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A Transmission-Line Transformer for Matching the Switched Oscillator to a Higher-Impedance Resistive Load

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Abstract

In connecting a switched oscillator to a high-impedance load one may wish to increase the voltage (peak) delivered to the load. One way to accomplish this is to insert a quarter-wavelength transformer (at the fundamental oscillator frequency). This transformer consists of a section of transmission line with characteristic impedance intermediate between that of the switched oscillator and the load impedance.

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1. Introduction

In [3] we consider a switched oscillator as a possible source for driving an antenna [1]. Modelling the antenna as a resistive load (appropriate for certain kinds of antennas), one may ask how well the switched oscillator is matched into this load for optimizing various performance parameters.

For a TEM antenna feed consisting of a conical transmission line, the region near the apex (connecting to the switched oscillator) may have very large electric fields. One may wish to have this region in some high-dielectric-strength medium such as oil. This in turn will affect the characteristic impedance of this portion of the conical transmission line, potentially introducing reflections back toward the source. Noting the resonant character of the source, such reflections can be beneficial if the length of this section and its characteristic impedance are chosen appropriately. Let us think of this section as a transmission-line transformer and make a corresponding mathematical model.

Consider an equivalent transmission-line network in Fig. 1.1 to represent this combination of switched oscillator, transformer, and load (e.g., antenna). This is characterized by:

1. switch

$$\begin{aligned} V_s(t) &\equiv \text{waveform from swich} \\ &= V_1(t) = \text{ for example } V_0 u(t) \end{aligned} \tag{1.0}$$

2. oscillator

$$t_1 \equiv \text{transit time}$$

$$Z_c^{(1)} \equiv \text{characteristic impedance (real, frequency independent)} \tag{1.2}$$

3. transformer

$$t_2 \equiv \text{transit time}$$

$$Z_c^{(2)} \equiv \text{characteristic impedance (real, frequency independent)} \tag{1.3}$$

4. load

$$Z_3 \equiv \text{load impedance (assumed real, frequency independent)} \tag{1.4}$$

For convenience we also introduce normalized parameters

$$
\begin{aligned}
\zeta_1 &= \frac{Z_c^{(1)}}{Z_c^{(2)}} \\
\zeta_2 &= \frac{Z_c^{(2)}}{Z_3} \\
\zeta_3 &= \frac{Z_c^{(1)}}{Z_3} = \zeta_1 \zeta_2
\end{aligned}
\tag{1.5}
$$

We can consider that ζ_3 is given a priori by the specified oscillator and load impedances, leaving only one of ζ_1 and ζ_2 to be varied.

