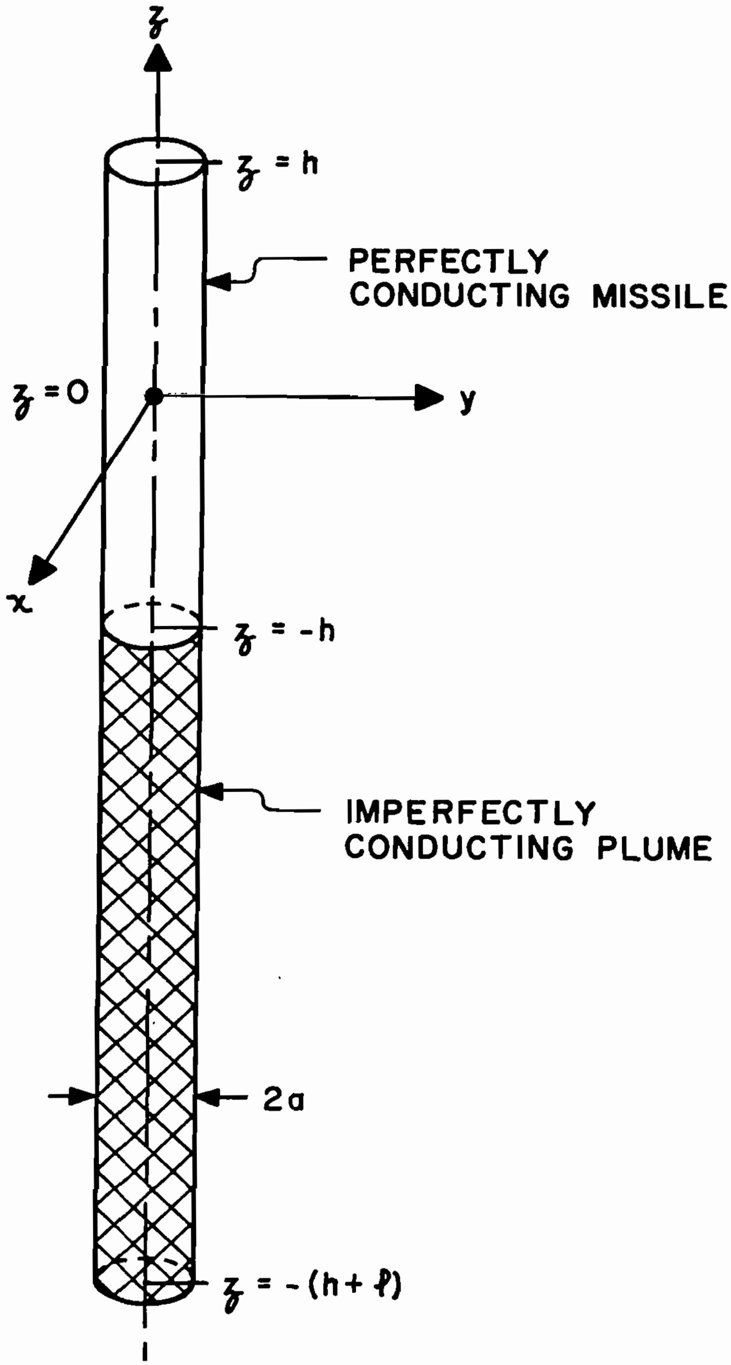

Figure 1. Two Section Cylindrical Antenna Representing a Missile with Plume (Ionized Trail) at Low Altitude

TABLE 1

h/a = 10.7

f (cps)	$k_o h$ (radians)	ℓ (meters)	σ (mhos/m)	$\frac{\omega\epsilon}{\sigma}$	ψ	$\lvert I_{zm}(0)\rvert$ (amperes)	k_o (radians/m)	Re k (radians/m)	Im k (radians/m)
6.9840+003	1.2500-003	0.0000+000	1.0000+001	3.8835-008	4.303+000	4.135-005	1.464-004	2.0640-004	-1.3670-004
6.9840+004	1.2500-002	0.0000+000	1.0000+001	3.8835-007	4.303+000	4.135-004	1.464-003	1.5559-003	-1.9241-004
6.9840+005	1.2500-001	0.0000+000	1.0000+001	3.8835-006	4.306+000	4.159-003	1.464-002	1.5037-002	-4.4479-004
6.9840+006	1.2500+000	0.0000+000	1.0000+001	3.8835-005	4.607+000	1.073-001	1.464-001	1.4756-001	-1.2316-003
6.9840+003	1.2500-003	0.0000+000	1.0000+000	3.8835-007	4.303+000	4.135-005	1.464-004	5.4237-004	-5.1986-004
6.9840+004	1.2500-002	0.0000+000	1.0000+000	3.8835-006	4.303+000	4.135-004	1.464-003	2.0640-003	-1.3670-003
6.9840+005	1.2500-001	0.0000+000	1.0000+000	3.8835-005	4.306+000	4.159-003	1.464-002	1.5558-002	-1.9231-003
6.9840+006	1.2500+000	0.0000+000	1.0000+000	3.8835-004	4.607+000	1.073-001	1.464-001	1.5011-001	-4.1642-003
6.9840+003	1.2500-003	0.0000+000	1.0000-001	3.8835-006	4.303+000	4.134-005	1.464-004	1.6827-003	-1.6756-003
6.9840+004	1.2500-002	0.0000+000	1.0000-001	3.8835-005	4.303+000	4.135-004	1.464-003	5.4237-003	-5.1986-003
6.9840+005	1.2500-001	0.0000+000	1.0000-001	3.8835-004	4.306+000	4.159-003	1.464-002	2.0636-002	-1.3664-002
6.9840+006	1.2500+000	0.0000+000	1.0000-001	3.8835-003	4.607+000	1.073-001	1.464-001	1.5493-001	-1.8048-002
6.9840+003	1.2500-003	0.0000+000	1.0000-002	3.8835-005	4.303+000	4.135-005	1.464-004	5.3111-003	-5.3088-003
6.9840+004	1.2500-002	0.0000+000	1.0000-002	3.8835-004	4.303+000	4.135-004	1.464-003	1.6827-002	-1.6756-002
6.9840+005	1.2500-001	0.0000+000	1.0000-002	3.8835-003	4.306+000	4.159-003	1.464-002	5.4222-002	-5.1970-002
6.9840+006	1.2500+000	0.0000+000	1.0000-002	3.8835-002	4.607+000	1.073-001	1.464-001	2.0194-001	-1.3050-001
6.9840+003	1.2500-003	0.0000+000	1.0000-003	3.8835-004	4.303+000	4.135-005	1.464-004	1.6792-002	-1.6791-002
6.9840+004	1.2500-002	0.0000+000	1.0000-003	3.8835-003	4.303+000	4.135-004	1.464-003	5.3111-002	-5.3088-002
6.9840+005	1.2500-001	0.0000+000	1.0000-003	3.8835-002	4.306+000	4.159-003	1.464-002	1.6822-001	-1.6751-001
6.9840+006	1.2500+000	0.0000+000	1.0000-003	3.8835-001	4.607+000	1.073-001	1.464-001	5.2488-001	-5.0175-001
3.4920+003	6.2500-004	1.7080+001	1.0000+001	1.9418-008	5.599+000	4.766-005	7.319-005	1.1891-004	-9.1134-005
3.4920+004	6.2500-003	1.7080+001	1.0000+001	1.9418-007	5.599+000	4.767-004	7.319-004	7.7662-004	-1.4175-004
3.4920+005	6.2500-002	1.7080+001	1.0000+001	1.9418-006	5.602+000	4.795-003	7.319-003	7.5341-003	-2.5200-004
