EMP Interaction Notes

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A Study of Some Factors Affecting the Interior Field of a Semi-Infinite Pipe Exposed to a Low-Frequency Magnetic Field

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Abstract

Various conditions near the end of a semi-infinite, perfectly conducting pipe with a perfectly conducting flange are critically examined with regard to their effects on the interior field when the pipe is exposed to a low-frequency magnetic field. This is accomplished by solving several particular boundary-value problems. The results of this investigation indicate that (1) the effect of removing the flange is about 34%, that (2) the effect of inserting a resistive cap is to yield a "shielding ratio" similar to that in other low-frequency shielding problems, and that (3) the effect of an extremely narrow annular slot in an otherwise perfectly conducting cap is surprisingly large.

I INTRODUCTION

An important problem in EMP shielding theory is that of calculating the leakage field inside a perfectly conducting, semi-infinite circular pipe imbedded in a conducting dielectric half-space when the pipe is exposed to a slowly time-varying magnetic field. The various effects which contribute to this leakage field all involve particular geometries or materials near the end of the pipe. To make this general problem mathematically tractable these various effects will be considered separately and, in each case, an idealized problem will be solved under the quasi-static approximation to estimate the magnitude of the effect. First, the effect of the conducting dielectric half-space on the leakage field will be estimated by considering two extreme cases, namely the case of a pipe with a perfectly conducting flange and the case of an unflanged pipe in free space. The effects of different kinds of caps will then be investigated and, for mathematical convenience, this investigation will be carried out for the case in which the pipe has a perfectly conducting flange. Two particular caps will be studied separately - a resistive cap and a perfectly conducting cap with a narrow annular slot.

In Section II the problem of calculating the field inside a perfectly conducting semi-infinite circular pipe with a perfectly conducting flange is solved for the case in which the pipe is exposed to a uniform magnetic field. The method of solution is first to deduce, by Green's theorem, an appropriate integral equation for the aperture field and to solve the integral equation by the technique of eigenfunction expansion.

The effect of the flange on the leakage field is investigated in Section III by solving the problem of calculating the magnetic field inside a perfectly conducting unflanged pipe in free space. The method of solution is to employ the Wiener-Hopf technique.

The calculation of the effect of inserting a resistive cap at the mouth of the pipe is presented in Section IV. First an appropriate set of boundary conditions at the mouth of the pipe is obtained by integrating the Maxwell equations through the thickness of the cap. These boundary conditions, in combination with Green's theorem, give rise to a coupled set of integro-differential equations which are then solved by the method of eigenfunction expansion.

In Section V the field leaking into the pipe through a narrow annular slot in an otherwise perfectly conducting cap is calculated. First it is observed that the kernel of the integral equation for the normal component of the field in the slot can be split up into two parts. The first part is twice the Green's function of an infinite plane with vanishing normal derivative at the plate. For a narrow annular slot this Green's function resembles the kernel of Carleman's integral equation[1]. The second part corresponds to the regular part of the Green's function of an infinitely long cylinder with vanishing normal derivative at the cylindrical surface. This part is estimated numerically for the case where the distance of the slot from the cylindrical surface is at least of the order of the width of the slot. Then the reduced integral equation is solved according to Carleman.

A discussion of all the results obtained is given in Section VI.

An appendix is devoted to some methods of evaluating certain integrals which appear in the solutions of the above four problems. These integrals are difficult to evaluate and have not been reported in any published literature.

II FIELD INSIDE A FLANGED CIRCULAR PIPE

The geometry of the problem is depicted in figure 1 where a uniform magnetic field falls into a perfectly conducting circular pipe with a perfectly conducting flange. The problem is to determine the magnetic field inside the pipe. In mathematical language the following equation is to be solved

$$\nabla^2\varphi = 0 ,$$

with

$$\frac{\partial\varphi}{\partial n} = 0 \quad \text{for} \quad z = 0,\ \rho > a \ ;\ z < 0\ ,\ \rho = a\ , \tag{1}$$

and

$$\varphi \to B_o x \quad \text{for} \quad z \to \infty \ .$$

The magnetic field is obtained from $\underline{B} = \nabla\varphi$.

In region I $(\rho \le a\ ,\ \ z \le 0)$ one has, by applying Green's theorem,

$$\varphi_I = \int_S \left(\frac{\partial\varphi_I}{\partial n'} G_I - \varphi_I \frac{\partial G_I}{\partial n'}\right) dS' \ , \tag{2}$$

where $\underline{n}'$ is the outward unit normal to the surface S . The Green's function G_I satisfies

$$\nabla^2 G_I = -\delta(z-z')\ \delta(\rho-\rho')\ \frac{\delta(\phi-\phi')}{\rho} \tag{3}$$

with

$$\frac{\partial G_I}{\partial n} = 0 \quad \text{for} \quad \rho = a\ ,\ z < 0 \quad \text{and} \quad \rho \le a\ ,\ z = 0\ .$$

Solving equation (3) by the standard technique of separation of variables one obtains, for $z < z' \le 0$

$$G_I = \sum_{n=0}^{\infty} \sum_{r=1}^{\infty} \frac{2-\delta_n^0}{\pi} \cos n(\phi-\phi') \frac{\mu_r J_n(\mu_r \rho') \; J_n(\mu_r \rho)}{(a^2\mu_r^2-n^2)[J_n(\mu_r a)]^2} e^{\mu_r z} \cosh \mu_r z' \; . \tag{4}$$

Here J_n is the Bessel function of the first kind of order n and for $z' < z \le 0$ one simply interchanges z and z' in (4). δ_n^0 is the Kronecker delta. The μ_r's are determined by

$$J_n'(\mu_r a) = 0 \; , \tag{5}$$

where the prime denotes differentiation with respect to the argument*. Equation (2) now becomes

$$\varphi_I(\rho,z,\phi) = \int_0^{2\pi} \int_0^{a} \left(\frac{\partial \varphi_I}{\partial z'}\right) G_I(\rho,\rho',z,\phi-\phi') \; \rho' d\rho' d\phi' \; , \tag{6}$$

where

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* There are a few remarks which should be made, concerning equations (3) and (4), to those interested in following the details of the calculation. One is that equation (3), because of the nature of Neumann's problem, implies that the integral of the flux of the Green's function over a cross section of the tube for $z \succ z'$ is unity. Another is that this behavior is implicit in equation (4) if one allows for a term with $n = 0$ and $\mu_r = 0$ by formally letting μ_r approach zero in equation ( ) for $n = 0$. The last remark is to the effect that there is actually an invisible "n" index on the μ_r's of equation (4) as is obvious from equation (5), but that since shortly one will only need the $n = 1$ sum, simplicity of notation dictates a slight ambiguity in the equations for the time being.

$$G_I(\rho,\rho',z,\phi-\phi') = \sum_{n=0}^{\infty} \sum_{r=1}^{\infty} \frac{2-\delta_n^0}{\pi} \cos n(\phi-\phi') \frac{\mu_r J_n(\mu_r\rho')\, J_n(\mu_r\rho)}{(a^2\mu_r^2-n^2)[J_n(\mu_r a)]^2} e^{\mu_r z} . \tag{7}$$

In region II $(z > 0)$ one has, after employing Green's theorem,

$$\varphi_{II} = \varphi_i + \int_S \left(\frac{\partial\varphi_{II}}{\partial n'} G_{II} - \varphi_{II} \frac{\partial G_{II}}{\partial n'}\right) dS' \tag{8}$$

where

S is the infinite plane at $z = 0$,

$\varphi_i = B_o x = B_o \rho \cos \phi$.

