EMP Interaction Notes

Note IV

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The Theoretical Basis of the MARS2 Cable Current Code

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Abstract

An expression to calculate the current induced in cables in the earth by CW surface electromagnetic (EM) fields is presented. A few simple approximations, pertaining to bare cables in good electrical contact with the earth, are performed on this expression to permit the use of Laplace Transforms in analyzing the current induced in these cables by pulsed EM fields. In particular, the response of a bare cable in the earth to a unit step function field (either magnetic or electric) is determined. The convolution integral of a general field pulse with this step function response is the basis of the MARS2 Cable Current Code.

where x is the distance along the cable, measured from the end nearest the field source (meters),

d is the length of the cable (meters),

$K_1 e^{-\Gamma x}$ and $K_2 e^{-\Gamma(d-x)}$ are the currents reflected from the cable ends $x = 0$ and $x = d$, respectively (amps),

Γ is the propagation constant for the current in the cable-earth transmission system (meters^{-1}),

Z_o is the characteristic impedance for the equivalent transmission line approximating the cable-earth transmission system (ohms),

v is a dummy variable which ranges from 0 to d (meters), and $E_o(v)$ is the component of the electric field along the wire axis that would exist at the spatial position of the wire's axis if the wire were absent.

The propagation constant is

$$\Gamma = (Z\ Y)^{1/2} \tag{2}$$

where Z is the longitudinal impedance per unit length of the equivalent transmission line (ohms/meter), and Y is the transverse admittance per unit length of the equivalent transmission line (mhos/meter).

Furthermore, the characteristic impedance of the system is

$$Z_o = (Z/Y)^{1/2} \tag{3}$$

For a cable in good electrical contact with the earth (either a bare cable or a cable with a high conductivity covering), the propagation constant Γ becomes large (see Figure 1). For $f > 10^3$, the current induced at a specific point on the wire will attenuate to a negligible value before it travels more than a few hundred meters. Examining equation (1), we see that, as Γ becomes large, practical electric fields will be of approximately constant magnitude over the range of v where $e^{-\Gamma|x-v|}$ has appreciable value. Therefore, the integral term of equation (1) is

$$\frac{1}{2Z_o}\int_o^d E_o(v)e^{-\Gamma|x-v|}d\,v \approx \frac{E_o(x)}{2Z_o}\int_o^d e^{-\Gamma|x-v|}d\,v$$

$$= \frac{E_o(x)}{2Z_o}\ \frac{1}{\Gamma}\ (2 - e^{-\Gamma x} - e^{-\Gamma(d-x)})$$

$$= \frac{E_o(x)}{Z}\ (1 - \frac{e^{-\Gamma x}}{2} - \frac{e^{\Gamma(d-x)}}{2}) \tag{4}$$