3.4920+006	6.2500-001	1.7080+001	1.0000+001	1.9418-005	5.909+000	1.228-001	7.319-002	7.3843-002	-6.8935-004
3.4920+003	6.2500-004	1.7080+001	1.0000+000	1.9418-007	5.599+000	4.766-005	7.319-005	3.3363-004	-3.2477-004
3.4920+004	6.2500-003	1.7080+001	1.0000+000	1.9418-006	5.599+000	4.767-004	7.319-004	1.1891-003	-9.1134-004
3.4920+005	6.2500-002	1.7080+001	1.0000+000	1.9418-005	5.602+000	4.795-003	7.319-003	7.7660-003	-1.4169-003
3.4920+006	6.2500-001	1.7080+001	1.0000+000	1.9418-004	5.909+000	1.262-001	7.319-002	7.5228-002	-2.3926-003
3.4920+003	6.2500-004	1.7080+001	1.0000-001	1.9418-006	5.599+000	4.766-005	7.319-005	1.0423-003	-1.0395-003
3.4920+004	6.2500-003	1.7080+001	1.0000-001	1.9418-005	5.599+000	4.767-004	7.319-004	3.3363-003	-3.2477-003
3.4920+005	6.2500-002	1.7080+001	1.0000-001	1.9418-004	5.602+000	4.794-003	7.319-003	1.1889-002	-9.1108-003
3.4920+006	6.2500-001	1.7080+001	1.0000-001	1.9418-003	5.909+000	1.207-001	7.319-002	7.7378-002	-1.3481-002
3.4920+003	6.2500-004	1.7080+001	1.0000-002	1.9418-005	5.599+000	4.767-005	7.319-005	3.2921-003	-3.2912-003
3.4920+004	6.2500-003	1.7080+001	1.0000-002	1.9418-004	5.599+000	4.766-004	7.319-004	1.0423-002	-1.0395-002
3.4920+005	6.2500-002	1.7080+001	1.0000-002	1.9418-003	5.602+000	4.750-003	7.319-003	3.3355-002	-3.2469-002
3.4920+006	6.2500-001	1.7080+001	1.0000-002	1.9418-002	5.909+000	4.074-002	7.319-002	1.1652-001	-8.8131-002
3.4920+003	6.2500-004	1.7080+001	1.0000-003	1.9418-004	5.599+000	4.766-005	7.319-005	1.0409-002	-1.0409-002
3.4920+004	6.2500-003	1.7080+001	1.0000-003	1.9418-003	5.599+000	4.722-004	7.319-004	3.2921-002	-3.2912-002
3.4920+005	6.2500-002	1.7080+001	1.0000-003	1.9418-002	5.602+000	3.151-003	7.319-003	1.0421-001	-1.0393-001
3.4920+006	6.2500-001	1.7080+001	1.0000-003	1.9418-001	5.909+000	2.236-002	7.319-002	3.2499-001	-3.1592-001
1.1640+003	2.0833-004	8.5400+001	1.0000+001	6.4726-009	7.735+000	4.217-005	2.440-005	5.4318-005	-4.8134-005
1.1640+004	2.0833-003	8.5400+001	1.0000+001	6.4726-008	7.735+000	4.217-004	2.440-004	2.6978-004	-9.7085-005
1.1640+005	2.0833-002	8.5400+001	1.0000+001	6.4726-007	7.737+000	4.241-003	2.440-003	2.5140-003	-1.2038-004
1.1640+006	2.0833-001	8.5400+001	1.0000+001	6.4726-006	8.037+000	1.058-001	2.440-002	2.4673-002	-3.0381-004
1.1640+003	2.0833-004	8.5400+001	1.0000+000	6.4726-008	7.735+000	4.217-005	2.440-005	1.6268-004	-1.6072-004
1.1640+004	2.0833-003	8.5400+001	1.0000+000	6.4726-007	7.735+000	4.217-004	2.440-004	5.4318-004	-4.8133-004
1.1640+005	2.0833-002	8.5400+001	1.0000+000	6.4726-006	7.737+000	4.241-003	2.440-003	2.6977-003	-9.7057-004
1.1640+006	2.0833-001	8.5400+001	1.0000+000	6.4726-005	8.037+000	1.103-001	2.440-002	2.5112-002	-1.1603-003
1.1640+003	2.0833-004	8.5400+001	1.0000-001	6.4726-007	7.735+000	4.217-005	2.440-005	5.1163-004	-5.1101-004