Here G_{II} satisfies the following equation

$$\nabla^2 G_{II} = -\delta(x-x')\,\delta(y-y')\,\delta(z-z') \tag{9}$$

with

$$\frac{\partial G_{II}}{\partial n} = 0 \quad \text{at} \quad z = 0 .$$

Equation (9) can be easily solved and the solution is

$$G_{II} = \frac{1}{4\pi\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} + \frac{1}{4\pi\sqrt{(x-x')^2+(y-y')^2+(z+z')^2}} \tag{10}$$

Substitution of (10) into (8) with $\frac{\partial}{\partial n'} = -\frac{\partial}{\partial z'}$ gives

$$\varphi_{II} = B_o\rho \cos \phi - \frac{1}{2\pi} \int_0^{2\pi} \int_0^{a} \left(\frac{\partial\varphi_{II}}{\partial z'}\right) \frac{\rho' d\rho' d\phi'}{\sqrt{\rho^2+\rho'^2-2\rho\rho' \cos(\phi-\phi')}} \tag{11}$$

For $z = 0$, $\rho \leq a$ one has the following boundary conditions:

$$\varphi_I = \varphi_{II}$$

$$\frac{\partial\varphi_I}{\partial z} = \frac{\partial\varphi_{II}}{\partial z} = \frac{\partial\varphi}{\partial z} \quad . \tag{12}$$

By equating (6) and (11) at $z = 0$, $\rho \leq a$ one obtains the following integral equation for $\frac{\partial\varphi}{\partial z'}$:

$$B_o\rho \cos \phi - \frac{1}{2\pi} \int_0^{2\pi} \int_0^a \left(\frac{\partial\varphi}{\partial z'}\right) \frac{\rho' d\rho' d\phi}{\sqrt{\rho^2+\rho'^2-2\rho\rho' \cos(\phi-\phi')}}$$

$$= \int_0^{2\pi} \int_0^a \left(\frac{\partial\varphi}{\partial z'}\right) G_I(\rho,\rho',0,\phi-\phi') \rho' d\rho' d\phi' \quad . \tag{13}$$

To simplify this integral equation it is to be noted that φ_I and φ_{II} vary as $\cos \phi$ and so will $\partial\varphi_I/\partial z$ and $\partial\varphi_{II}/\partial z$. Accordingly, one substitutes

$$\frac{\partial\varphi}{\partial z} = B_o \cos \phi \; f(\rho) \tag{14}$$

into (13) and obtains, with $\rho = ax'$, $\rho' = ax'$ and $\zeta_r = au_r$, the following integral equation in dimensionless form

$$x - \int_0^1 f(x')x'dx' \frac{1}{2\pi} \int_0^{2\pi} \frac{\cos \theta \; d\theta}{\sqrt{x^2+x'^2-2xx' \cos \theta}}$$

$$= \int_0^1 f(x')x'dx' \sum_r \frac{2\zeta_r J_1(\zeta_r x) \; J_1(\zeta_r x')}{(\zeta_r^2-1)[J_1(\zeta_r)]^2} \quad . \tag{15}$$

The kernel on the left-hand side can be transformed as follows[2]:

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{\cos\theta\, d\theta}{\sqrt{x^2+x'^2-2xx'\cos\theta}} = \frac{1}{2\pi}\int_0^{2\pi} \cos\theta\, d\theta \int_0^\infty J_o(k\sqrt{x^2+x'^2-2xx'\cos\theta})dk$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \cos\theta\, d\theta \sum_{n=0}^{\infty} (2-\delta_n^o)\cos n\theta \int_0^\infty J_n(kx)\, J_n(kx')dk$$

$$= \int_0^\infty J_1(kx)\, J_1(kx')dk\ . \tag{16}$$

To solve the integral equation (15) for $f(x)$ one substitutes the expansion

$$f(x) = \sum_s C_s\, J_1(\zeta_s x) \tag{17}$$

into (15), and then multiplying both sides of the equation by $xJ_1(\zeta_r x)$ and integrating with respect to x from 0 to 1, one obtains the following matrix equation

$$C_r\left[\frac{\zeta_r^2-1}{2\zeta_r^3} + \int_0^\infty \frac{u^2[J_1'(u)]^2}{(u^2-\zeta_r^2)^2}\, du\right]$$

$$= \frac{J_2(\zeta_r)}{\zeta_r[J_1(\zeta_r)]^2} - \sum_{s\neq r} C_s \frac{J_1(\zeta_s)}{J_1(\zeta_r)} \int_0^\infty \frac{u^2[J_1'(u)]^2}{(u^2-\zeta_r^2)(u^2-\zeta_s^2)}\, du\ . \tag{18}$$

The integrals in (18) are evaluated in the appendix for $\zeta_1 = 1.84118$ and $\zeta_2 = 5.33144$. Solving equation (18) one finds that, if the couplings of C_1 with all other C's are neglected, C_1 is given by

$$C_1 = 1.6491 \qquad \text{(no coupling)}, \tag{19}$$

and that, if only the coupling between C_1 and C_2 is taken into account, C_1 is given by

$$C_1 = 1.6758 \qquad \text{(coupling between 1st and 2nd mode)}. \tag{20}$$

It is felt that the value given by equation (20) for C_1 is rather accurate.

To obtain the field inside the pipe (region I), one simply substitutes (14) together with (17) into (6) and carries out the integration. It is found that

$$\varphi_I = B_o a \cos \phi \sum_r C_r \zeta_r^{-1} J_1(\zeta_r \rho/a) e^{\zeta_r z/a}, \quad z<0 \tag{21}$$

The principal (first) mode is, by virtue of (20), given by

$$\varphi_I^{(p)} = 0.9102\, B_o a \cos \phi\, J_1(\zeta_1 \rho/a) e^{\zeta_1 z/a}, \quad z<0, \tag{22}$$

whence the magnetic field of this mode along the axis of the pipe is

$$B_x = 0.838\, B_o \exp(\zeta_1 z/a)\ , \quad z<0\ , \tag{23}$$

where $\zeta_1 = 1.84118$.

III FIELD INSIDE AN UNFLANGED PIPE

An explicit solution may be exhibited to the problem of a perfectly conducting, semi-infinite, hollow circular cylinder immersed in a static external magnetic field which is uniform and perpendicular to the axis of the cylinder. The physical situation is diagrammed in figure 2 where B_o represents the external magnetic field.

The method of solution of this problem is based on the Wiener-Hopf technique, and the only work which will be referred to in the following exposition is the book by Noble[3]. The following two theorems from his book will shorten the exposition considerably:

Theorem A (Liouville's theorem)

If $f(\alpha)$ is an integral function such that $|f(\alpha)| \leq M|\alpha|^p$ as $\alpha \to \infty$ where M,p are constants, then $f(\alpha)$ is a polynomial of degree less than or equal to $[p]$ where $[p]$ is the integral part of p .

Theorem B

Let $f(\alpha)$ be regular and non-zero in a strip $\tau_- < \tau < \tau_+$, $-\infty < \sigma < \infty$ where $\alpha = \sigma + i\tau$. Furthermore let $f(\alpha) \to +1$ as $\sigma \to \pm\infty$ in the strip. Then one can write $f(\alpha) = f_+(\alpha)\, f_-(\alpha)$ where $f_+(\alpha)$, $f_-(\alpha)$ are regular, bounded, and non-zero in $\tau > \tau_-$, $\tau < \tau_+$ respectively. Moreover one may write explicitly:

$$\ln f_+(\alpha) = \frac{1}{2\pi i} \int_{ic-\infty}^{ic+\infty} \frac{\ln f(\alpha')}{\alpha'-\alpha}\, d\alpha'$$

$$\ln f_-(\alpha) = \frac{1}{2\pi i} \int_{id-\infty}^{id+\infty} \frac{\ln f(\alpha')}{\alpha'-\alpha}\, d\alpha'$$

where c, d are any numbers such that $\tau_- < c < \tau < d < \tau_+$.

