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Theory of Inductive Shielding

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Abstract

A mathematical theory of low-frequency electromagnetic shielding is constructed on the basis that an appropriate set of boundary conditions can be derived to duplicate the effect of the shield's wall on the fields within the shield. Shields with electrically thin shells are considered in detail; mathematical methods that are best suited for computational purposes are presented for calculating the shielding effectiveness of such a shield of arbitrary shape. Shells with arbitrary electrical-thickness are also treated but in less detail, since the shielding problem involving this kind of shell is shown to be different from but no more general than the shielding problem involving electrically thin shells. Explicit results are given for shields of particular shape.

1. Introduction

The problem of computing the low-frequency electromagnetic shielding effectiveness of a metallic enclosure has been perplexing designers for some time. At these frequencies the geometrical shape of a shield becomes critical while the familiar attenuation and scattering losses of the shield's wall, although accounting for a great deal of shielding at higher frequencies, become irrelevant. As a consequence, reliable data are available in the literature only on a few particular shapes, such as a spherical shell[1,2,3,4], an infinite circular-cylindrical shell[5,6,7], and two-parallel plates[8]. These shapes have been chosen for study mainly because of the mathematical ease with which they may be handled by the classical technique of separation of variables, but they are not very closely related to any realistic shield which one may imagine, such as a rectangular box or a finite cylindrical box.

What is required now is a general formulation of the low-frequency shielding problem taking into account the present availability of rapid digital computers. Two attacks have already been made on the general formulation of this problem. The first of these[9,10], based on what may be termed the circuit approach, is very approximate in nature although it predicts correct results for a spherical shell. The reliability of the results derived from this approach depends on a judicious guess of the induced current density at each point on the shell. A more recent, and more accurate, approach[11] is to formulate the problem as a pair of coupled vector integral equations which one may write for a three-medium problem, and then to make certain simplifications in these equations appropriate for a thin shell. This approach, which was developed primarily to treat scattering from thin shells, does not take advantage of special factors which greatly simplify the low-frequency shielding problem. The present paper reports an attempt to formulate the shielding problem in a manner free from arbitrary guesses and yet in the simplest way sufficient to account for the phenomenon involved.

It is common knowledge that the shielding of electric fields by closed conducting shells increases as the frequency is decreased; hence only the shielding of magnetic fields need be considered at low frequencies. When the shell is electrically thin, i.e., when the skin depth of the shell is larger than its thickness, the only way to shield the magnetic field is to induce in the shell sheet currents which vary in such a manner on the surface as to cancel most of the external field inside the enclosure. Thus, the only phenomenon involved is the induction of sheet currents, and that is the reason the words "inductive shielding" are used as a label for the problem under study.

In sections 2 and 3 are given special techniques for the treatment of two-dimensional and axially symmetric shields, respectively. In section 4 the general problem is formulated in terms of a single scalar integral equation which, for a shield of arbitrary shape, must be solved numerically. Section 5 is devoted to the shielding problem involving shells of arbitrary electrical-thickness. The purpose of including this problem is to show how the attenuation loss of the shield's wall can be taken into account, this loss factor being of an exponential form. It is shown that different assumptions are needed for the formulation of this problem than for the one discussed in the previous sections; in this sense the problem in section 5 is not at all a generalization of the previous one.

The assumptions sufficient to account for inductive shielding are that the displacement current may be neglected and that the conduction current is confined to a surface and is proportional to the component of the electric field parallel to the surface, this tangential electric field being continuous through the surface. Also assumed are that the relative permeability of the shell is unity and that the source of the external field may be specified by some quasi-static magnetic-field distributions. These assumptions imply that we must write Maxwell's equations in the form

$$\nabla \times \underline{E} = i\omega\mu_o \underline{H} \tag{1a}$$

$$\nabla \times \underline{H} = g\underline{E}_s \, \delta(\underline{r}-\underline{r}') \, , \tag{1b}$$

with the time-dependence factor $\exp(-i\omega t)$ suppressed throughout, where the subscript s denotes the component parallel to the surface defined by $\underline{r} = \underline{r}'(u,v)$, u and v being the two variables specifying the surface. The function g , which equals the product of the conductivity σ and the thickness Δ of the shell, defines the sheet conductance as a function of surface position, while the delta function in (1b) has the dimension of reciprocal length. In addition to equations (1), the external field must be specified.

Alternatively, one may look upon equations (1) as the definition of the inductive-shielding problem. The following sections, except section 5, will be devoted to methods available for solving these equations.