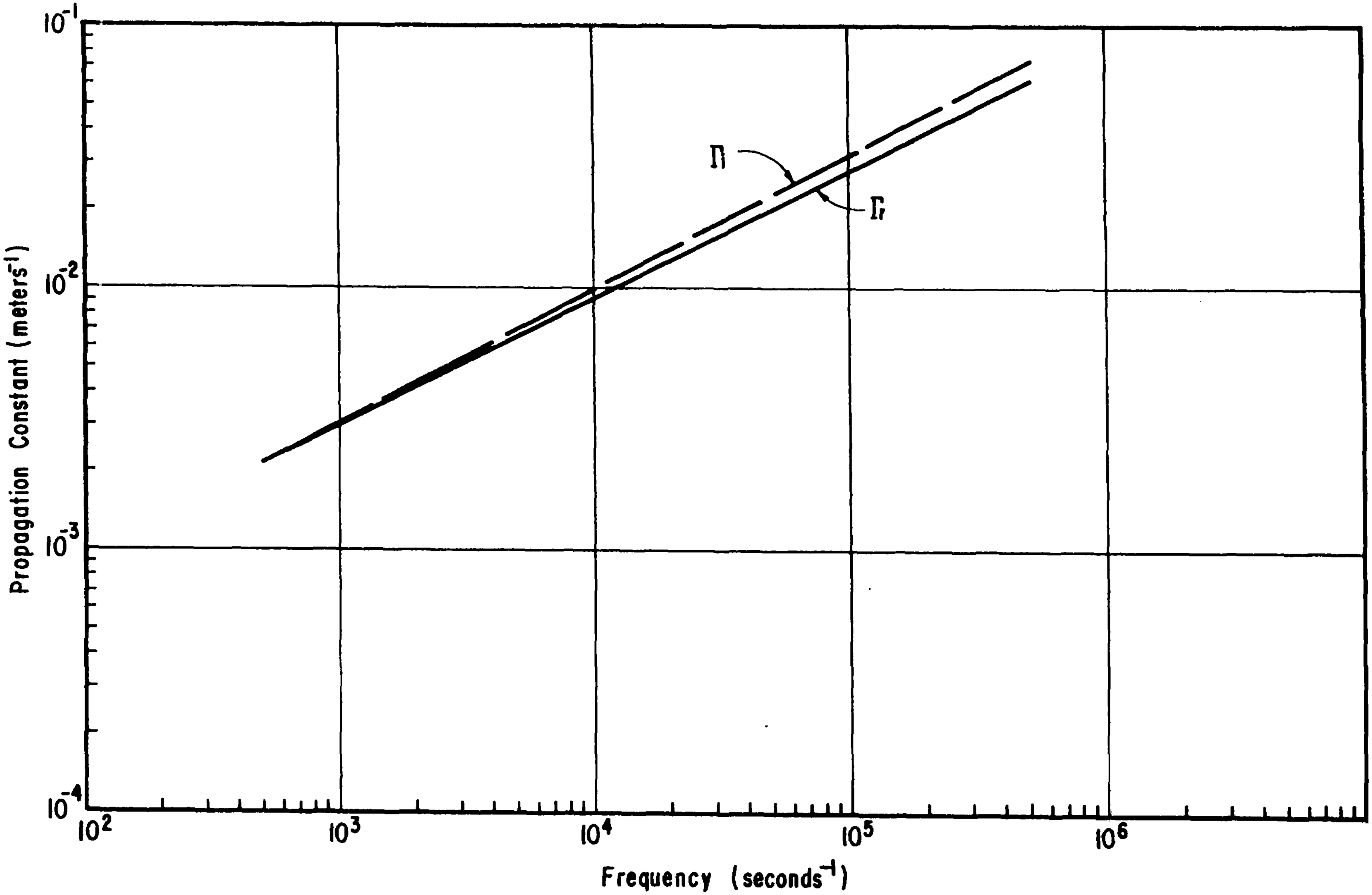

Figure 1. Current Propagation Constant vs Frequency

To further simplify equation (1) we must examine the reflection terms. Using the expressions for the reflection coefficients in AFWL-TR-65-94, we see that.[6]

$$K_1 = \frac{(Z_2 + Z_o)(Z_o - Z_1)\, e^{\Gamma d}\, F(0) + (Z_1 - Z_o)(Z_2 - Z_o)\, F(d)}{(Z_1 + Z_o)(Z_2 + Z_o)\, e^{\Gamma d} - (Z_1 - Z_o)(Z_2 - Z_o)\, e^{-\Gamma d}}$$

$$K_2 = \frac{(Z_2 - Z_o)(Z_o - Z_1)\, F(0) + (Z_1 + Z_o)(Z_2 - Z_o)\, e^{\Gamma d}\, F(d)}{(Z_1 - Z_o)(Z_2 - Z_o)\, e^{-\Gamma d} - (Z_1 + Z_o)(Z_2 + Z_o)\, e^{\Gamma d}}$$

where Z_1 and Z_2 are the respective termination impedances of the cable to the ground (ohms)

and F(x) is the integral term of equation (1). For practical fields and cables, F(x) and Z_o are finite for all x and f. From this fact it may be shown that K_1 and K_2 remain finite as Γd becomes large, independent of what Z_1 and Z_2 may be.

If we consider points on the cable that are more than 2 to 3 hundred meters from the ends of the cable, at frequencies $f > 10^3$, $e^{-\Gamma x}$ and $e^{-\Gamma(d-x)} \ll 1$. This means that $K_1\, e^{-\Gamma x}$ and $K_2\, e^{-\Gamma(d-x)}$ become negligible, and the integral term (equation (4)) becomes $\frac{E_o(x)}{Z}$. Therefore, along most of the length of the cable, the current is

$$I(x) \cong \frac{E_o(x)}{Z} \tag{5}$$

If the frequency range of interest is $f > 10^4$, equation (5) may be used to calculate currents as close as 100 meters from the cable ends (Figure 1). This frequency range is sufficient for many practical calculations of induced currents from lightning and EMP.