The solution of the particular problem posed above will be obtained, for convergence reasons, from that of the corresponding scalar wave problem in which the medium has a slight loss, i.e., the potential φ in

$$\underline{B} = \nabla\varphi \tag{24}$$

will be obtained from φ_k^{total} where

$$\varphi_k^{total} = \varphi_k + e^{ikx} \tag{25}$$

and

$$(\nabla^2 + k^2)\varphi_k = 0 \,, \qquad \text{Im } k > 0 \tag{26}$$

with

$$\frac{\partial\varphi_k^{total}}{\partial\rho} = 0 \quad \text{for} \quad \rho = a \,, \quad z < 0 \,. \tag{27}$$

Since φ must vary as $\cos\phi$ from the symmetry of the problem, one will first identify the corresponding term in φ_k^{total} and then take the limit $k \to 0$ to obtain φ , i.e.,

$$\varphi(\rho,z,\phi) = B_0 \cos\phi \lim_{k\to 0} \frac{1}{ik}\left[\frac{1}{\pi}\int_0^{2\pi} \varphi_k^{total}(\rho,z,\phi')\cos\phi'\, d\phi'\right] . \tag{28}$$

The scalar wave problem is an extension of work presented by Noble in Section 3.4. Defining

$$\Phi(\rho,\alpha,n) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty}\int_0^{2\pi} \varphi_k(\rho,z,\phi)\, e^{in\phi}\, e^{i\alpha z}\, d\phi\, dz \tag{29}$$

one can easily obtain from (26)

$$\begin{aligned} \Phi(\rho,\alpha,n) &= A_n(\alpha)\, K_n(\gamma\rho) & \rho \geq a \\ &= B_n(\alpha)\, I_n(\gamma\rho) & 0 \leq \rho \leq a \end{aligned} \tag{30}$$

where I_n and K_n are modified Bessel functions of order n, while

$$\gamma = \sqrt{\alpha^2 - k^2} .$$

The cuts are shown in figure 3 and the branch is chosen in such a way that for $\alpha = 0$, $\gamma = -ik$.

Now write

$$\Phi_+(a+0,\alpha,n) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{i\alpha z} \int_0^{2\pi} \varphi_k(a+0,z,\phi) e^{in\phi}\, d\phi\, dz$$

$$\Phi_-(a+0,\alpha,n) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{i\alpha z} \int_0^{2\pi} \varphi_k(a+0,z,\phi) e^{in\phi}\, d\phi\, dz$$

with similar obvious definitions for $\Phi_+(a-0,\alpha,n)$ and $\Phi_-(a-0,\alpha,n)$. Further, introduce

$$\Phi_+'(a,\alpha,n) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{i\alpha z} \int_0^{2\pi} \frac{\partial\varphi_k}{\partial\rho}\, e^{in\phi}\, d\phi\, dz \qquad (31)$$

with another obvious definition for $\Phi_-'(a,\alpha,n)$. Then it may readily be shown, from the nature of $\varphi_k(\rho,z,\phi)$ and the theory of Fourier integrals, that $\Phi_+(a+0,\alpha,n)$, $\Phi_+(a-0,\alpha,n)$ and $\Phi_+'(a,\alpha,n)$ are regular in the upper half plane $\operatorname{Im} \alpha > -\operatorname{Im} k$, while $\Phi_-(a+0,\alpha,n)$, $\Phi_-(a-0,\alpha,n)$ and $\Phi_-'(a,\alpha,n)$ are regular in the lower half plane $\operatorname{Im} \alpha < \operatorname{Im} k$.

Employing the above notation and invoking the boundary condition (27) one may write

$$\Phi_+(a+0,\alpha,n) + \Phi_-(a+0,\alpha,n) = A_n(\alpha)\; K_n(\gamma a)$$

$$\Phi_+(a-0,\alpha,n) + \Phi_-(a-0,\alpha,n) = B_n(\alpha)\; I_n(\gamma a)$$

$$\Phi_+'(a,\alpha,n) + \Phi_-'(a,\alpha,n) = \gamma A_n(\alpha)\; K_n'(\gamma a) = \gamma B_n(\alpha)\; I_n'(\gamma a)\,. \qquad (32)$$

By algebraic manipulation of equation (32) it is easily shown that, by defining

$$D_-(\alpha,n) \equiv \Phi_-(a+0,\alpha,n) - \Phi_-(a-0,\alpha,n)\;,$$

$$D_-(\alpha,n) = \frac{1}{\gamma^2 a I_n'(\gamma a) K_n'(\gamma a)}\,[\Phi_+'(a,\alpha,n) + \Phi_-'(a,\alpha,n)]\;, \qquad (33)$$

where

$$\Phi_-'(a,\alpha,n) = (-i)^n i\sqrt{2\pi}\,\frac{kJ_n'(ka)}{\alpha}\;.$$

The latter expression is obtained from equations (25) and (27) and the definition of $\Phi_-'(a,\alpha,n)$ analogous to equation (31).

Now the Wiener-Hopf technique is employed. That is to say, equation (33) is rewritten in a form such that the left hand side of the equation is regular in the upper half α-plane while the right hand side is regular in the lower half α-plane. When this is done it is found that both sides approach zero as $\alpha \to \infty$. Since the rewritten equation is true everywhere in the strip $-\text{Im}\; k < \alpha < 0$, it is clear, from complex variable theory, that both sides define the same function of α in the whole plane. From Theorem A, and the asymptotic behavior, this function is identically zero. In this manner one finds that

$$D_-(\alpha,n) = \frac{-2i(-i)^n}{\alpha K_-(\alpha,n)}\,\frac{\sqrt{2\pi}\; J_n'(ka)}{(\alpha-k)aK_+(0,n)}\;. \qquad (34)$$

where $K_+(\alpha,n)$ and $K_-(\alpha,n)$ are given by

$$\ln K_\pm(\alpha,n) = \frac{\pm 1}{2\pi i}\int_{-\infty+ic}^{\infty+ic} \frac{\ln[-2K_n'(\gamma'a)I_n'(\gamma'a)]}{\alpha'-\alpha}\,d\alpha' \ , \quad \mathrm{Im}\ \alpha \gtrless c$$

which are obtained from Theorem B by setting

$$f(\alpha) = K(\alpha,n) = -2K_n'(\gamma a)\ I_n'(\gamma a)\ .$$

Now from equations (25), (29) and (30) one has

$$\varphi_k^{total}(\rho,z,\phi) = e^{ikx} + \frac{1}{(2\pi)^{3/2}}\sum_{n=-\infty}^{\infty} e^{-in\phi}\int_{-\infty+ib}^{\infty+ib} e^{-i\alpha z}\,B_n(\alpha)\ I_n(\gamma\rho)\,d\alpha \tag{35}$$

for $-\mathrm{Im}\ k < b < 0$ and $0 \le \rho \le a$.

Elimination of B_n from this equation in favor of D_- by means of equations (32) and (33) gives

$$\varphi_k^{total}(\rho,z,\phi) = e^{ikx} + \frac{1}{(2\pi)^{3/2}}\sum_{n=-\infty}^{\infty} e^{-in\phi}\int_{-\infty+ib}^{\infty+ib} e^{-i\alpha z}\gamma a K_n'(\gamma a)D_-(\alpha,n)I_n(\gamma\rho)\,d\alpha\ . \tag{36}$$

Substituting (34) into (36) and then into (28) one finds that the potential φ of the magneto-static problem is given by

$$\varphi = B_o x - \frac{2\cos\phi}{\pi}\lim_{k\to 0}\frac{J_1'(ka)}{ik\ K_+(0,1)}\int_{-\infty+ib}^{\infty+ib}\frac{e^{-i\alpha z}\gamma K_1'(\gamma a)I_1(\gamma\rho)}{\alpha(\alpha-k)\ K_-(\alpha,1)}\,d\alpha$$

$$= B_o x + B_o a\cos\phi\lim_{k\to 0}\frac{1}{ka\ K_+(0,1)}\ \frac{1}{2\pi i}\int_{-\infty+ib}^{\infty+ib}\frac{e^{-i\alpha z}\gamma I_1'(\gamma\rho)K_+(\alpha,1)}{\alpha(\alpha-k)I_1'(\gamma a)}\,d\alpha \tag{37}$$

where $K_+(\alpha,1)\ K_-(\alpha,1) = -2K_1'(\gamma a)\ I_1'(\gamma a)$ has been used.

In evaluating equation (37) it is expedient to state three facts without proof. The demonstration of their validity is quite straightforward but unilluminating.

(i). $K_+(0,1) = \sqrt{K(0,1)}$

(ii). For $z < 0$ one may close the contour of the integral in equation (37) by a large semicircle in the upper half-plane. The only singularities of the integrand in this half-plane are poles at 0 and the zeros of $I_1'(\gamma a)$.