1.1640+004	2.0833-003	8.5400+001	1.0000-001	6.4726-006	7.735+000	4.217-004	2.440-004	1.6267-003	-1.6072-003

TABLE 1 (Cont.)

h/a = 10.7

f (cps)	$k_o h$ (radians)	ℓ (meters)	σ (mhos/m)	$\frac{\omega\epsilon}{\sigma}$	ψ	$\lvert I_{zm}(0) \rvert$ (amperes)	k_o (radians/m)	Re k (radians/m)	Im k (radians/m)
1.1640+005	2.0833-002	8.5400+001	1.0000-001	6.4726-005	7.737+000	4.235-003	2.440-003	5.4310-003	-4.8124-003
1.1640+006	2.0833-001	8.5400+001	1.0000-001	6.4726-004	8.037+000	6.087-002	2.440-002	2.6839-002	-9.3920-003
1.1640+003	2.0833-004	8.5400+001	1.0000-002	6.4726-006	7.735+000	4.217-005	2.440-005	1.6170-003	-1.6168-003
1.1640+004	2.0833-003	8.5400+001	1.0000-002	6.4726-005	7.735+000	4.212-004	2.440-004	5.1163-003	-5.1101-003
1.1640+005	2.0833-002	8.5400+001	1.0000-002	6.4726-004	7.737+000	3.753-003	2.440-003	1.6265-002	-1.6069-002
1.1640+006	2.0833-001	8.5400+001	1.0000-002	6.4726-003	8.037+000	1.167-002	2.440-002	5.3405-002	-4.7116-002
1.1640+003	2.0833-004	8.5400+001	1.0000-003	6.4726-005	7.735+000	4.211-005	2.440-005	5.1132-003	-5.1132-003
1.1640+004	2.0833-003	8.5400+001	1.0000-003	6.4726-004	7.735+000	3.738-004	2.440-004	1.6170-002	-1.6168-002
1.1640+005	2.0833-002	8.5400+001	1.0000-003	6.4726-003	7.737+000	1.170-003	2.440-003	5.1154-002	-5.1092-002
1.1640+006	2.0833-001	8.5400+001	1.0000-003	6.4726-002	8.037+000	5.547-003	2.440-002	1.5962-001	-1.5763-001
1.3694+002	2.4510-005	8.5400+002	1.0000+001	7.6148-010	1.199+001	2.940-005	2.870-006	1.4241-005	-1.3936-005
1.3694+003	2.4510-004	8.5400+002	1.0000+001	7.6148-009	1.199+001	2.939-004	2.870-005	4.9594-005	-4.0019-005
1.3694+004	2.4510-003	8.5400+002	1.0000+001	7.6148-008	1.199+001	2.955-003	2.870-004	3.0031-004	-6.6237-005
1.3694+005	2.4510-002	8.5400+002	1.0000+001	7.6148-007	1.226+001	7.349-002	2.870-003	2.9231-003	-8.0221-005
1.3694+002	2.4510-005	8.5400+002	1.0000+000	7.6148-009	1.199+001	2.940-005	2.870-006	4.4597-005	-4.4501-005
1.3694+003	2.4510-004	8.5400+002	1.0000+000	7.6148-008	1.199+001	2.939-004	2.870-005	1.4241-004	-1.3936-004
1.3694+004	2.4510-003	8.5400+002	1.0000+000	7.6148-007	1.199+001	2.954-003	2.870-004	4.9590-004	-4.0014-004
1.3694+005	2.4510-002	8.5400+002	1.0000+000	7.6148-006	1.226+001	5.614-002	2.870-003	2.9988-003	-6.4848-004
1.3694+002	2.4510-005	8.5400+002	1.0000-001	7.6148-008	1.199+001	2.940-005	2.870-006	1.4089-004	-1.4086-004
1.3694+003	2.4510-004	8.5400+002	1.0000-001	7.6148-007	1.199+001	2.938-004	2.870-005	4.4597-004	-4.4501-004
1.3694+004	2.4510-003	8.5400+002	1.0000-001	7.6148-006	1.199+001	2.831-003	2.870-004	1.4239-003	-1.3935-003
1.3694+005	2.4510-002	8.5400+002	1.0000-001	7.6148-005	1.226+001	1.070-002	2.870-003	4.9146-003	-3.9472-003
1.3694+002	2.4510-005	8.5400+002	1.0000-002	7.6148-007	1.199+001	2.939-005	2.870-006	4.4550-004	-4.4549-004