2. Two-Dimensional Geometry

2.1 Longitudinal Case

In this section we shall consider the magnetic field inside an infinitely long, cylindrical shell of any cross section when the shell is exposed to a uniform external magnetic field parallel to the axis of the shell (Fig. 1). Since the external field H^{ex} is parallel to the z-axis, it is obvious from the geometry of the problem that there is only a H_z everywhere. From (1b) it is seen that, off the surface, $\underline{H}$ is irrotational; consequently H_z is constant everywhere inside (H_z^{in}) and outside (H_z^{out}) the shell. Integrating (1a) over any cross section of area A of the cylindrical shell we obtain

$$\int E_\ell \, d\ell = i\omega\mu_o \, A \, H_z^{in} \; . \tag{2}$$

Substituting into (2) the expression

$$E_\ell = \frac{1}{g} \left(H_z^{in} - H_z^{out} \right) ,$$

which follows directly from (1b), and solving the resulting equation for H_z^{in} we finally get

$$H_z^{in} = H_z^{ex} \left[1 - \frac{i\omega\mu_o A}{\int \frac{d\ell}{g(\ell)}} \right]^{-1} , \tag{3}$$

where we have used $H_z^{out} = H_z^{ex}$.

When $\sigma\Delta$ is constant, equation (3) gives

$$H_z^{in} = H_z^{ex}\left[1 - \frac{i\omega\mu_o\sigma\Delta a}{2}\right]^{-1} \tag{4}$$

for a circular-cylindrical shell of radius a , and

$$H_z^{in} = H_z^{ex}\left[1 - i\omega\mu_o\sigma\Delta d\right]^{-1} \tag{5}$$

for two parallel plates of separation $2d$.

2.2 Transverse Case

When the external magnetic field is perpendicular to the axis of the cylindrical shell of any cross section (Fig. 2), the problem of finding the magnetic field inside the shell can be solved most effectively by means of the vector potential $\underline{A}$. Since $\nabla\cdot\underline{H} = 0$ everywhere, we can define $\underline{A}$ by

$$\nabla \times \underline{A} = \mu_o\underline{H}$$

together with the Coulomb gauge

$$\nabla \cdot \underline{A} = 0 \quad .$$

Then from (1b) it follows that

$$\underline{A}(\underline{r}) = \underline{A}^{ex}(\underline{r}) + \mu_o\int G(\underline{r},\underline{r}')\; g(\underline{r}')\; \underline{E}_s\,(\underline{r}')d\ell' \ , \tag{6}$$

where $2\pi G(\underline{r},\underline{r}') = -\ln|\underline{r}-\underline{r}'|$. Let us now eliminate the surface component $\underline{E}_s$ in favor of $\underline{A}$ and the scalar potential ϕ by means of

$$\underline{E}_s = i\omega\underline{A}_s - \nabla_s\phi$$

which is the direct consequence of (1a). Substituting this equation into (6) one gets

$$\underline{A}(\underline{r}) = \underline{A}^{ex}(\underline{r}) + i\omega\mu_o \int G(\underline{r},\underline{r}')g(\underline{r}')\underline{A}_s(\underline{r}')d\ell'$$

$$- \mu_o \int G(\underline{r},\underline{r}')g(\underline{r}')\nabla_s'\phi(\underline{r}')d\ell' \ . \tag{7}$$

Let us pause for a moment and examine under what conditions $\nabla_s\phi$ will vanish on the surface. Taking the surface divergence of (1b) we have[12]

$$\nabla_s \cdot g\underline{E}_s = \nabla_s \cdot \nabla \times \underline{H} = \nabla\cdot\nabla \times \underline{H} - \frac{\partial}{\partial n}\underline{n}\cdot\nabla \times \underline{H}$$

$$= -\frac{\partial}{\partial n} g\ \underline{n}\cdot\underline{E}_s = 0 \ , \tag{8}$$

whence

$$g\nabla_s \cdot \underline{E}_s + \underline{E}_s \cdot \nabla_s g = 0 \ .$$

In the present case g is independent of z and $\underline{E}_s$ is parallel to the z-axis. Hence the equation (8) implies that $\nabla_s \cdot \underline{E}_s = 0$. This, in turn, means that

$$\nabla_s^2\phi = i\omega\nabla_s \cdot \underline{A}_s \ .$$

Thus, on a closed surface $\nabla_s \phi \equiv 0$ if and only if $\nabla_s \cdot \underline{A}_s \equiv 0$. It is to be noted that the Coulomb gauge does not in general imply that $\nabla_s \cdot \underline{A}_s = 0$. In some special cases, e.g., the case we are considering in this section and the case we shall consider in the next section, $\nabla_s \cdot \underline{A}_s = 0$ holds true and we then obtain a pure integral equation for $\underline{A}$ by omitting the last integral in (7) and by letting $\underline{r}$ lie on the surface.