(iii). The contribution of the pole at $\alpha = 0$ in the integral (37) just cancels the potential of the external field, the term given by $B_o x$.

Using all these facts in evaluating equation (37) and going to the limit $k \to 0$ of equation (28) one obtains for the desired potential function

$$\varphi(\rho,z,\phi) = B_o a \cos\phi \sum_r C_r \zeta_r^{-1} J_1(\zeta_r \rho/a) e^{\zeta_r z/a} , \quad z < 0 \qquad (38)$$

where the sum is over all ζ_r's .

In equation (38) C_r is given by

$$C_r = -\frac{1}{J_1''(\zeta_r)} \exp\left[\frac{\zeta_r}{\pi} \int_0^\infty \frac{\ln[-2K_1'(x)I_1'(x)]}{x^2 + \zeta_r^2} dx\right] . \qquad (39)$$

Equation (39) may be evaluated numerically, the result for the coefficient of the first mode being

$$C_1 = 2.245 .$$

Hence, the magnetic field of the first mode along the axis of the pipe is given by

$$B_x = 1.12\, B_o \exp(\zeta_1 z/a) , \qquad z < 0 \qquad (40)$$

where $\zeta_1 = 1.84118$.

IV FIELD INSIDE A FLANGED CIRCULAR PIPE WITH A RESISTIVE CAP

In this section the effect of a resistive cap on the slowly time-varying magnetic field leaking into a circular pipe with a perfectly conducting flange is calculated. The geometry of the problem is shown in figure 4.

The first step in the solution of the problem is to derive a set of boundary conditions for φ_I and φ_{II} at the cap $(z = 0, \rho \leq a)$, by treating the thickness d of the cap as vanishingly small and the product σd as finite, σ being the conductivity of the cap. In addition, the cap is taken to be non-magnetic and its skin depth is assumed much larger than d. Under these conditions the tangential component of the electric field (or, equivalently, the normal component of the magnetic field) does not vary appreciably through the thickness of the cap. On the other hand, the tangential component of the magnetic field is discontinuous through the cap due to the presence of a current sheet. To find such a discontinuity one integrates the equation

$$\nabla \times \underline{B} = \mu_0 \sigma \underline{E} \tag{41}$$

through the cap as indicated in figure 4, μ_0 being the free-space permeability. By means of Stoke's theorem one obtains

$$B_{II,x} - B_{I,x} = \mu_0 \sigma d \; E_y \; , \tag{42a}$$

$$B_{II,y} - B_{I,y} = -\mu_0 \sigma d \; E_x \; . \tag{42b}$$

Differentiating (42a) with respect to x and (42b) with respect to y and then adding the resulting equations one has

$$\frac{\partial}{\partial x}\left[B_{II,x} - B_{I,x}\right] + \frac{\partial}{\partial y}\left[B_{II,y} - B_{I,y}\right]$$

$$= \mu_o \sigma d \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right) \qquad (43a)$$

$$= i\omega\mu_o \sigma d \; B_{I,z} \qquad (43b)$$

$$= i\omega\mu_o \sigma d \; B_{II,z} \; . \qquad (43c)$$

The last two steps follow from the equation

$$\nabla \times \underline{E} = i\omega \underline{B} \qquad (44)$$

and the continuity of the normal component of $\underline{B}$. Here and henceforth the time harmonic factor $\exp(-i\omega t)$ is suppressed throughout all discussions. Since $\underline{B} = \nabla\varphi$, equations (43) yield the following boundary conditions for φ when $z = 0,\ \rho \le a$:

$$\frac{\partial\varphi_I}{\partial z} = \frac{\partial\varphi_{II}}{\partial z} = \frac{\partial\varphi}{\partial z} \qquad (45a)$$

$$\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\right)\left(\varphi_{II} - \varphi_I\right) = i\omega\mu_o\sigma d\,\frac{\partial\varphi}{\partial z} \; . \qquad (45b)$$

Since $\varphi_I \neq \varphi_{II}$ at the mouth of the pipe, the integral equation will differ from the one given by equation (13) where the resistive cap is absent. Subtracting equation (6) from equation (11) and making use of (45a) one gets

$$\left(\varphi_{II} - \varphi_I\right) - \int_0^{2\pi}\int_0^a \left(\frac{\partial\varphi}{\partial z'}\right) G_I(\rho,\rho',0,\phi-\phi')\,\rho'd\rho'd\phi'$$

$$= B_o\rho\cos\phi - \frac{1}{2\pi}\int_0^{2\pi}\int_0^a \left(\frac{\partial\varphi}{\partial z'}\right)\frac{\rho'd\rho'd\phi'}{\sqrt{\rho^2 + \rho'^2 - 2\rho\rho'\cos(\phi-\phi')}} \; , \text{ for } z=0,\ \rho < a \; . \qquad (46)$$

where G_I is given by equation (7).

Equations (45b) and (46) constitute a coupled set of integro-differential equations for the present problem. First solve equation (45b) for $\varphi_{II} - \varphi_I$ when $\partial\varphi/\partial z$ is given by

$$\frac{\partial\varphi}{\partial z} = B_o \cos\phi \sum_s C_s J_1(\zeta_s \rho/a) \ , \text{ for } z = 0, \ \rho < a \ . \tag{47}$$

It is easily found that the solution of (45b) is given by

$$\varphi_{II} - \varphi_I = i\omega\mu_o\sigma d a^2 B_o \cos\phi \sum_s \frac{C_s}{\zeta_s^2} J_1(\zeta_s \rho/a) \ , \tag{48}$$

which satisfies the condition of regularity at $\rho = 0$ and the boundary condition $\partial(\varphi_{II} - \varphi_I)/\partial\rho = 0$ at $\rho = a$. Substituting (47) and (48) into (46) and following the same procedure as in Section II one finds that the C_r's satisfy the following matrix equation

$$C_r\left[\frac{\zeta_r^2-1}{2\zeta_r^3}\left(1 - \frac{i\omega\mu_o\sigma d a}{\zeta_r}\right) + \int_0^\infty \frac{u^2[J_1'(u)]^2}{(u^2-\zeta_r^2)^2}\, du\right]$$

$$= \frac{J_2(\zeta_r)}{\zeta_r[J_1(\zeta_r)]^2} - \sum_{s\neq r} C_s \frac{J_1(\zeta_s)}{J_1(\zeta_r)} \int_0^\infty \frac{u^2[J_1'(u)]^2}{(u^2-\zeta_r^2)(u^2-\zeta_s^2)}\, du \ . \tag{49}$$

Solving equation (49) one finds that (see appendix) if the couplings of C_1 with all the other $C's$ are neglected. C_1 is given by

$$C_1 = \frac{1.649}{1 - i\ 0.338\ \omega/\omega_o} \qquad \text{(no coupling)} \ . \tag{50}$$

and that if only the coupling between C_1 and C_2 is taken into account, C_1 is given by

$$C_1 = \frac{1.6769}{1 - i\ 0.3392\ \omega/\omega_o} - \frac{0.0011}{1 - i\ 0.0098\ \omega/\omega_o} \quad \begin{pmatrix}\text{coupling between} \\ \text{1st and 2nd mode}\end{pmatrix} \tag{51}$$

which, upon combining the two fractions together, becomes, to a very good accuracy, the following simpler form:

$$C_1 = \frac{1.6758}{1 - i\ 0.349\ \omega/\omega_o} \ . \tag{52}$$

In equations (50), (51) and (52), $\omega_o^{-1} = (\mu_o a)(\sigma d)$. Comparison of equations (50) and (51) shows that the effect of the coupling of the second mode, and perhaps of all other higher modes, on the first mode, is indeed negligibly small. From equations (21) and (52) one can easily find that the magnetic field of the first mode along the axis of the pipe is given by

$$B_x = \frac{0.838}{1 - i\ 0.349\ \omega/\omega_o} B_o \exp(\zeta_1\ z/a)\ , \qquad z < 0\ , \tag{53}$$

where $\zeta_1 = 1.84118$.