1.3694+003	2.4510-004	8.5400+002	1.0000-002	7.6148-006	1.199+001	2.818-004	2.870-005	1.4089-003	-1.4086-003
1.3694+004	2.4510-003	8.5400+002	1.0000-002	7.6148-005	1.199+001	1.096-003	2.870-004	4.4593-003	-4.4497-003
1.3694+005	2.4510-002	8.5400+002	1.0000-002	7.6148-004	1.226+001	3.212-003	2.870-003	1.4082-002	-1.3775-002
1.3694+002	2.4510-005	8.5400+002	1.0000-003	7.6148-006	1.199+001	2.819-005	2.870-006	1.4088-003	-1.4088-003
1.3694+003	2.4510-004	8.5400+002	1.0000-003	7.6148-005	1.199+001	1.096-004	2.870-005	4.4550-003	-4.4549-003
1.3694+004	2.4510-003	8.5400+002	1.0000-003	7.6148-004	1.199+001	3.245-004	2.870-004	1.4088-002	-1.4085-002
1.3694+005	2.4510-002	8.5400+002	1.0000-003	7.6148-003	1.226+001	9.981-004	2.870-003	4.4092-002	-4.3995-002
9.1895+001	1.6447-005	1.2810+003	1.0000+001	5.1099-010	1.278+001	2.764-005	1.926-006	1.1262-005	-1.1089-005
9.1895+002	1.6447-004	1.2810+003	1.0000+001	5.1099-009	1.278+001	2.765-004	1.926-005	3.8162-005	-3.2724-005
9.1895+003	1.6447-003	1.2810+003	1.0000+001	5.1099-008	1.279+001	2.780-003	1.926-004	2.0552-004	-6.0819-005
9.1895+004	1.6447-002	1.2810+003	1.0000+001	5.1099-007	1.306+001	6.886-002	1.926-003	1.9618-003	-6.8761-005
9.1895+001	1.6447-005	1.2810+003	1.0000+000	5.1099-009	1.278+001	2.764-005	1.926-006	3.5366-005	-3.5311-005
9.1895+002	1.6447-004	1.2810+003	1.0000+000	5.1099-008	1.278+001	2.765-004	1.926-005	1.1262-004	-1.1089-004
9.1895+003	1.6447-003	1.2810+003	1.0000+000	5.1099-007	1.279+001	2.777-003	1.926-004	3.8159-004	-3.2721-004
9.1895+004	1.6447-002	1.2810+003	1.0000+000	5.1099-006	1.306+001	4.447-002	1.926-003	2.0512-003	-5.9677-004
9.1895+001	1.6447-005	1.2810+003	1.0000-001	5.1099-008	1.278+001	2.764-005	1.926-006	1.1176-004	-1.1174-004
9.1895+002	1.6447-004	1.2810+003	1.0000-001	5.1099-007	1.278+001	2.763-004	1.926-005	3.5366-004	-3.5311-004
9.1895+003	1.6447-003	1.2810+003	1.0000-001	5.1099-006	1.279+001	2.566-003	1.926-004	1.1261-003	-1.1088-003
9.1895+004	1.6447-002	1.2810+003	1.0000-001	5.1099-005	1.306+001	8.279-003	1.926-003	3.7820-003	-3.2330-003
9.1895+001	1.6447-005	1.2810+003	1.0000-002	5.1099-007	1.278+001	2.761-005	1.926-006	3.5339-004	-3.5338-004
9.1895+002	1.6447-004	1.2810+003	1.0000-002	5.1099-006	1.278+001	2.555-004	1.926-005	1.1176-003	-1.1174-003
9.1895+003	1.6447-003	1.2810+003	1.0000-002	5.1099-005	1.279+001	8.477-004	1.926-004	3.5362-003	-3.5308-003
9.1895+004	1.6447-002	1.2810+003	1.0000-002	5.1099-004	1.306+001	2.572-003	1.926-003	1.1145-002	-1.0971-002
9.1895+001	1.6447-005	1.2810+003	1.0000-003	5.1099-006	1.278+001	2.554-005	1.926-006	1.1175-003	-1.1175-003
9.1895+002	1.6447-004	1.2810+003	1.0000-003	5.1099-005	1.278+001	8.478-005	1.926-005	3.5339-003	-3.5338-003
9.1895+003	1.6447-003	1.2810+003	1.0000-003	5.1099-004	1.279+001	2.597-004	1.926-004	1.1175-002	-1.1173-002
9.1895+004	1.6447-002	1.2810+003	1.0000-003	5.1099-003	1.306+001	7.857-004	1.926-003	3.4995-002	-3.4940-002