We now return to the case we have started out to consider at the beginning of this section. From the nature of the problem we see that $\underline{A}$ has only a z-component and is independent of the coordinate z. Then we have $\nabla_s \cdot \underline{A}_s = 0$; consequently we obtain from (7) the integral equation

$$A_z(\ell) = A_z^{ex}(\ell) + i\omega\mu_o \int G(\ell,\ell')g(\ell')A_z(\ell')d\ell' \ . \tag{9}$$

A_z^{ex} is not necessarily the potential of a uniform field and can be any function of the transverse coordinates x and y , e.g., it can be the potential of a current-carrying wire. We now introduce the surface current K which is related to A_z by

$$K = i\omega g A_z \ .$$

Substitution into (9) gives

$$K = i\omega g A_z^{ex} + i\omega\mu_o g \int G\ K(\ell')d\ell' \tag{10}$$

where K , A_z^{ex} and g are functions of ℓ . Once K has been found from this integral equation the magnetic field inside can be obtained immediately from

$$\underline{H}_t^{in}(\underline{r}) = \underline{H}_t^{ex}(\underline{r}) - \underline{e}_z \times \nabla \int G\, K(\ell')d\ell' \ , \tag{11}$$

where the subscript t denotes components transverse to the z-axis. The expression (11) is, however, not convenient to use especially in the case of highly conducting shields, since most of the contribution from the integral will be cancelled by the external field. To obtain a representation best suited for computational purposes we proceed as follows.

When the shield is perfectly conducting, $\underline{H}_t^{in}$ is identically zero; therefore from (11) we have

$$\underline{H}_t^{ex}(\underline{r}) = \underline{e}_z \times \nabla \int G\, K_\infty(\ell')d\ell'$$

for $\underline{r}$ inside the enclosure. K_∞ is the surface current on the perfectly conducting shield. The equation (11) can now be rewritten in the following useful form:

$$\underline{H}_t^{in}(\underline{r}) = \underline{e}_z \times \nabla \int G\, K_d(\ell')d\ell' \tag{12}$$

where $K_d = K_\infty - K$.

The next step is to obtain an integral equation directly for K_d . To do this we first express A_z^{ex} on the surface in terms of K_∞ . The required expression for A_z^{ex} can be easily obtained from (10) by letting g go to infinity and by keeping K finite. Then we eliminate A_z^{ex} in (10) in favor of K_{∞} , and by adding K_{∞} on both sides of the resulting equation we finally obtain the required integral equation for K_d:

$$K_d(\ell) = K_{\infty}(\ell) + i\omega\mu_o g(\ell) \int G(\ell,\ell') \, K_d(\ell')d\ell' \quad . \tag{13}$$

Here K is well known to satisfy the equation

$$\frac{1}{2} K_{\infty}(\ell) = K^{ex}(\ell) - \int \frac{\partial G}{\partial n} K_{\infty}(\ell')d\ell' \quad , \tag{14}$$

where $K^{ex} = \underline{e}_z \cdot (\underline{n} \times \underline{H}_t^{ex})$. The equations (14), (13) and (12) constitute the formulation of the problem posed at the beginning of this section. For a cylindrical shield of arbitrary cross section these equations must be solved numerically on a digital computer.

In the case of constant g and a circular-cylindrical shell of radius a one can readily solve (14) and (13) by the method of eigen-function expansion. When the external field is uniform one finds that the interior field is also uniform and is given by

$$H_t^{in} = H_t^{ex} \left[1 - \frac{i\omega\mu_o \sigma\Delta a}{2}\right]^{-1} \tag{15}$$

which is identical to (5). Thus the magnetic shielding effectiveness of a circular-cylindrical shell is the same for both polarizations.

3. Three-Dimensional Geometry with Axial Symmetry

The low-frequency shielding problem can be formulated as a one-dimensional integral equation if axial symmetry obtains in the problem, i.e., if the external field and the shield have the same axis of symmetry. For example, when the external field is uniform and parallel to the axis of the shield, or when the axis of a uniform current-loop which produces the external field coincides with that of the shield, one will then have a case of axial symmetry. In such cases the vector potential has only a ϕ-component and is independent of ϕ (Fig. 3). Hence $\nabla_s \cdot \underline{A}_s = 0$, and we obtain from (7), with the line integral replaced by a surface integral,

$$A_\phi(s) = A_\phi^{ex}(s) + i\omega\mu_o \int \cos(\phi-\phi')G(s,\phi;s',\phi')g(s')A_\phi(s')\rho' d\phi' ds' \qquad (16)$$

where $4\pi G = |\underline{r}-\underline{r}'|^{-1}$ and (s,ϕ) are the coordinates of the shield's surface.