V FIELD INSIDE A FLANGED PIPE WITH A SLOTTED CAP

This section is devoted to the calculation of the magnetic field leaking into a flanged pipe through an annular slot at the top of the pipe. The cylindrical wall, the flange, and the slotted cap are all perfectly conducting and hence the normal component of the magnetic field vanishes there. The geometry of the problem is depicted in figure 5.

The integral equation for the normal component of the magnetic field $\partial\varphi/\partial z$ in the slot is similar to the one given by equation (13) except that now the limits of the ρ'-integration vary from c to b. It is given by

$$B_o \cos\phi = \int_0^{2\pi}\int_c^b \left(\frac{\partial\varphi}{\partial z'}\right)\left[G_I(\rho,\rho',0,\phi-\phi') + G_{II}(\rho,\rho',0,\phi-\phi')\right]\rho' d\rho' d\phi' \quad \text{for } c \le \rho \le b, \tag{54}$$

where G_I and G_{II} are respectively given by (7) and (10).

When one attempts to solve equation (54) a serious difficulty which he will encounter is that G_I and G_{II} have seemingly different forms. For the case of a narrow slot G_{II} can be approximated as a logarithmic kernel which is well known in the theory of integral equations. However, G_I as given by equation (7) does not seem to reduce to any familiar kernels even in the case of a narrow slot. Thus the first logical step in solving (54) is to find a desirable representation for G_I rather than that given by (7). To achieve this end one solves equation (3) anew, keeping in mind that the inverse distance can be expressed in terms of modified Bessel's functions as follows[4]:

$$\frac{1}{4\pi}\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}$$

$$= \frac{1}{2\pi^2}\sum_{m=0}^{\infty}(2-\delta_m^0)\cos m(\phi-\phi')\int_0^{\infty} K_m(k\rho)I_m(k\rho')\cos k(z-z')dk\ ,$$

$$\rho > \rho'\ . \qquad (55)$$

If $\rho < \rho'$ interchange ρ and ρ' in (55). Solving equation (3) one easily obtains

$$G_I = \frac{1}{4\pi\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} + \frac{1}{4\pi\sqrt{(x-x')^2+(y-y')^2+(z+z')^2}}$$

$$-\frac{1}{2\pi^2}\sum_{m=0}^{\infty}(2-\delta_m^0)\cos(\phi-\phi')\int_0^{\infty}\frac{K_m'(ka)I_m(k\rho')I_m(k\rho)}{I_m'(ka)}\left[\cos k(z+z')+\cos k(z-z')\right]dk\ ,$$

$$\rho > \rho'\ . \qquad (56)$$

The first two terms combine to give G_{II} as given by (10).

Insert (56) in (54) and set $\partial\varphi/\partial z' = B_o \cos\phi' f(\rho')$, $\rho = ax$, $\rho' = ax'$. Then equation (54) reduces to the following dimensionless form

$$x = \int_{c/a}^{b/a} f(x')\, x'\left[K(x,x') + \frac{2}{\pi} K_c(x,x')\right]dx'\ , \qquad (57)$$

where

$$K(x,x') = 2G_{II}(x,x') = \frac{1}{\pi}\int_0^{2\pi}\frac{\cos\theta\, d\theta}{\sqrt{x^2+x'^2-2xx'\cos\theta}} \qquad (58)$$

$$K_c(x,x') = -\int_0^\infty \frac{K_1'(\lambda)\ I_1'(\lambda x')\ I_1(\lambda x)}{I_1'(\lambda)}\, d\lambda \ . \tag{59}$$

K_c is always positive for real and positive x and x'. In the case of an annular slot in an infinite plane the integral equation for $f(x)$ can be obtained from (57) by setting $K_c = 0$. Thus, K_c accounts for the effect of the cylindrical wall on the aperture field.

In the following consider only the case where the annular slot is narrow, i.e., $\Delta = (b-c)/a \ll 1$. Then the kernel K takes a simpler form, as is shown below[5]:

$$K(x,x') = \frac{1}{\pi}\int_0^{2\pi} \frac{\cos\theta\, d\theta}{\sqrt{x^2+x'^2-2xx'\cos\theta}} = \frac{4}{\pi}\int_0^{\pi/2} \frac{\sin^2\varphi-\cos^2\varphi}{\sqrt{(x+x')^2-4xx'\sin^2\varphi}}\, d\varphi$$

$$= \frac{4}{\pi(x+x')}\int_0^{\pi/2} \frac{\sin^2\varphi-\cos^2\varphi}{\sqrt{1-k^2\sin^2\varphi}}\, d\varphi \ , \qquad k^2 = \frac{4xx'}{(x+x')^2}$$

$$= \frac{x+x'}{\pi xx'}\left[(2-k^2)\mathbf{K}-2\mathbf{E}\right] , \tag{60}$$

where $\mathbf{K}$ and $\mathbf{E}$ are complete elliptic integrals of the first and second kind. For $|x-x'| \ll 1$, equation (60) has the following approximate form[5]:

$$K(x,x') \approx \frac{2}{\pi\bar{x}}\left[\ln 8\bar{x} - 2 - \ln|x-x'|\right] , \tag{61}$$

where $\bar{x}$ is the arithmetic mean of b/a and c/a. $K_c(x,x')$ is well behaved for $0 \le (x,x') \le 1-\epsilon\ (\epsilon > 0)$ and becomes logarithmically singular only at $x = x' = 1$. Thus, when the distance of the slot from the cylindrical wall is at least of the order of the slot's width and $|x-x'| << 1$, K_c can be taken out of the integral sign in (57) and replaced by

$$K_c(x,x') \approx K_c(\bar{x},\bar{x}) = -\int_0^\infty \frac{K_1'(\lambda)[I_1(\lambda\bar{x})]^2}{I_1'(\lambda)} d\lambda \ . \tag{62}$$

For a given value of $\bar{x}$ equation (62) will be evaluated numerically.

Substitution of (61) and (62) into the integral equation (57) gives

$$\int_{c/a}^{b/a} f(x') \ln|x-x'|dx'$$

$$= [\ln 8\bar{x} - 2 + \bar{x} K_c(\bar{x},\bar{x})] \int_{c/a}^{b/a} f(x')dx' - \frac{\pi}{2} x \ . \tag{63}$$

Let

$$x = \bar{x} + \frac{\Delta}{2} u, \qquad x' = \bar{x} + \frac{\Delta}{2} u', \qquad f(\bar{x} + \frac{\Delta}{2} u') = F(u') \ .$$

Then equation (63) becomes

$$\int_{-1}^{1} F(u') \ln|u-u'|du'$$

$$= [\ln \frac{16\bar{x}}{\Delta} -2 + \bar{x} K_c] \int_{-1}^{1} F(u')du' - \frac{\pi\bar{x}}{\Delta} - \frac{\pi}{2} u \ . \tag{64}$$

The last term on the right-hand side is small compared to Δ^{-1} and can therefore be neglected. The remaining integral equation has the solution[1]

$$F(u) = \frac{1}{\pi \ln 2 \sqrt{1-u^2}} \left[\frac{\pi\bar{x}}{\Delta} - (\ln \frac{16\bar{x}}{\Delta} - 2 + \bar{x} K_c) \int_{-1}^{1} F(u')du'\right]. \quad (65)$$

Integrating this equation from -1 to 1 and solving the resulting equation one obtains

$$\int_{-1}^{1} F(u)du = \frac{\pi\bar{x}}{\Delta} \frac{1}{5 \ln 2 - 2 - \ln \Delta + \ln \bar{x} + \bar{x} K_c} . \quad (66)$$

One can now proceed to evaluate the coefficients C_r which appear in equation (21). It is easily seen that

$$C_r = \frac{2\zeta_r^2}{(\zeta_r^2-1)[J_1(\zeta_r)]^2} \int_{c/a}^{b/a} f(x') J_1(\zeta_r x')x'dx'$$

$$\approx \frac{\zeta_r^2}{(\zeta_r^2-1)[J_1(\zeta_r)]^2} \bar{x} J_1(\zeta_r\bar{x}) \Delta \int_{-1}^{1} F(u')du' .$$

Upon substituting (66) into this equation it gives

$$C_r \approx \frac{\pi\zeta_r^2}{(\zeta_r^2-1)[J_1(\zeta_r)]^2} \frac{\bar{x} J_1(\zeta_r\bar{x})}{5 \ln 2 - 2 - \ln \Delta + \ln \bar{x} + \bar{x} K_c(\bar{x})} . \quad (67)$$

Equation (67) for C_r is accurate up to the order of $\Delta/(1-\bar{x})$, i.e., the ratio of the slot's width to its distance from the pipe's wall.