TABLE 2

h/a = 85.4

f (cps)	$k_o h$ (radians)	ℓ (meters)	σ (mhos/m)	$\frac{\omega\epsilon}{\sigma}$	ψ	$\|I_{zm}(0)\|$ (amperes)	k_o (radians/m)	Re k (radians/m)	Im k (radians/m)
6.9840+003	1.2500-003	0.0000+000	1.0000+001	3.8835-008	8.284+000	2.148-005	1.464-004	9.7409-004	-9.6235-004
6.9840+004	1.2500-002	0.0000+000	1.0000+001	3.8835-007	8.284+000	2.148-004	1.464-003	3.2525-003	-2.8822-003
6.9840+005	1.2500-001	0.0000+000	1.0000+001	3.8835-006	8.286+000	2.161-003	1.464-002	1.6153-002	-5.8107-003
6.9840+006	1.2500+000	0.0000+000	1.0000+001	3.8835-005	8.582+000	5.762-002	1.464-001	1.5044-001	-6.8618-003
6.9840+003	1.2500-003	0.0000+000	1.0000+000	3.8835-007	8.284+000	2.148-005	1.464-004	3.0636-003	-3.0599-003
6.9840+004	1.2500-002	0.0000+000	1.0000+000	3.8835-006	8.284+000	2.148-004	1.464-003	9.7408-003	-9.6235-003
6.9840+005	1.2500-001	0.0000+000	1.0000+000	3.8835-005	8.286+000	2.161-003	1.464-002	3.2521-002	-2.8817-002
6.9840+006	1.2500+000	0.0000+000	1.0000+000	3.8835-004	8.582+000	5.762-002	1.464-001	1.6077-001	-5.6365-002
6.9840+003	1.2500-003	0.0000+000	1.0000-001	3.8835-006	8.284+000	2.148-005	1.464-004	9.6826-003	-9.6814-003
6.9840+004	1.2500-002	0.0000+000	1.0000-001	3.8835-005	8.284+000	2.148-004	1.464-003	3.0636-002	-3.0599-002
6.9840+005	1.2500-001	0.0000+000	1.0000-001	3.8835-004	8.286+000	2.161-003	1.464-002	9.7394-002	-9.6220-002
6.9840+006	1.2500+000	0.0000+000	1.0000-001	3.8835-003	8.582+000	5.762-002	1.464-001	3.2019-001	-2.8258-001
6.9840+003	1.2500-003	0.0000+000	1.0000-002	3.8835-005	8.284+000	2.148-005	1.464-004	3.0617-002	-3.0617-002
6.9840+004	1.2500-002	0.0000+000	1.0000-002	3.8835-004	8.284+000	2.148-004	1.464-003	9.6826-002	-9.6814-002
6.9840+005	1.2500-001	0.0000+000	1.0000-002	3.8835-003	8.286+000	2.161-003	1.464-002	3.0631-001	-3.0594-001
6.9840+006	1.2500+000	0.0000+000	1.0000-002	3.8835-002	8.582+000	5.762-002	1.464-001	9.5717-001	-9.4526-001
6.9840+003	1.2500-003	0.0000+000	1.0000-003	3.8835-004	8.284+000	2.148-005	1.464-004	9.6820-002	-9.6820-002
6.9840+004	1.2500-002	0.0000+000	1.0000-003	3.8835-003	8.284+000	2.148-004	1.464-003	3.0617-001	-3.0617-001
6.9840+005	1.2500-001	0.0000+000	1.0000-003	3.8835-002	8.286+000	2.161-003	1.464-002	9.6811-001	-9.6799-001
6.9840+006	1.2500+000	0.0000+000	1.0000-003	3.8835-001	8.582+000	5.762-002	1.464-001	3.0098+000	-3.0061+000
3.4920+003	6.2500-004	1.7080+001	1.0000+001	1.9418-008	9.658+000	2.763-005	7.319-005	6.3626-004	-6.3182-004
3.4920+004	6.2500-003	1.7080+001	1.0000+001	1.9418-007	9.658+000	2.763-004	7.319-004	2.0764-003	-1.9360-003
3.4920+005	6.2500-002	1.7080+001	1.0000+001	1.9418-006	9.661+000	2.780-003	7.319-003	8.7895-003	-4.5743-003
3.4920+006	6.2500-001	1.7080+001	1.0000+001	1.9418-005	9.948+000	7.396-002	7.319-002	7.5161-002	-5.3920-003