Although dependent on the distance variation of the field magnitude and the length of the wire, equation (5) will be valid through most of the frequency range of interest ($f > 10^3$) for many practical cable systems and electromagnetic (EM) field distributions. This expression may now be used to determine the current induced in a cable by a pulsed electromagnetic field.

Approximations on Frequency Dependent Factors.

Equation (5) may be rewritten as

$$I(x,\omega) = \frac{E_o(x,\omega)}{Z(\omega)} \tag{6}$$

6. Comparing this text with the given references, $K_1 = K$ and $K_2 = Le^{\Gamma d}$.

where $\omega = 2\pi f$ (radians per second) is the radian frequency of the EM wave. By using Laplace Transforms, analytic expressions for the current induced by a given pulsed field may be derived. First, however, we must determine the longitudinal impedance $Z(\omega)$.

The longitudinal impedance may be written as

$$Z(\omega) = R(\omega) + i\,\omega L(\omega)$$

where $R(\omega)$ is the effective longitudinal resistance per unit length of the cable-earth system (ohms/meter), and $L(\omega)$ is the effective inductance per unit length (henries/meter). CW calculations have shown that the inductive impedance outweighs the resistance ($\omega L >> R$) for the higher frequencies (see Figure 2);[7,8] that is, the frequencies at which the resistance is frequency dependent. Therefore, to a first approximation,

$$R(\omega) = R_{DC} = \text{a constant}$$

$$= \frac{1}{\sigma A}$$

where σ is the conductivity of the wire or cable sheath metal (mhos/meter) and A is the cross-sectional area of the wire or cable sheath (meter^2).

Also from CW calculations, we see that the inductance is relatively insensitive to frequency changes (varies as $\ln\omega$ - see Figure 2).[8,9] Therefore, to a first approximation, the constant value of the inductance will be taken as the value at the frequency which corresponds to the rise time of the induced current. This will necessitate a preliminary rough calculation of the pulse current to determine its rise time, but to use a pure guess would result in an error in the inductance of less than 20%.[10] The calculation of the inductance is discussed in AFWL-TR-65-94.

Using the above approximations and equation (6),

$$I(x,\omega) \cong \frac{E_o(x,\omega)}{R + i\omega L} \tag{7}$$

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7. AFWL-TR-65-94.
8. Figure 2 shows the calculated impedances and inductance for a copper cable sheath of length 1000 meters, of outside diameter 2 centimeters, and of sheath thickness 1 millimeter.
9. W. R. Graham, Electric Field-Induced Cable Currents, EMP Interaction Note I, AFWL.
10. For a more accurate determination of L, see SGC 726TM-6, Part II, Theoretical Studies Relating to the Space General Corp. Pulse Model, A. Stogryn, Space General Corp. (11 July 1966).