We now introduce the magnetic flux Φ and the surface current K_ϕ related to A_ϕ by

$$\Phi = 2\pi\rho A_\phi \ , \qquad K_\phi = i\omega g A_\phi \ . \qquad (17)$$

The equation (16) then becomes

$$\Phi(z) = \Phi^{ex}(z) + \int M(z,z')K_\phi(z')dz' \quad , \qquad (18)$$

where the kernel M has the meaning of mutual inductance and is given by

$$M(z,z') = \mu_o \rho\rho' \frac{ds'}{dz'} \int_o^{\pi} \frac{\cos\theta \; d\theta}{\sqrt{\rho^2 + \rho'^2 - 2\rho\rho' \cos\theta + (z-z')^2}} , \tag{19}$$

ρ and ρ' being functions of z and z' respectively. For a given Φ^{ex} one first solves (18) to obtain K_ϕ from which one can calculate the field everywhere inside the shield. Or the field inside can be calculated by the procedure similar to (12) - (14). In the case of constant g and a spherical shell of radius a in a uniform field, one finds that the interior field is also uniform and is given by

$$H^{in} = H^{ex} \left[1 - \frac{i\omega\mu_o \sigma\Delta a}{3}\right]^{-1} . \tag{20}$$

The induced e.m.f. in a loop of radius b situated at $z = z_o$ inside an axially symmetric shield can also be calculated by the method leading to (18), provided that the presence of the loop does not destroy the axial symmetry of the problem. Let R_b and L_b be the resistance and inductance of the loop when it is in free space, and let I_b be the induced current in the loop. Then the total flux linking the loop is given by

$$\Phi_b(z_o) = \Phi_b^{ex}(z_o) + L_b I_b + \int_{shell} M_{bs}(z_o,z')K_\phi(z')dz'$$

whence, by virtue of the relation $R_b I_b = i\omega\Phi_b$,

$$\left(1 - \frac{i\omega L_b}{R_b}\right)\Phi_b(z_o) = \Phi_b^{ex}(z_o) + \int_{shell} M_{bs}(z_o,z')K_\phi(z')dz' , \tag{21}$$

where K_ϕ is the induced surface current in the shell; M_{bs} is the mutual inductance between the loop and a differential loop-element of the shell and is given by

$$M_{bs}(z_o,z') = \mu_o b\rho' \frac{ds'}{dz'} \int_o^{\pi} \frac{\cos\theta \;\; d\theta}{\sqrt{b^2 + \rho'^2 - 2b\rho' \cos\theta + (z_o-z')^2}} \quad . \tag{22}$$

Another equation that relates Φ_b and K_ϕ is provided by (18) with an added term due to the flux from the loop:

$$\Phi(z) = \Phi^{ex}(z) + \int_{shell} M(z,z')K_\phi(z')dz' + \frac{i\omega M_{bs}(z,z_o)}{R_b}\Phi_b(z_o) \quad . \tag{23}$$

The equations (21) and (23) constitute the mathematical formulation of the problem. To find the induced e.m.f. $(i\omega\Phi_b)$ in the loop one first eliminates Φ_b in (23) by means of (21) and then solves the resulting integral equation for Φ. With Φ one calculates Φ_b directly from (21).

4. Three-Dimensional Geometry - The General Case

In the previous two sections we discussed special methods that are best suited for problems involving shields of particular shape. We now go on to consider a shield of general shape and we shall see that this general problem can be formulated as a scalar integral equation.

First, we note that $\underline{H}$ is irrotational off the surface of the shield; hence we can write

$$\underline{H} = \nabla\Omega \quad . \tag{24}$$

Moreover, $\nabla\cdot\underline{H} = 0$ holds true everywhere. Thus

$$\nabla^2\Omega = 0 \tag{25}$$

for points off the surface. Next, we shall derive the boundary conditions for Ω and $\partial\Omega/\partial n$, where $\underline{n}$ is always taken to be pointing into the exterior region of the shield. To do this we recall that the equation (8), i.e., $\nabla_s\cdot g\underline{E}_s = 0$, implies that

$$\underline{E}_s = \frac{1}{g}\,\underline{n} \times \nabla_s\psi \quad , \tag{26}$$

where ψ is often called the stream function of the current density. The connection between Ω and ψ is provided by equations (1) and one can easily show that

$$\Omega(\underline{r}) = \Omega^{ex}(\underline{r}) + \int \frac{\partial G}{\partial n'}\psi(\underline{r}')\,dS' \tag{27}$$

for $\underline{r}$ inside or outside the shield. Here G is, as before, given by $4\pi G = |\underline{r}-\underline{r}'|^{-1}$. From (27) we immediately deduce the boundary conditions:

$$\frac{\partial\Omega^{out}}{\partial n} = \frac{\partial\Omega^{in}}{\partial n} = \frac{\partial\Omega}{\partial n} \tag{28a}$$

$$\Omega^{out} - \Omega^{in} = \psi \ . \tag{28b}$$

It now remains to find a relation between ψ and $\partial\Omega/\partial n$ on the surface. Scalarly multiplying (1a) by the outward unit normal $\underline{n}$ we have[12]

$$\begin{aligned} i\omega\mu_o H_n &= \underline{n} \cdot \nabla \times \underline{E} = \underline{n} \cdot \left(\nabla_s \times \underline{E} + \frac{\partial}{\partial n} \underline{n} \times \underline{E}\right) \\ &= \underline{n} \cdot \nabla_s \times \underline{E} = \underline{n} \cdot \left(\nabla_s \times \underline{E}_s + \nabla_s \times \underline{n}E_n\right) \\ &= \underline{n} \cdot \nabla_s \times \underline{E}_s \ . \end{aligned}$$