3.4920+003	6.2500-004	1.7080+001	1.0000+000	1.9418-007	9.658+000	2.763-005	7.319-005	2.0057-003	-2.0043-003
3.4920+004	6.2500-003	1.7080+001	1.0000+000	1.9418-006	9.658+000	2.763-004	7.319-004	6.3626-003	-6.3182-003
3.4920+005	6.2500-002	1.7080+001	1.0000+000	1.9418-005	9.661+000	2.776-003	7.319-003	2.0762-002	-1.9358-002
3.4920+006	6.2500-001	1.7080+001	1.0000+000	1.9418-004	9.948+000	4.466-002	7.319-002	8.7318-002	-4.4713-002
3.4920+003	6.2500-004	1.7080+001	1.0000-001	1.9418-006	9.658+000	2.763-005	7.319-005	6.3406-003	-6.3401-003
3.4920+004	6.2500-003	1.7080+001	1.0000-001	1.9418-005	9.658+000	2.760-004	7.319-004	2.0057-002	-2.0043-002
3.4920+005	6.2500-002	1.7080+001	1.0000-001	1.9418-004	9.661+000	2.488-003	7.319-003	6.3618-002	-6.3174-002
3.4920+006	6.2500-001	1.7080+001	1.0000-001	1.9418-003	9.948+000	1.522-002	7.319-002	2.0480-001	-1.9057-001
3.4920+003	6.2500-004	1.7080+001	1.0000-002	1.9418-005	9.658+000	2.760-005	7.319-005	2.0050-002	-2.0050-002
3.4920+004	6.2500-003	1.7080+001	1.0000-002	1.9418-004	9.658+000	2.477-004	7.319-004	6.3406-002	-6.3401-002
3.4920+005	6.2500-002	1.7080+001	1.0000-002	1.9418-003	9.661+000	1.248-003	7.319-003	2.0054-001	-2.0040-001
3.4920+006	6.2500-001	1.7080+001	1.0000-002	1.9418-002	9.948+000	1.194-002	7.319-002	6.2697-001	-6.2247-001
3.4920+003	6.2500-004	1.7080+001	1.0000-003	1.9418-004	9.658+000	2.476-005	7.319-005	6.3404-002	-6.3404-002
3.4920+004	6.2500-003	1.7080+001	1.0000-003	1.9418-003	9.658+000	1.246-004	7.319-004	2.0050-001	-2.0050-001
3.4920+005	6.2500-002	1.7080+001	1.0000-003	1.9418-002	9.661+000	1.012-003	7.319-003	6.3398-001	-6.3393-001
3.4920+006	6.2500-001	1.7080+001	1.0000-003	1.9418-001	9.948+000	1.106-002	7.319-002	1.9762+000	-1.9748+000
1.1640+003	2.0833-004	8.5400+001	1.0000+001	6.4726-009	1.185+001	2.753-005	2.440-005	3.3098-004	-3.3004-004
1.1640+004	2.0833-003	8.5400+001	1.0000+001	6.4726-008	1.185+001	2.753-004	2.440-004	1.0601-003	-1.0304-003
1.1640+005	2.0833-002	8.5400+001	1.0000+001	6.4726-007	1.185+001	2.768-003	2.440-003	3.8020-003	-2.8728-003
1.1640+006	2.0833-001	8.5400+001	1.0000+001	6.4726-006	1.213+001	6.075-002	2.440-002	2.5251-002	-4.2456-003
1.1640+003	2.0833-004	8.5400+001	1.0000+000	6.4726-008	1.185+001	2.753-005	2.440-005	1.0453-003	-1.0450-003
1.1640+004	2.0833-003	8.5400+001	1.0000+000	6.4726-007	1.185+001	2.753-004	2.440-004	3.3098-003	-3.3004-003
1.1640+005	2.0833-002	8.5400+001	1.0000+000	6.4726-006	1.185+001	2.703-003	2.440-003	1.0600-002	-1.0303-002
1.1640+006	2.0833-001	8.5400+001	1.0000+000	6.4726-005	1.213+001	1.284-002	2.440-002	3.7700-002	-2.8313-002
1.1640+003	2.0833-004	8.5400+001	1.0000-001	6.4726-007	1.185+001	2.752-005	2.440-005	3.3052-003	-3.3051-003
1.1640+004	2.0833-003	8.5400+001	1.0000-001	6.4726-006	1.185+001	2.689-004	2.440-004	1.0453-002	-1.0450-002

TABLE 2 (Cont.)