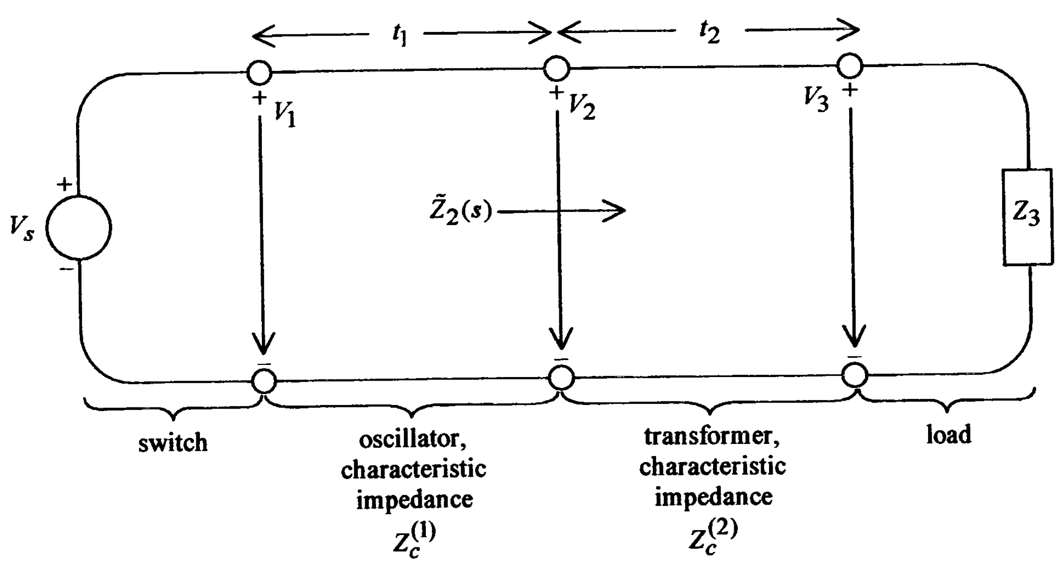

Fig 1.1. Equivalent Transmission-Line Network

2. Initial Transient Step Up

We can gain some insight by looking at the initial transient performance of the transformer. We have transmission coefficients

$$T_2 = \frac{2\,Z_c^{(2)}}{Z_c^{(1)} + Z_c^{(2)}} = 2\left[1+\zeta_1\right]^{-1}$$

$\equiv$ step up into transformer

$$T_3 = \frac{2\,Z_3}{Z_c^{(2)} + Z_3} = 2\left[1+\zeta_2\right]^{-1} \tag{2.1}$$

$\equiv$ step up into load

This gives a net step up of

$$T = T_2\,T_3 = 4\left[1+\zeta_1\right]^{-1}\left[1+\zeta_2\right]^{-1} \tag{2.2}$$

with the transformer section included. Without the transformer section ($Z_c^{(2)} = Z_3$) we have

$$T_0 = \frac{2\,Z_3}{Z_c^{(1)} + Z_3} = 2\left[1+\zeta_3\right]^{-1} \tag{2.3}$$

which is the case in [3].

As $\zeta_3 \to 0$ without the transformer section we have $T_0 \to 2$ as a maximum voltage gain for the initial transient. With the transformer this gain can be increased. If both ζ_1 and ζ_2, then $T \to 4$. With ζ_3 fixed by the oscillator characteristic impedance and load impedance we can vary ζ_1 to maximize the transient gain

$$T = 4\left[1+\zeta_1\right]^{-1}\left[1+\frac{\zeta_3}{\zeta_1}\right]^{-1} = \frac{4}{1+\zeta_3+\zeta_1+\zeta_3\zeta_1^{-1}} \tag{2.4}$$

by noting that the minimum of $\zeta_1 + \zeta_3\,\zeta_1^{-1}$ occurs at

$$0 = \frac{d}{d\zeta_1}\left[\zeta_1 + \zeta_3\zeta_1^{-1}\right] = 1 - \zeta_3\zeta_1^{-2}$$
$$\zeta_1 = \zeta_3^{\frac{1}{2}} = \zeta_2 \tag{2.5}$$

for which case the transient gain is

$$T = 4\left[1+\zeta_3^{\frac{1}{2}}\right]^{-2} = 4\left[1+\left[\frac{Z_c^{(1)}}{Z_3}\right]^{\frac{1}{2}}\right]^{-2} \tag{2.6}$$

We have some selected values in Table 2.1. Note that the transformer characteristic impedance is constrained as the geometric mean

$$Z_c^{(2)} = \left[Z_c^{(1)}Z_3\right]^{\frac{1}{2}} \tag{2.7}$$

for optimum results.

$\frac{Z_3}{Z_1} = \zeta_3^{-1}$	T
∞	4.0
100	3.31
50	3.07
25	2.78
10	2.31
5	1.91

Table 2.1. Selected Values of Transient Gain

The transmission-line model has limitations. In particular, as a conical transmission line extends from the apex, the cross section dimensions (on a sphere centered at the apex) increase. When such dimension become appreciable in wavelength units (in the local dielectric media) there can be non-TEM modes generated at the transformer output, depending on the details of the cross-section geometry.