Upon substituting (24) and (26) into this equation we get

$$\frac{\partial\Omega}{\partial n} = \frac{1}{i\omega\mu_o g} \nabla_s^2 \psi \quad , \tag{29}$$

where, for reason of simplicity, g has been assumed constant.

The equation (25) and the boundary conditions (28) and (29) are the basic equations for the general problem. Wherever the method of separation of variables applies the equation (25) can readily be solved together with (28) and (29). For example, the cases of a circular-cylindrical shell and a spherical shell become trivially simple and the results derived by the present method agree respectively with (15) and (20).

For a shield of general shape one must translate the differential equation (25) into an integral equation, taking into account the boundary conditions (28) and (29). The standard procedure of doing this is to express, with the aid of Green's theorem, the magnetic scalar potentials inside and outside the shield in terms of their values and their normal derivatives on the surface. Thus we have

$$\Omega^{in}(\underline{r}) = \int \left(G\frac{\partial\Omega^{in}}{\partial n'} - \frac{\partial G}{\partial n'}\Omega^{in}\right) dS' \quad , \tag{30a}$$

$$\Omega^{out}(\underline{r}) = \Omega^{ex}(\underline{r}) - \int \left(G\frac{\partial\Omega^{out}}{\partial n'} - \frac{\partial G}{\partial n'}\, \Omega^{out}\right) dS' \quad , \tag{30b}$$

where $\underline{r}$ is off the surface.

Let us consider briefly the case of a highly conducting shield. In this case the field inside the shield can be obtained by the perturbation method. We first substitute (28a) and (29) into (30) and then bring the point $\underline{r}$ onto the surface to obtain

$$\frac{1}{2}\Omega^{in} = - \int \Omega^{in} \frac{\partial G}{\partial n'}\, dS' + \frac{1}{i\omega\mu_o g} \int G\, \nabla_s'^2 \psi\, dS' \tag{31a}$$

$$\frac{1}{2}\Omega^{out} = \Omega^{ex} + \int \Omega^{out} \frac{\partial G}{\partial n'}\, dS' - \frac{1}{i\omega\mu_o g} \int G\, \nabla_s'^2 \psi\, dS' \quad . \tag{31b}$$

In the zeroth approximation, we assume the shield to be perfectly conducting so that the last terms on the right-hand sides of (31) can be neglected. Thus, in this approximation, Ω_o^{in} is identically zero and Ω_o^{out} satisfies an integral equation of the second kind, the subscript denoting the order of approximation. In the next approximation, we insert, according to (28b), $\psi = \Omega_0^{out}$ in (31a) and obtain an integral equation of the second kind for Ω_1^{in} . It is worth pointing out that the integral equations for Ω_0^{out} and Ω_1^{in} have the same form and the same kernel. Thus the same method of solution applies to both equations. With a knowledge of the value of Ω_1^{in} on the surface one can compute the first-order potential everywhere inside the shield by (30a). For example, in the case of a circular-cylindrical shell of radius a in a uniform external transverse field we find

$$H_1^{in} = \frac{2i}{\omega\mu_0\sigma\Delta a} H^{ex} \quad , \tag{32}$$

and in the case of a spherical shell of radius a in a uniform external field we find

$$H_1^{in} = \frac{3i}{\omega\mu_0\sigma\Delta a} H^{ex} \quad . \tag{33}$$

Clearly, (32) and (33) can also be obtained directly from (15) and (20) by neglecting the term unity in the denominator of the latter two expressions.