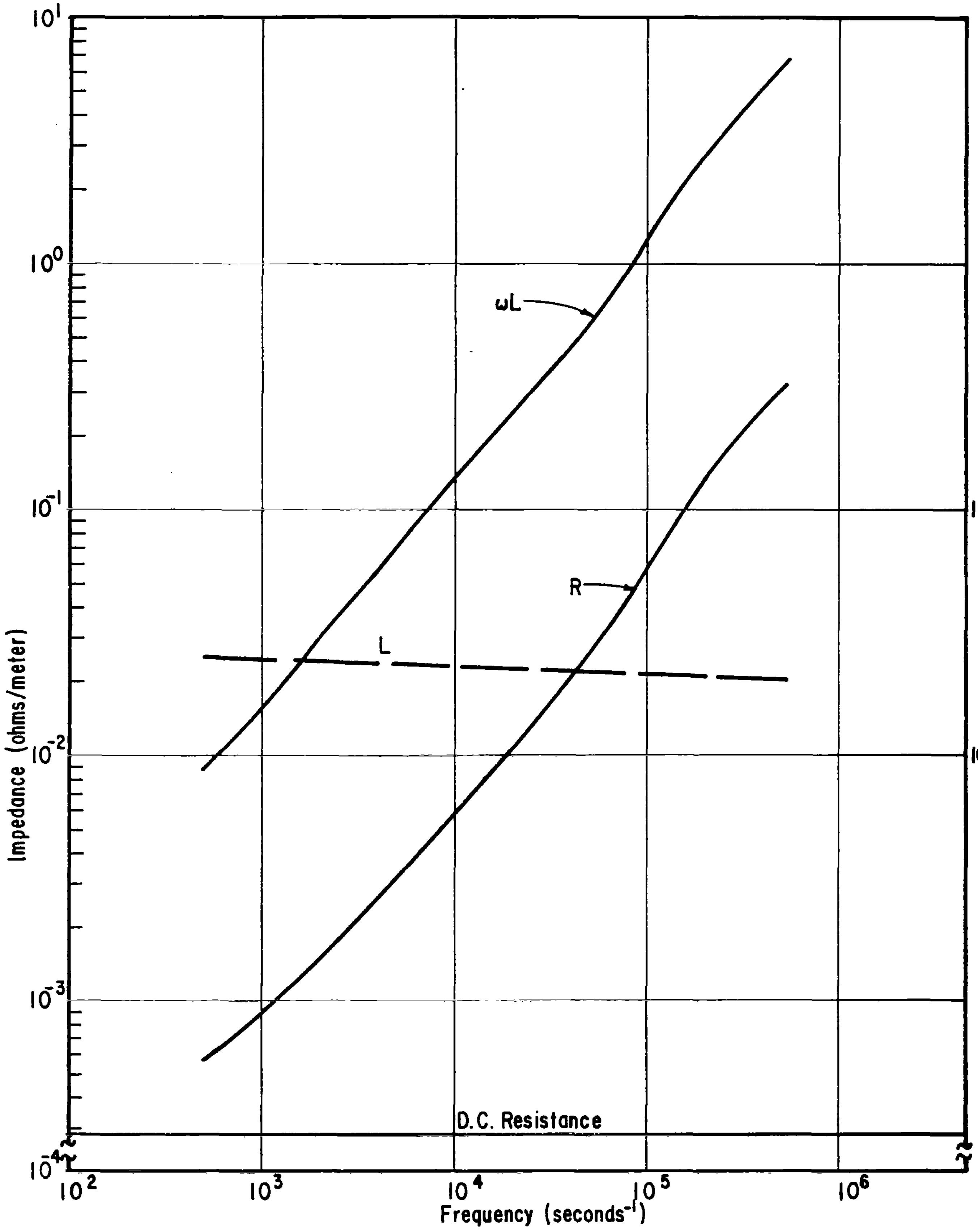

Figure 2. Longitudinal Impedance Per Unit Length and Inductance Per Unit Length vs Frequency for a Typical Bare Cable Sheath

where R is the DC resistance of the wire or cable sheath and L is the constant value of the inductance discussed above.

The driving electric field $E_o(x,\omega)$ for this problem is a component of the electric field tangential to the earth's surface. However, these tangential electric fields are often difficult to measure or determine because of the presence of a much larger vertical electric field component. Fortunately, there is a simple approximate relation between the tangential electric field and the tangential magnetic field. For points on the surface of the earth greater than a skin depth in the earth away from the field source,[11]

$$|E_{tangential}| \approx \eta_g \, |H_{tangential}|$$

where $\eta_g = \left(\dfrac{i\omega\mu}{\sigma+i\omega\varepsilon}\right)^{1/2}$

μ is the magnetic permeability of the soil (henries/meter)

σ is the conductivity of the soil (mhos/meter)

ε is the dielectric permittivity of the soil (farads/meter)

and $H_{tangential}$ is perpendicular to $E_{tangential}$.

The earth conductivity normally ranges from 10^{-4} to 3×10^{-2} mhos/meter and ε is normally around 10^{-10} to 5×10^{-9} farads/meter. Therefore, $\sigma >> \omega\varepsilon$, except for very limited ranges of ground conductivities and frequencies (primarily, for $\omega > 10^6$). For most problems of interest[12]

$$\eta_g \approx \left(\frac{i\omega\mu}{\sigma}\right)^{1/2}$$

For pulse current problems, the maximum deviation from this approximation will come at times less than 10^{-6} seconds. For currents of rise time slower than 10^{-6} seconds, the approximation should be good.