h/a = 85.4

f (cps)	k_oh (radians)	ℓ (meters)	σ (mhos/m)	$\frac{\omega\epsilon}{\sigma}$	ψ	$\lvert I_{zm}(0)\rvert$ (amperes)	k_o (radians/m)	Re k (radians/m)	Im k (radians/m)
1.1640+005	2.0833-002	8.5400+001	1.0000-001	6.4726-005	[illegible]	1.253-003	2.440-003	3.3095-002	-3.3001-002
1.1640+006	2.0833-001	8.5400+001	1.0000-001	6.4726-004	1.213+001	4.538-003	2.440-002	1.0482-001	-1.0183-001
1.1640+003	2.0833-004	8.5400+001	1.0000-002	6.4726-006	1.185+001	2.689-005	2.440-005	1.0452-002	-1.0452-002
1.1640+004	2.0833-003	8.5400+001	1.0000-002	6.4726-005	1.185+001	1.252-004	2.440-004	3.3052-002	-3.3051-002
1.1640+005	2.0833-002	8.5400+001	1.0000-002	6.4726-004	1.185+001	4.497-004	2.440-003	1.0452-001	-1.0449-001
1.1640+006	2.0833-001	8.5400+001	1.0000-002	6.4726-003	1.213+001	2.998-003	2.440-002	3.2718-001	-3.2623-001
1.1640+003	2.0833-004	8.5400+001	1.0000-003	6.4726-005	1.185+001	1.252-005	2.440-005	3.3051-002	-3.3051-002
1.1640+004	2.0833-003	8.5400+001	1.0000-003	6.4726-004	1.185+001	4.497-005	2.440-004	1.0452-001	-1.0452-001
1.1640+005	2.0833-002	8.5400+001	1.0000-003	6.4726-003	1.185+001	2.997-004	2.440-003	3.3048-001	-3.3047-001
1.1640+006	2.0833-001	8.5400+001	1.0000-003	6.4726-002	1.213+001	2.639-003	2.440-002	1.0333+000	-1.0330+000
1.3694+002	2.4510-005	8.5400+002	1.0000+001	7.6148-010	1.612+001	2.186-005	2.870-006	9.7196-005	-9.7153-005
1.3694+003	2.4510-004	8.5400+002	1.0000+001	7.6148-009	1.612+001	2.185-004	2.870-005	3.0798-004	-3.0660-004
1.3694+004	2.4510-003	8.5400+002	1.0000+001	7.6148-008	1.613+001	2.175-003	2.870-004	9.9377-004	-9.5008-004
1.3694+005	2.4510-002	8.5400+002	1.0000+001	7.6148-007	1.638+001	1.385-002	2.870-003	3.8028-003	-2.4440-003
1.3694+002	2.4510-005	8.5400+002	1.0000+000	7.6148-009	1.612+001	2.186-005	2.870-006	3.0730-004	-3.0729-004
1.3694+003	2.4510-004	8.5400+002	1.0000+000	7.6148-008	1.612+001	2.164-004	2.870-005	9.7196-004	-9.7153-004
1.3694+004	2.4510-003	8.5400+002	1.0000+000	7.6148-007	1.613+001	1.304-003	2.870-004	3.0796-003	-3.0658-003
1.3694+005	2.4510-002	8.5400+002	1.0000+000	7.6148-006	1.638+001	3.522-003	2.870-003	9.8629-003	-9.4227-003
1.3694+002	2.4510-005	8.5400+002	1.0000-001	7.6148-008	1.612+001	2.164-005	2.870-006	9.7175-004	-9.7174-004
1.3694+003	2.4510-004	8.5400+002	1.0000-001	7.6148-007	1.612+001	1.303-004	2.870-005	3.0730-003	-3.0729-003
1.3694+004	2.4510-003	8.5400+002	1.0000-001	7.6148-006	1.613+001	3.547-004	2.870-004	9.7190-003	-9.7146-003
1.3694+005	2.4510-002	8.5400+002	1.0000-001	7.6148-005	1.638+001	1.070-003	2.870-003	3.0555-002	-3.0416-002
1.3694+002	2.4510-005	8.5400+002	1.0000-002	7.6148-007	1.612+001	1.303-005	2.870-006	3.0729-003	-3.0729-003
1.3694+003	2.4510-004	8.5400+002	1.0000-002	7.6148-006	1.612+001	3.548-005	2.870-005	9.7175-003	-9.7174-003