3. Transmission-Line Solution in Frequency Domain

The voltage transfer through the transformer is given by

$$\frac{\tilde{V}_3(s)}{\tilde{V}_2(s)} e^{st_2} = \frac{1+\xi_3}{1+\xi_3 e^{-2st_2}} \tag{3.1}$$

$\sim$ $\equiv$ Laplace transform (two sided) over time t

$s = \Omega + j\omega$ $\equiv$ Laplace-transform variable or complex frequency

where the delay through the transformer has been moved to the left side for convenience. The reflection coefficient at the load is

$$\xi_3 = \frac{Z_3 - Z_c^{(2)}}{Z_3 + Z_c^{(2)}} = \frac{1-\zeta_2}{1+\zeta_2} \tag{3.2}$$

The result in (3.1) has the same form as that for the switched oscillator without transformer in [1]. The impedance $\tilde{Z}_2(s)$ looking into the transformer (as indicated in Fig. 1.1) is

$$\tilde{Z}_2(s) = Z_c^{(2)} \frac{1+\xi_3 e^{-2st_2}}{1-\xi_3 e^{-2st_2}} \tag{3.3}$$

Now consider driving into the switched oscillator from the switch giving (similar to (3.1) and (3.2))

$$\begin{aligned}
\frac{\tilde{V}_2(s)}{\tilde{V}_1(s)} e^{st_1} &= \frac{1+\tilde{\xi}_2(s)}{1+\tilde{\xi}_2(s) e^{-2st_1}} \\
\tilde{\xi}_2(s) &= \frac{\dfrac{\tilde{Z}_2(s)}{Z_c^{(1)}} - 1}{\dfrac{\tilde{Z}_2(s)}{Z_c^{(1)}} + 1} = \frac{\dfrac{1+\xi_3 e^{-2st_2}}{1-\xi_3 e^{-2st_2}} - \zeta_1}{\dfrac{1+\xi_3 e^{-2st_2}}{1-\xi_3 e^{-2st_2}} + \zeta_1} \\
&= \frac{[1-\zeta_1] + [1+\zeta_1]\xi_3 e^{-2st_2}}{[1+\zeta_1] + [1-\zeta_1]\xi_3 e^{-2st_2}}
\end{aligned} \tag{3.4}$$

These combine to give

$$\frac{\tilde{V}_2(s)}{\tilde{V}_1(s)}\, e^{st_1} = 2\, \frac{1+\xi_3\, e^{-2st_2}}{[1+\zeta_1] + [1-\zeta_1]\xi_3\, e^{-2st_2} + \Big[[1-\zeta_1] + [1+\zeta_1]\xi_3\, e^{-2st_2}\Big] e^{-2st_1}} \tag{3.5}$$

The total transfer function from switch to load (with delay removed) is then

$$\frac{\tilde{V}_3(s)}{\tilde{V}_1(s)}\, e^{s[t_1+t_2]} = 2\, \frac{1+\xi_3}{[1+\zeta_1] + [1-\zeta_1]\xi_3\, e^{-2st_2} + \Big[[1-\zeta_1] + [1+\zeta_1]\xi_3\, e^{-2st_2}\Big] e^{-2st_1}} \tag{3.6}$$

Including a choice for the source voltage we have the voltage delivered to the load. For an ideal switch we have

$$\begin{gathered}\tilde{V}_1(s) = \frac{V_0}{s} \\ \frac{\tilde{V}_3(s)}{V_0}\, e^{s[t_1+t_2]} = \frac{2}{s}\, \frac{1+\xi_3}{[1+\zeta_1] + [1-\zeta_1]\xi_3\, e^{-2st_2} + \Big[[1-\zeta_1] + [1+\zeta_1]\xi_3\, e^{-2st_2}\Big] e^{-2st_1}}\end{gathered} \tag{3.7}$$

4. Quarter-Wave Choice for Transformer

Equation (3.7) is a very general result in which one can insert choices for t_1, t_2, ζ_1, and ζ_2. This gives the frequency spectrum at the load, and by numerical Fourier inversion gives the time-domain waveform at the load. Let us now simplify the result for greater analytic understanding.