Step - Function Response.

Now that the frequency dependent impedance factors have been simplified, we can use Laplace Transforms to solve for the current response of a bare wire or cable sheath to a simple pulsed electromagnetic field. In particular, we want to find the response of the conductor to a unit step-function field, since this normalized response may be used to calculate the conductor response to an arbitrary (in time) field through the well-known associated convolution integral.

11. AFWL-TR-65-94.
12. Further borne out by detailed analysis in SGC726TM-6.

Assume a driving electric field at the surface of the earth of the form:

$$E_o(t) = u(t) = \begin{cases} 0, & t<0 \\ 1, & t>0 \end{cases}$$

The Laplace Transform of this field is[13]

$$\mathcal{L}[E_o\ (t)] = E_o(s) = \frac{1}{s}$$

where $s = i\omega$. From equation (7), we see that

$$I_o{}^E(s) = \frac{E_o(s)}{R+sL} = \frac{1}{s(R+sL)}$$

Therefore, for a unit step function electric field as the driving field,

$$I_o{}^E(t) = \mathcal{L}^{-1}\ [I_o{}^E(s)] = \frac{1}{R} \left[1 - e^{-\frac{R}{L}t}\right] u(t) \qquad (9)$$

where $I_o{}^E(t)$ is the response to a unit step electric driving field. For early times, when $\frac{R}{L}t \ll 1$,

$$I_o{}^E(t) \simeq \frac{1}{R}\ [1 - 1 + \frac{R}{L}t]\ u(t)$$

$$= \frac{t}{L}\ u(t) \qquad (10)$$

For practical cables, this approximation corresponds to times less than 50 microseconds.

If the field of interest is a pulsed tangential magnetic field, then it is necessary to determine the cable response to a unit step magnetic field. That is

$$H(t) = u(t)$$

$$H(s) = \frac{1}{s}$$

From equations (7) and (8)

13. Handbook of Mathematical Functions, National Bureau of Standards, U.S. Dept of Commerce (June 1964).

$$I_o^H(s) = \left(\frac{s\mu}{\sigma}\right)^{1/2} \frac{1}{s(R+sL)}$$

Therefore, for a unit step function magnetic field as a driving field,

$$I_o^H(t) = \mathcal{L}^{-1}\, I_o^H(s) = 2\left(\frac{\mu}{\pi\sigma\ RL}\right)^{1/2} u(t)\ D\left(\sqrt{\frac{R}{L}\, t}\right) \qquad (11)$$

wnere $D(x) = e^{-x^2} \int_0^x e^{\lambda^2} d\lambda$, known as Dawson's Integral, may be found in tables,[13]

and $I_o^H(t)$ is the response to a unit step magnetic driving field. For early times, as previously defined,

$$I_o^H(t) = 2\left(\frac{\mu}{\pi\sigma\ RL}\right)^{1/2} \left(\frac{R}{L}\, t\right)^{1/2} u(t)$$

$$= \frac{2}{L}\left(\frac{\mu}{\pi\sigma}\right)^{1/2} \sqrt{t}\ \ u(t) \qquad (12)$$

Now that we have derived the response of a cable's current for a unit step electric or magnetic field, we can use the convolution integral theorum to determine the current induced by an arbitrary pulsed electromagnetic field F(t).

$$I(t) = \int_0^t I_o(t-\tau)\, F'(\tau)\, d\tau \qquad (13)$$

where $I_o\ (t - \tau)$ is the unit step response for the field of interest

$$F'(\tau) = \left.\frac{dF(t)}{dt}\right]_{t=\tau}$$

and τ is a dummy variable ranging from 0 to t. For a few special field waveforms, this integral may be analytic. In general, it is not. The following section will explain the formation of the MARS2 computer code from this integral equation.

The MARS2 Code.

Assuming a practical electromagnetic field, with no gross discontinuities or singularities,

$$I(t) = \lim_{\Delta\tau \to 0} \sum_{n=1}^{k} I_o(t - [n - \tfrac{1}{2}]\Delta\tau)\ \frac{F(n\Delta\tau)-F([n-1]\Delta\tau)}{\Delta\tau}\ \Delta\tau$$

where $k = \frac{t}{\Delta\tau}$. Using a finite $\Delta\tau$ that is sufficiently small so that the F(t) of interest varies approximately linearly over the range of each $\Delta\tau$ from 0 to t, we may say that

$$I(t) \cong \sum_{n=1}^{k} I_o(t-[n-\tfrac{1}{2}]\Delta\tau) \left\{ F(n\Delta\tau) - F([n-1]\Delta\tau) \right\}$$

This is the relation used in the MARS2 program, with equations (9), and (11) used for $I_o(t)$.[14] $F(n\Delta\tau)$ is read in as data at each point $n\Delta\tau$.