1.3694+004	2.4510-003	8.5400+002	1.0000-002	7.6148-005	1.613+001	1.078-004	2.870-004	3.0728-002	-3.0726-002
1.3694+005	2.4510-002	8.5400+002	1.0000-002	7.6148-004	1.638+001	4.016-004	2.870-003	9.6425-002	-9.6381-002
1.3694+002	2.4510-005	8.5400+002	1.0000-003	7.6148-006	1.612+001	3.552-006	2.870-006	9.7175-003	-9.7175-003
1.3694+003	2.4510-004	8.5400+002	1.0000-003	7.6148-005	1.612+001	1.078-005	2.870-005	3.0729-002	-3.0729-002
1.3694+004	2.4510-003	8.5400+002	1.0000-003	7.6148-004	1.613+001	4.059-005	2.870-004	9.7168-002	-9.7168-002
1.3694+005	2.4510-002	8.5400+002	1.0000-003	7.6148-003	1.638+001	2.590-004	2.870-003	3.0486-001	-3.0485-001
9.1895+001	1.6447-005	1.2810+003	1.0000+001	5.1099-010	1.692+001	2.088-005	1.926-006	7.7717-005	-7.7692-005
9.1895+002	1.6447-004	1.2810+003	1.0000+001	5.1099-009	1.692+001	2.089-004	1.926-005	2.4611-004	-2.4533-004
9.1895+003	1.6447-003	1.2810+003	1.0000+001	5.1099-008	1.692+001	2.058-003	1.926-004	7.8938-004	-7.6481-004
9.1895+004	1.6447-002	1.2810+003	1.0000+001	5.1099-007	1.718+001	1.004-002	1.926-003	2.8557-003	-2.0828-003
9.1895+001	1.6447-005	1.2810+003	1.0000+000	5.1099-009	1.692+001	2.088-005	1.926-006	2.4573-004	-2.4572-004
9.1895+002	1.6447-004	1.2810+003	1.0000+000	5.1099-008	1.692+001	2.048-004	1.926-005	7.7717-004	-7.7692-004
9.1895+003	1.6447-003	1.2810+003	1.0000+000	5.1099-007	1.692+001	1.003-003	1.926-004	2.4610-003	-2.4532-003
9.1895+004	1.6447-002	1.2810+003	1.0000+000	5.1099-006	1.718+001	2.836-003	1.926-003	7.8369-003	-7.5895-003
9.1895+001	1.6447-005	1.2810+003	1.0000-001	5.1099-008	1.692+001	2.047-005	1.926-006	7.7705-004	-7.7704-004
9.1895+002	1.6447-004	1.2810+003	1.0000-001	5.1099-007	1.692+001	1.003-004	1.926-005	2.4573-003	-2.4572-003
9.1895+003	1.6447-003	1.2810+003	1.0000-001	5.1099-006	1.692+001	2.856-004	1.926-004	7.7712-003	-7.7687-003
9.1895+004	1.6447-002	1.2810+003	1.0000-001	5.1099-005	1.718+001	8.599-004	1.926-003	2.4427-002	-2.4349-002
9.1895+001	1.6447-005	1.2810+003	1.0000-002	5.1099-007	1.692+001	1.002-005	1.926-006	2.4572-003	-2.4572-003
9.1895+002	1.6447-004	1.2810+003	1.0000-002	5.1099-006	1.692+001	2.856-005	1.926-005	7.7704-003	-7.7704-003
9.1895+003	1.6447-003	1.2810+003	1.0000-002	5.1099-005	1.692+001	8.659-005	1.926-004	2.4571-002	-2.4570-002
9.1895+004	1.6447-002	1.2810+003	1.0000-002	5.1099-004	1.718+001	2.986-004	1.926-003	7.7134-002	-7.7110-002
9.1895+001	1.6447-005	1.2810+003	1.0000-003	5.1099-006	1.692+001	2.846-006	1.926-006	7.7705-003	-7.7705-003
9.1895+002	1.6447-004	1.2810+003	1.0000-003	5.1099-005	1.692+001	8.660-006	1.926-005	2.4572-002	-2.4572-002
9.1895+003	1.6447-003	1.2810+003	1.0000-003	5.1099-004	1.692+001	3.014-005	1.926-004	7.7699-002	-7.7699-002
9.1895+004	1.6447-002	1.2810+003	1.0000-003	5.1099-003	1.718+001	1.743-004	1.926-003	2.4389-001	-2.4388-001