Let us now return to the general case where the shield is not necessarily highly conducting. First, we bring $\underline{r}$ in (30) onto the surface. Then, addition and subtraction of the resulting equations give

$$\Omega_+ = \Omega^{ex} + 2 \int \frac{\partial G}{\partial n'} \Omega_- \, dS' - 2 \int G \frac{\partial \Omega_-}{\partial n'} \, dS' \tag{34a}$$

$$\Omega_- = \Omega^{ex} + 2 \int \frac{\partial G}{\partial n'} \Omega_+ \, dS' - 2 \int G \frac{\partial \Omega_+}{\partial n'} \, dS' \tag{34b}$$

where $2\Omega_+ = \Omega^{out} + \Omega^{in}$, $2\Omega_- = \Omega^{out} - \Omega^{in}$. Inserting (34a) in (34b) and then using (28) and (29) we obtain

$$\frac{1}{4}\psi(\underline{r}) = \int G(\underline{r},\underline{r}') \frac{\partial \Omega^{ex}}{\partial n'} dS' + \iint \psi(\underline{r}'') \frac{\partial}{\partial n''} G(\underline{r}'',\underline{r}') \frac{\partial}{\partial n'} G(\underline{r}',\underline{r}) dS' dS''$$

$$- \frac{1}{i\omega\mu_o g} \int G(\underline{r},\underline{r}') \nabla_s'^2 \psi \, dS' \quad . \tag{35}$$

The last term on the right-hand side presents some difficulties in computation. It is necessary to transform this term into a form suitable for computational purposes. This can be achieved by the formula

$$\int_S G(\underline{r},\underline{r}') \nabla_s'^2 \psi \, dS' = \lim_{\sigma \to o} \int_{S-\sigma} \left[\psi(\underline{r}') - \psi(\underline{r})\right] \nabla_s'^2 G \, dS' \quad , \tag{36}$$

where S is a closed surface and σ is the area of a very small circular disc centered at $\underline{r}$. The proof of (36) is omitted here, since it follows essentially the same procedure that leads to the familiar factor one-half in (14), (31a) and (31b).

Insertion of (36) in (35) gives the required integral equation for the general problem. Although the integral equation thus obtained looks quite complicated from the viewpoint of analytical treatment, it can readily be solved with the aid of a present-day computer. Having found the numerical solution for ψ from (35) one then substitutes it into (31a), solves the resulting integral equation numerically, and finally calculates the field inside via (30a). The reason for taking this seemingly roundabout way instead of the direct way via (27) is mainly for numerical accuracy, for most of the contribution from the integral in (27) cancels the external potential Ω^{ex} if the shell under consideration has any shielding effect at all.

5. Shells with Arbitrary Electrical-Thickness

So far we have been considering the shielding of a shell against a low-frequency magnetic field. The assumptions that we have used are the following:

(a) the quasi-static approximation is applicable,

(b) the shell under consideration is geometrically thin, i.e., $\kappa\Delta \ll 1$, κ being the sum of the two principal curvatures of the shell's surface,

(c) the shell is electrically thin, i.e., $\delta \gg \Delta$, δ being the skin depth of the shell.

When these three conditions are met the results derived in the previous sections are acceptable regardless of the relative magnitude of κ and δ. In this section we wish to treat a shell of arbitrary electrical-thickness and, thus, the condition (c) no longer applies. Although the condition (c) will be discarded, two conditions must be added in the present case. These added conditions, as will be shown below, are the following:

(c') $\kappa\delta \ll 1$,

(d') the source of the external field must be far away from the shield.

The condition (c') places quite a stringent restriction on the geometrical shape of the shell; in this sense the results obtained below, although taking into account the attenuation loss of the shield's wall, are not as general as those given in the previous sections.

Let us now proceed to consider a shell of arbitrary electrical-thickness and we shall assume the conditions (a), (b), (c') and (d') to hold. The shell can be magnetic as well as resistive. Inside the shield's wall we have

$$\nabla \times \underline{E} = i\omega\mu\underline{H} , \qquad \nabla \times \underline{H} = \sigma\underline{E} \tag{37a}$$

and outside the shield's wall

$$\nabla \times \underline{E} = i\omega\mu_o\underline{H} , \qquad \nabla \times \underline{H} = 0 . \tag{37b}$$

In the following the permeability μ and the conductivity σ will be assumed constant mainly for reason of simplicity.

We now integrate the second equation of (37a) around the rectangle a b c d inside the wall (Fig. 4) to obtain

$$H_y^{out} - H_y^{in} = - \sigma \int E_x \, dz , \tag{38a}$$

$$H_x^{out} - H_x^{in} = \sigma \int E_y \, dz , \tag{38b}$$

where the integrals are over the thickness of the shell. In deriving these equations we have used the continuity of the tangential $\underline{H}$ across the air-shell interface. Also, because of the conditions (b) and (d') we have treated H_y, H_z, E_y, etc. to be constant along ab or cd . Consequently, the integral of H_z over bc cancels the integral of H_z over da.