Two numerical examples are now considered in order.

Example 1:

For $\Delta^{-1} = 2304$ and $\bar{x} = 0.9$, it is found that $K_c(\bar{x}) = 1.1343$ by numerically evaluating (62). Then equation (67) gives $C_1 = 0.6051$ from which one obtains the magnetic field of the first mode along the axis of the pipe to be

$$B_x = 0.303\ B_o \exp(\zeta_1\ z/a)\ , \qquad z < 0\ . \tag{68}$$

Example 2:

For $\Delta^{-1} = 2304$ and $\bar{x} = 71/72$, equation (62) gives $K_c = 2.091$ and equation (67) gives $C_1 = 0.6698$. In this case the magnetic field of the first mode along the axis of the pipe is given by

$$B_x = 0.335\ B_o \exp(\zeta_1\ z/a)\ , \qquad z < 0 \tag{69}$$

where $\zeta_1 = 1.84118$.
The results given by equations (68) and (69) are accurate within 1%.

VI CONCLUDING REMARKS

The results of this report can be summarized in the following chart:

The magnetic field of the first mode along the axis of the pipe is given by

$$B_x = \frac{C_1}{2} B_o \exp(\zeta_1 z/a) , \qquad z < 0 ,$$

where $\zeta_1 = 1.84118$, B_o = external field , a = pipe radius.

	Perfectly conducting, semi-infinite pipe	$C_1/2$
1.	with a perfectly conducting flange	0.838
2.	without a flange	1.120
3.	with a perfectly conducting flange and a resistive cap	$\frac{0.838}{1-i0.349\ \omega/\omega_o}$
4.	with perfectly conducting flange and cap having a slot of width a/2304 at a distance of (a) a/10 and (b) a/72 from the pipe's wall	(a) 0.303 (b) 0.335

It is not the purpose of these concluding remarks to discuss the various mathematical manipulations that were necessary to produce the data exhibited in this table. Rather a few thoughts of an intuitive nature will be offered which may help in interpreting the above results and in using them to get some rough ideas of the behavior of similar physical system. .

A comparison of items 1 and 2 shows that the absence of a perfectly conducting flange increases the coefficient by 34%. This increase may be ascribed to the fact that in the case of an unflanged pipe the incident field exists in the whole space, whereas in the case of a flanged pipe the incident field below the flange is almost annulled by the field of the flange currents alone. This means that the pressure of the external field in the former case pushes about twice as much energy as in the latter case into the vicinity of the end of the pipe. This is equivalent to roughly 40% more field lines inside the pipe in the former case. In dealing with problems of this kind it is often helpful to think of the analogous ones in potential flow. When the flange is resistive the coefficient should lie between those in items 1 and 2 since the latter represent the two extreme cases. It is expected that the effect of the absence of a flange on the coefficients in items 3 and 4 would also be 30% or 40%.

The presence of a resistive cap not only decreases the field strength inside the pipe but also introduces a phase shift, as can be seen by comparing items 1 and 3 in the chart. The decrease in field strength is clearly due to the fact that the slowly time-varying incident field induces in the cap conduction currents which cancel part of the magnetic field inside the pipe. The dependence of the coefficient on frequency, as given in item 3, is well known in low-frequency shielding problems. The physics of this behavior can be more easily understood in the time domain. To illustrate this point consider a step function incident pulse. According to Lenz's law, currents will initially be set up in the cap to render the field zero inside the pipe. These currents will gradually dissipate due to the finite resistance of the cap and as they decay the field will slowly approach the value that it would have if the cap were not present. This gradual growth of the interior field resembles the gradual transfer of the current of a current source from a resistance to a parallel inductance. This circuit is well known to have a frequency behavior similar to that given by the coefficient in item 3.

The results in item 4 are indeed startling when compared with the one in item 1. The comparison shows that a perfectly conducting cap with an annular slot, even though the slot is extremely narrow, seems to have not much effectiveness in shielding against the external magnetic field. This strange behavior becomes comprehensible when one thinks in terms of the current flow on the flange and cap. Since the annular slot isolates part of the cap the currents there must flow in closed loops. This means that by cutting the annular slot in the infinite plane on which the currents would be uniform in direction and magnitude in order to annul the external field below the plane, the current flow will be greatly disturbed, especially on the disc. In fact the currents at both edges of the slot are flowing in opposite directions and thus they enhance the normal component of magnetic field in the slot. For a narrow slot the integral of this component of the field is directly proportional to the coefficient of each mode inside the pipe. As the slot gets narrower and narrower the disturbed currents will be confined in a smaller and smaller area around either side of the slot and, consequently, only in the slot's immediate neighborhood will a large field appear. Eventually the field below the cap goes to zero with the slot's width.

ACKNOWLEDGMENT

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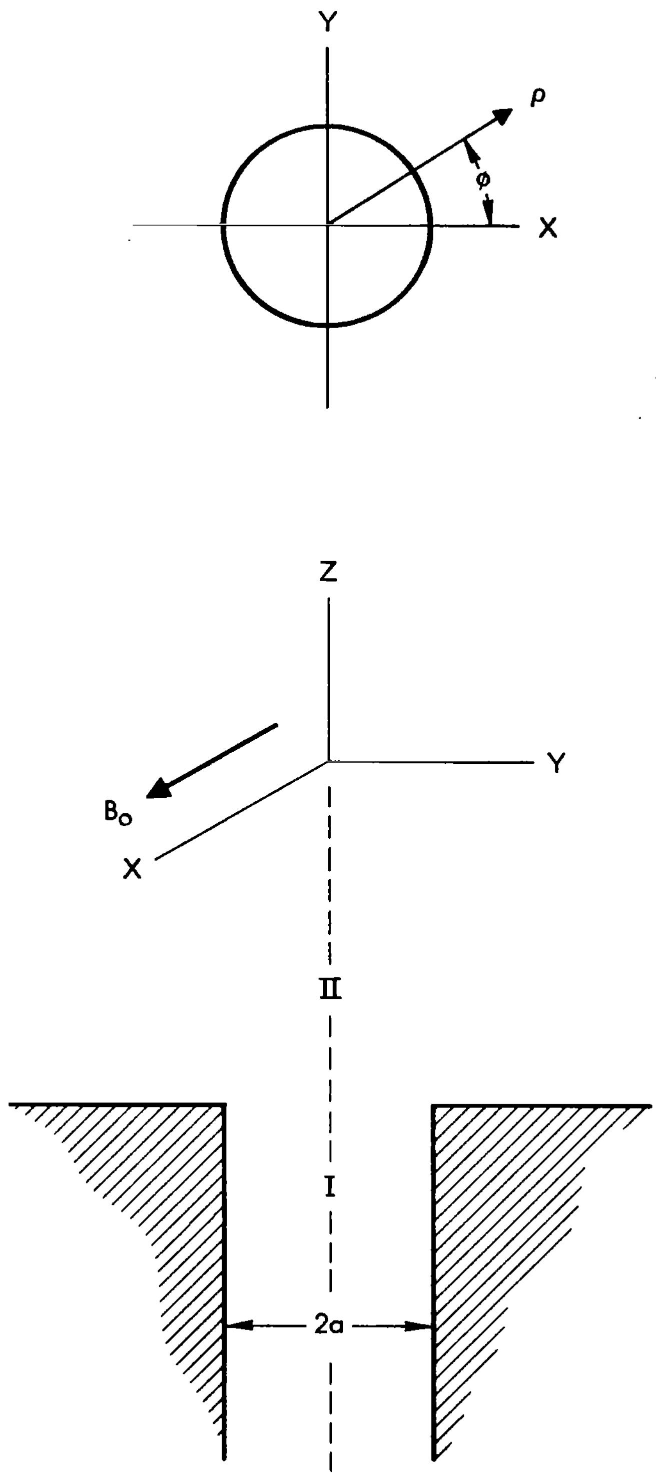

FIGURE 1.