If $F(t)$ is the tangential electric field, $I_o(t)$ (from equation (9)) may be readily calculated by the computer. However, if $F(t)$ is the tangential magnetic field, it is necessary to calculate Dawson's Integral (equation (11)). It is possible, of course, to read in this quantity point by point from a set of tables. It is much easier to derive a simple expression which will approximate Dawson's Integral within a percent or so. The following approximations were used in MARS2.

$$D(x) = e^{-x^2} \int_o^x e^{\lambda^2} d\lambda \approx x, \ x < 0.1 \qquad \text{(error 0.7\%)}$$

$$\approx \frac{x}{1 + 0.668\, x^2 + 0.192x^4}, \ 0.1 \le x \le 1.9 \qquad \text{(error 1\%)}$$

$$\approx \frac{0.5}{x} + \frac{0.254}{x^3} + \frac{0.5}{x^5} + \frac{0.5}{x^7}, \ x > 1.9 \qquad \text{(error 0.8\%)}$$

The final thing that must be considered in writing the computer program is the size of the differential $\Delta\tau$. $\Delta\tau$ must be small enough so that $F(t)$ is approximately linear within the interval. Yet, it should be as large as possible to minimize the number of calculations performed by the computer. Therefore, a simple system of regridding $\Delta\tau$, based on electromagnetic fields of interest to the Air Force Weapons Laboratory, has been used. Two times are read into the program: one marks the change of $\Delta\tau$ from 10^{-8} seconds to 10^{-7} seconds, and the other marks the change from 10^{-7} seconds to 10^{-6} seconds.

There is a similar regridding scheme for Δt, defined by:

$$t_m = t_{m-1} + \Delta t$$

This mechanism determines at what times t_m the current will be calculated. Again, there are two times read in, corresponding to the changes $\Delta t = 10^{-8}$ to 10^{-7} seconds and $\Delta t = 10^{-7}$ to 10^{-6} seconds.

The MARS2 program, written in PORTRAN IV for a CDC 6600 Computer, uses the above approximations to calculate the current induced in bare wires and cable sheaths in the earth by an arbitrary pulsed electromagnetic field.

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14. The early time approximations, equations (10) and (12), were used in the original MARS program. Using this program is equivalent to neglecting the resistance of the wire.

III. Conclusions.

This note has discussed the MARS2 computer code, a computer program designed to calculate the current induced in a noninsulated wire or cable sheath in the earth by an impressed electromagnetic field. General limitations are inherent in this program. It is limited to cables or wires that are in good electrical contact with the earth throughout their length. It is limited to points on the cable more than 100 meters from the cable terminations, since it does not account for termination reflections. It is therefore limited to cables greater than 200 to 300 meters in length. The cable must be greater than a skin depth in the earth (for the lowest frequency of interest) away from the source of the external field if the magnetic field is used as the basis for calculating the induced currents.

At present, there is no dependable time-history data on pulsed cable currents induced by external fields with which we may compare the MARS2 calculations. However, the MARS2 calculations compare within a factor of 2 with peak current measurements made on a bare #10 copper wire and bare lead sheath cable during the Small Boy nuclear test, except at distances less than 200 meters from ground zero. Also, the MARS2 calculated waveshapes compared well with waveshapes calculated by Stanford Research Institute using a Fourier Integral computer code, a code which uses much more computer time than the MARS2 code.[15] For cables and wires in the earth which meet the above limitations, the MARS2 code should give an excellent first approximation to the current induced by a pulsed electromagnetic field at the surface of the earth.

I would like to thank Dr. William R. Graham for suggesting the use of Laplace Transforms on this problem.

15. DASIAC Special Report 41, "Electromagnetic Pulse Phenomenology and Effects," DASA Information and Analysis Center, Santa Barbara (April 1966 - SECRET RD).