To find E_x and E_y inside the wall we begin with the equation

$$(\nabla^2 + k^2) E_x = 0 \tag{39}$$

and a similar equation for E_y . Here $k^2 = i\omega\mu\sigma$. Clearly, the condition (c') implies that the equation

$$\left(\frac{d^2}{dz^2} + k^2\right) E_x = 0 \tag{40}$$

is a good approximation to (39). Substituting the solution of (40) into the integral in (38a) and using the continuity of E_x across the air-shell interface we obtain

$$H_y^{out} - H_y^{in} = - \frac{\alpha}{\omega\mu_o} (E_x^{out} + E_x^{in}) \ , \tag{41a}$$

and similarly

$$H_x^{out} - H_x^{in} = + \frac{\alpha}{\omega\mu_o} (E_y^{out} + E_y^{in}) \ , \tag{41b}$$

where

$$\alpha = \frac{\omega\mu_o\sigma\Delta}{2} \frac{\tan(k\Delta/2)}{k\Delta/2} \ . \tag{42}$$

We now differentiate (41a) with respect to y and (41b) with respect to x and add the resulting equations. Then, using the first equation of (37b) and introducing Ω through (24) we finally obtain the first boundary condition

$$\nabla_s^2 (\Omega^{out} - \Omega^{in}) = i\alpha \frac{\partial}{\partial n} (\Omega^{out} + \Omega^{in}) \ . \tag{43a}$$

In obtaining (43a) we have replaced $\frac{\partial}{\partial z}$ by $\frac{\partial}{\partial n}$ and $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ by the surface Laplacian ∇_s^2 . This is permissible in view of the condition (b) that $\kappa\Delta \ll 1$.

To find the second boundary condition we integrate the first equation of (37a) around the rectangle a b c d inside the wall (Fig. 4), then follow exactly the same procedure leading to (43a), and obtain

$$\frac{\partial}{\partial n}\left(\Omega^{out} - \Omega^{in}\right) = \beta\, \nabla_s^2 \left(\Omega^{out} + \Omega^{in}\right) \tag{43b}$$

with

$$\beta = -\frac{\mu_r \Delta}{2}\, \frac{\tan(k\Delta/2)}{k\Delta/2} \tag{44}$$

and $\mu_r = \mu/\mu_o$. Thus, we see that the effect of the shield's wall on the interior and the exterior field can be duplicated by (43a) and (43b), that is to say, a three-medium problem can be reduced to a two-medium one.

The Laplace equation, $\nabla^2\Omega = 0$, together with the boundary conditions (43a) and (43b) constitute the formulation of the shielding problem involving shells of arbitrary electrical-thickness. Wherever the method of separation of variables is applicable, solutions can be readily obtained. For example, one can easily find that

$$H_t^{in} = H_t^{ex} \left[\cos k\Delta + \frac{1}{2}\left(\frac{\mu_r}{ka} - \frac{ka}{\mu_r}\right)\sin k\Delta\right]^{-1} \tag{45}$$

for a circular-cylindrical shell of radius a in a uniform, external, transverse field, and that

$$H^{in} = H^{ex} \left[\cos k\Delta + \frac{1}{3}\left(\frac{2\mu_r}{ka} - \frac{ka}{\mu_r}\right) \sin k\Delta \right]^{-1} \tag{46}$$

for a spherical shell of radius a in a uniform, external field. The expressions (45) and (46) agree with those obtained by Kaden[13]. In the non-magnetic case, i.e., $\mu_r = 1$, the term proportional to $(ka)^{-1}$ in both expressions should be neglected, since the condition (c') implies that ka is much larger than unity.

The longitudinal case of a two-dimensional problem as described in section 2.1 can be treated by the technique employed in that section. One point to be noted is that, contrary to the case of an electrically thin shell where the tangential component E_ℓ is continuous across the shell, one must, in the present case, distinguish E_ℓ^{out} and E_ℓ^{in} immediately outside and inside the shell's surface. The relation between E_ℓ^{out} and E_ℓ^{in} can be obtained by integrating (37a) across the thickness of the shell. The final result is that

$$H_z^{in} = H_z^{ex} \left[\cos k\Delta - \frac{kA}{\mu_r p} \sin k\Delta \right]^{-1} , \tag{47}$$

where A is the cross-sectional area within the cylindrical shell and p is its perimeter. The results derived from (47) for a circular-cylindrical shell and for two-parallel plates agree with those obtained by Kaden[13].

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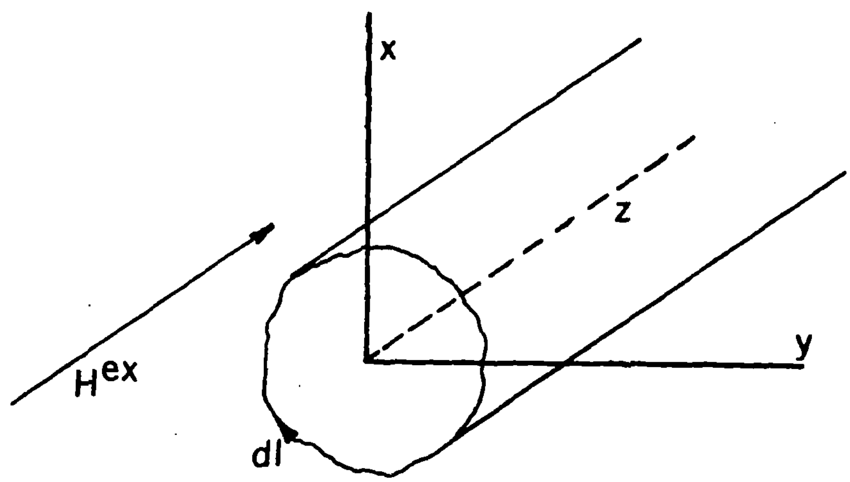

Figure 1.