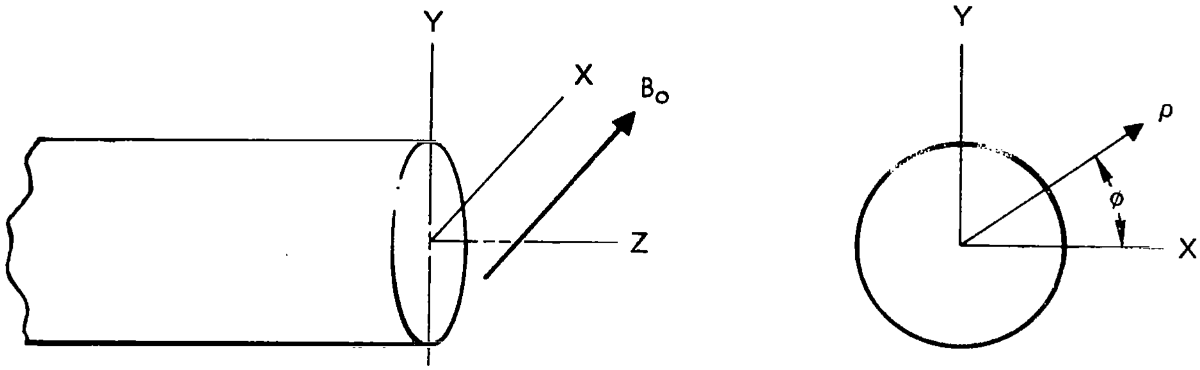

FIGURE 2.

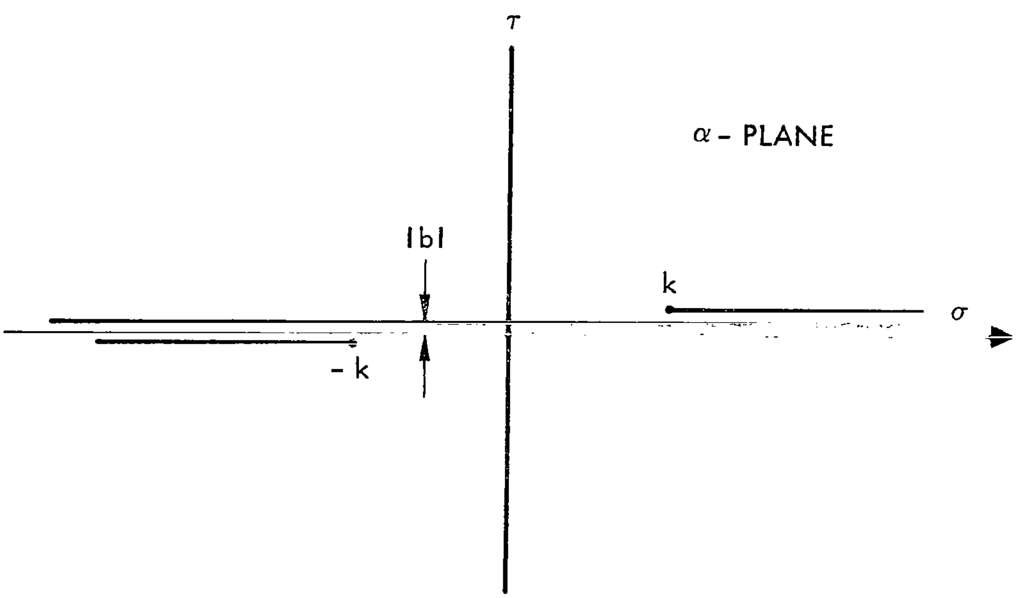

FIGURE 3.

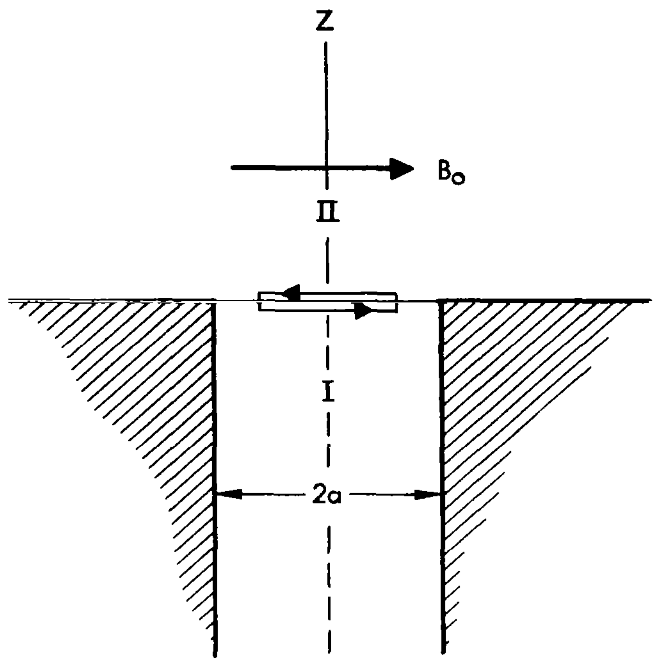

FIGURE 4.

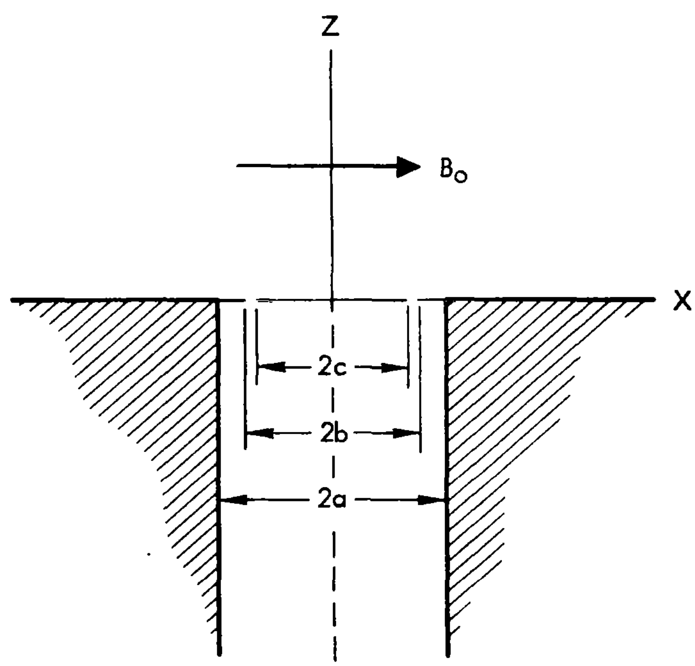

FIGURE 5.

APPENDIX

This appendix is devoted to the solution of equations (18) and (49). Particular attention will be directed to the evaluation of the integrals which occur in those equations.