Two-dimensional geometry - the longitudinal case.

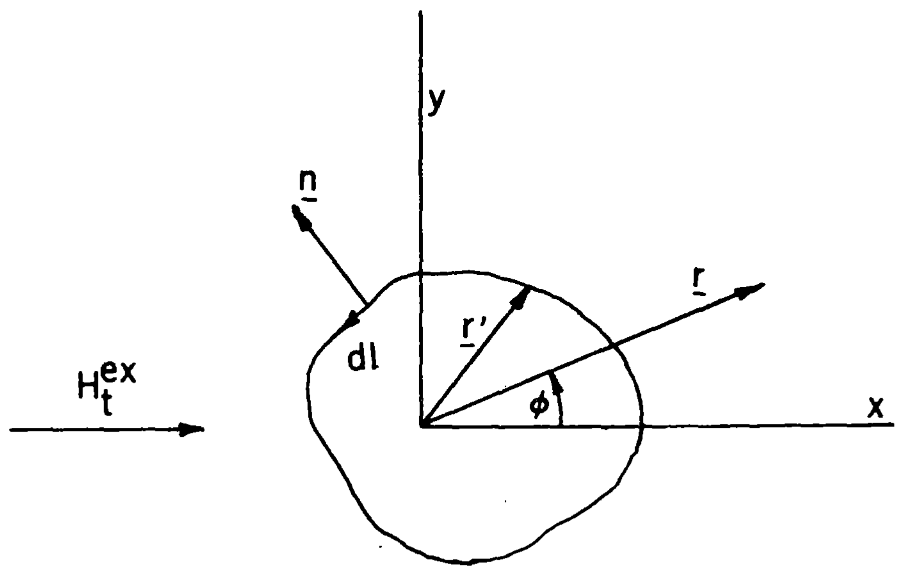

Figure 2.

Two-dimensional geometry - the transverse case.

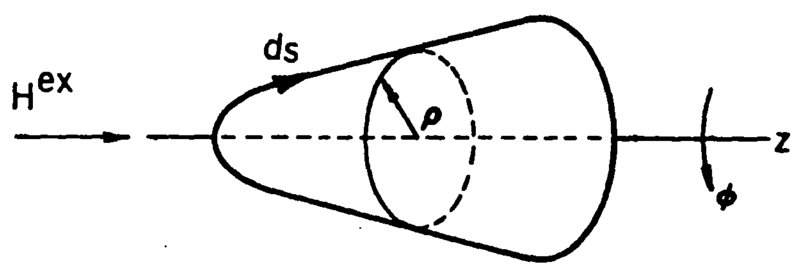

Figure 3.

Three-dimensional geometry with axial symmetry.

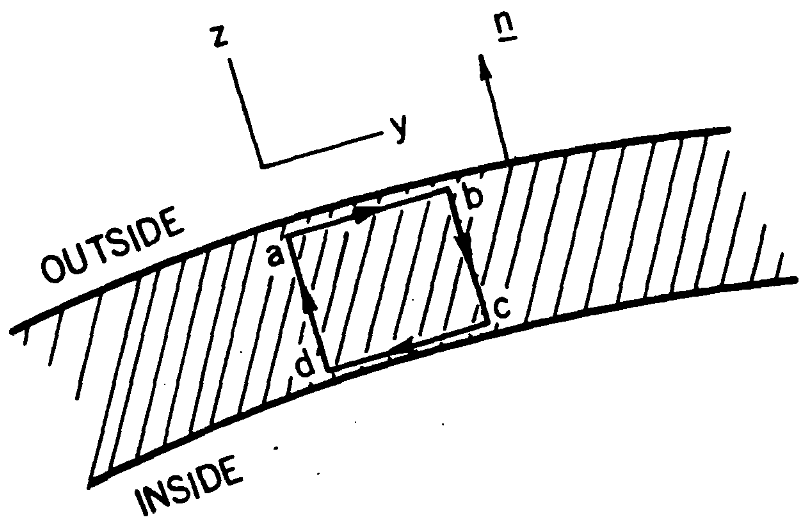

Figure 4.

Section of a shield's wall with the indicated path of integration.