First it is noted that because of the identity,

$$\int_0^\infty \frac{x^2[J_1'(x)]^2}{(x^2-\zeta_r^2)^2}\,dx = \frac{1}{4}\int_0^\infty \frac{x^2(J_0^2-2J_0J_2+J_2^2)}{(x^2-\zeta_r^2)^2}\,dx\ , \tag{A-1}$$

part of the problem can be reduced to the evaluation of several integrals of the form

$$I(m,n,k) = \int_0^\infty \frac{x^2\ J_m(x)\ J_n(x)}{(x^2+k^2)^2}\,dx\ , \tag{A-2}$$

where m,n are even integers and k is a complex number. Substituting[6]

$$J_m(x)\ J_n(x) = \frac{1}{2\pi i}\int_{-\frac{m+n}{2}-i\infty}^{-\frac{m+n}{2}+i\infty} \frac{\Gamma(-s)\Gamma(m+n+2s+1)(\frac{1}{2}x)^{m+n+2s}}{\Gamma(m+s+1)\Gamma(m+s+1)\Gamma(m+n+s+1)}\,ds$$

into (A-2) and interchanging the order of integration one obtains, after carrying out the integration with respect to x,

$$I(m,n,k) = \frac{1}{8i}\int_{-\frac{m+n}{2}-i\infty}^{-\frac{m+n}{2}+i\infty} \left(\frac{k}{2}\right)^{m+n+2s-1} \frac{(s+\frac{m+n+1}{2})\Gamma(m+n+2s+1)\Gamma(-s)}{\Gamma(m+s+1)\Gamma(n+s+1)\Gamma(m+n+s+1)\cos\pi(s+\frac{m+n}{2})}\,ds\ .$$

(A-3)

This integral can be evaluated by the method of residues by closing the contour in the right half s-plane. The singularities inside the contour come from the poles of $\Gamma(-s)$ and the zeros of $\cos \pi(s + \frac{m+n}{2})$. Thus, it is found that

$$I(m,n,k) = F(m,n,k)$$

$$- (-1)^{\frac{m+n}{2}} \frac{\pi}{4} \sum_{q=-\frac{m+n}{2}}^{\infty} \left(\frac{k}{2}\right)^{m+n+2q} \frac{(q+1 + \frac{m+n}{2})\Gamma(m+n+2q+2)}{\Gamma(q+\frac{3}{2})\Gamma(q+m+\frac{3}{2})\Gamma(q+n+\frac{3}{2})\Gamma(q+m+n+\frac{3}{2})} , \qquad \text{(A-4)}$$

where

$$F(m,n,k) = (-1)^{\frac{m+n}{2}} \frac{\pi}{4} \sum_{p=0}^{\infty} \left(\frac{k}{2}\right)^{m+n+2p-1} \frac{(p + \frac{m+n+1}{2})\Gamma(m+n+2p+1)}{\Gamma(p+1)\Gamma(p+m+1)\Gamma(p+n+1)\Gamma(p+m+n+1)} .$$

The integral (A-1) is now given by

$$\int_0^\infty \frac{x^2[J_1'(x)]^2}{(x^2-\zeta_r^2)^2} dx = \frac{1}{4} I(0,0,i\zeta_r) - \frac{1}{2} I(0,2,i\zeta_r) + \frac{1}{4} I(2,2,i\zeta_r) . \qquad \text{(A-5)}$$

It can be shown that, $F(0,0,i\zeta_r) - 2F(0,2,i\zeta_r) + F(2,2,i\zeta_r) = 0$ by making use of the fact that $J_1'(\zeta_r) = 0$. Hence, one obtains from (A-4) and (A-5)

$$\int_0^\infty \frac{x^2[J_1'(x)]^2}{(x^2-\zeta_r^2)^2} dx = \sum_{q=0}^{\infty} (-1)^{q+1} A_q\left(\frac{\zeta_r}{2}\right)^{2q} , \qquad \text{(A-6)}$$

where

$$A_q = \frac{\pi}{16}\,\frac{(q+1)\Gamma(2q+2)}{[\Gamma(q+\frac{3}{2})]^2}\left[1 + 2\left(\frac{2q+1}{2q+3}\right)^2 + \frac{(2q+1)(2q-1)}{(2q+5)(2q+3)}\right] ,$$

$A_o = 3.678249 \times 10^{-1}$

$A_1 = 1.362429$

$A_2 = 1.308725$

$A_3 = 5.543214 \times 10^{-1}$

$A_4 = 1.319129 \times 10^{-1}$

$A_5 = 2.012584 \times 10^{-2}$

$A_6 = 2.138086 \times 10^{-3}$

$A_7 = 1.673561 \times 10^{-4}$

$A_8 = 1.005684 \times 10^{-5}$

$A_9 = 4.787085 \times 10^{-7}$

$A_{10} = 1.849944 \times 10^{-8}$

$A_{11} = 5.920465 \times 10^{-10}$

$A_{12} = 1.595076 \times 10^{-11}$

$A_{13} = 3.642798 \times 10^{-13}$

$A_{14} = 7.283233 \times 10^{-15}$

$A_{15} = 1.261560 \times 10^{-15}$

$A_{16} = 1.833897 \times 10^{-18}$

For $\zeta_1 = 1.84118$ equation (A-6) gives

$$\int_0^\infty \frac{x^2[J_1'(x)]^2}{(x^2-\zeta_1^2)^2}\,dx = 0.115875 \; . \qquad \text{(A-7)}$$

For $\zeta_2 = 5.33144$ equation (A-6) gives

$$\int_0^\infty \frac{x^2[J_1'(x)]^2}{(x^2-\zeta_2^2)^2}\,dx = 0.084680 \; . \qquad \text{(A-8)}$$

Following the same procedure as above one finds

$$I(\zeta_r) = \int_0^\infty \frac{[J_1'(x)]^2}{x^2-\zeta_r^2}\,dx = \sum_{q=0}^{\infty} (-1)^{q+1}\,\frac{A_q}{q+1}\left(\frac{\zeta_r}{2}\right)^{2q} \qquad \text{(A-9)}$$

from which one can evaluate

$$\int_0^\infty \frac{x^2[J_1'(x)]^2}{(x^2-\zeta_r^2)(x^2-\zeta_s^2)}\,dx = \frac{1}{\zeta_s^2-\zeta_r^2}\,[\zeta_s^2\, I(\zeta_s) - \zeta_r^2\, I(\zeta_r)] \;. \qquad (A-10)$$

In particular, for $\zeta_1 = 1.84118$ and $\zeta_2 = 5.33144$,

$$\int_0^\infty \frac{x^2[J_1'(x)]^2}{(x^2-\zeta_1^2)(x^2-\zeta_2^2)}\,dx = -1.63200 \times 10^{-2} \;. \qquad (A-11)$$

After obtaining the values of the above integrals one can now go on to evaluate the matrix equation (18) of Section II which can be re-written as

$$\sum_{s=1}^{\infty} M_{rs}\, D_s = \frac{J_2(\zeta_r)}{\zeta_r J_1(\zeta_r)} \;, \qquad (A-12)$$

where $D_r = C_r\, J_1(\zeta_r)$. Keeping only D_1 and D_2 in (A-12) one has

$$\begin{aligned} 3.0733\, D_1 - 0.1632\, D_2 &= 2.9490 \\ -0.1632\, D_1 + 1.7516\, D_2 &= 0.3533 \;. \end{aligned} \qquad (A-13)$$

Solution of equation (A-13) gives

$$D_1 = 0.99363 \;, \qquad D_2 = 0.2911 \;,$$

whence

$$\begin{aligned} C_1 &= 1.6758 \;, \\ C_2 &= -0.8410 \;. \end{aligned} \qquad \left(\begin{array}{l}\text{coupling between} \\ \text{the 1st and 2nd mode}\end{array}\right) \qquad (A-14)$$

If the coupling between C_1 and C_2 were neglected, there would result

$$C_1 = 1.6491 \qquad \text{(no coupling)} . \tag{A-15}$$

Now matrix equation (49) of Section IV will be solved. Keeping only the first two elements one has

$$(3.0733 - i\ 1.04\ \omega/\omega_o)\ C_1 + 0.0971\ C_2 = 5.0680$$

$$0.2744\ C_1 + (1.7516 - i\ 0.017\ \omega/\omega_o)\ C_2 = -1.0207 . \tag{A-16}$$

where, as in Section IV, $\omega_o^{-1} = (\mu_o a)(\sigma d)$. Solution of equation (A-16) gives

$$C_1 = \frac{1.6769}{1-i0.3392\ \omega/\omega_o} - \frac{0.0011}{1-i0.0098\ \omega/\omega_o} \quad \left(\begin{array}{l}\text{coupling between}\\ \text{the 1st and 2nd mode}\end{array}\right) \tag{A-17}$$

If the coupling between C_1 and C_2 is neglected, there results

$$C_1 = \frac{1.649}{1 - i\ 0.338\ \omega/\omega_o} . \quad \text{(no coupling)} \tag{A-18}$$

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