An interesting choice for t_2 is t_1 so that at the basic oscillator frequency (unloaded) the transformer is a quarter wavelength long, maximizing the transformer gain at this frequency. For convenience then define

$$\tilde{X}(s) \equiv e^{-2st_1} = e^{-2st_2} \tag{4.1}$$

so that (3.6) becomes

$$\frac{\tilde{V}_3(s)}{\tilde{V}_1(s)}\tilde{X}^{-1}(s) = 2\frac{1+\xi_3}{[1+\zeta_1] + [1-\zeta_1]\xi_3\tilde{X}(s) + \left[[1-\zeta_1] + [1+\zeta_1]\xi_3\tilde{X}(s)\right]\tilde{X}(s)} \tag{4.2}$$

The denominator is a quadratic in $\tilde{X}(s)$ yielding two solutions for the denominator zeros, say X_1 and X_2, giving the factored form

$$\begin{aligned}\frac{\tilde{V}_3(s)}{\tilde{V}_1(s)}\tilde{X}^{-1}(s) &= 2\frac{1+\xi_3}{1+\zeta_1}\left[1-\frac{\tilde{X}(s)}{X_1}\right]^{-1}\left[1-\frac{\tilde{X}(s)}{X_2}\right]^{-1} \\ &= 2\frac{1+\xi_3}{1+\zeta_1}X_1X_2\left[X_1 - \tilde{X}(s)\right]^{-1}\left[X_2 - \tilde{X}(s)\right]^{-1}\end{aligned} \tag{4.3}$$

This can be inverse transformed into time domain, yielding a sequence of delta functions, using the technique discussed in Appendix A. Convolution with $V_1(t)$ is then merely a sequence of time-shifted $V_1(t)$ pulses. For the special case of step excitation as in (3.7) the time-domain voltage waveform on the load is described by a series of step functions.

As a special case suppose that $\zeta_2 = 1$ implying $\xi_3 = 0$. Then (4.2) reduces to

$$\begin{aligned}\frac{\tilde{V}_3(s)}{\tilde{V}_1(s)}\tilde{X}^{-1}(s) &= 2\left[[1+\zeta_1] + [1-\zeta_1]\tilde{X}(s)\right]^{-1} \\ \zeta_1 = \zeta_3 &= \frac{Z_c^{(1)}}{Z_3}\end{aligned} \tag{4.4}$$

This is the previous result in [1].

Another special case concerns the optimum transformer for the initial transient discussed in Section 2, for which we have

$$\zeta_1 = \zeta_2 = \zeta_3^{1/2} = \frac{Z_c^{(1)}}{Z_c^{(2)}} = \frac{Z_c^{(2)}}{Z_3} = \left[\frac{Z_c^{(1)}}{Z_3}\right]^{1/2}$$
$$\zeta_3 = \frac{1-\zeta_2}{1+\zeta_2} = \frac{1-\zeta_1}{1+\zeta_1} \tag{4.5}$$

This gives

$$\begin{aligned}\frac{\tilde{V}_3(s)}{\tilde{V}_1(s)}\tilde{X}^{-1}(s) &= 4\left[[1+\zeta_1]^2+[1-\zeta_1]^2\tilde{X}(s)+[1+\zeta_1][1-\zeta_1][1+\tilde{X}(s)]\tilde{X}(s)\right]^{-1}\\ &= 4[1+\zeta_1]^{-2}\left[1+2\frac{1-\zeta_1}{[1+\zeta_1]^2}\tilde{X}(s)+\frac{1-\zeta_1}{1+\zeta_1}\tilde{X}^2(s)\right]^{-1}\\ &= 4[1+\zeta_1]^{-2}\left[1-\frac{\tilde{X}(s)}{X_1}\right]^{-1}\left[1-\frac{\tilde{X}(s)}{X_2}\right]^{-1}\\ X_{\substack{1\\2}} &= -\frac{1}{1+\zeta_1}\pm j\left[\frac{1+\zeta_1}{1-\zeta_1}-[1+\zeta_1]^{-2}\right]^{1/2}\end{aligned} \tag{4.6}$$

Again, this can be put in time domain by the technique in Appendix A. There are limiting cases including

$$\zeta_1 = 1 \text{ (all impedances the same)} \tag{4.7}$$

giving a single delta function in time, and

$$\begin{aligned}\zeta_2 &= 0 \text{ (very large impedance ratios)}\\ X_{\substack{1\\2}} &= -1\\ \frac{\tilde{V}_3(s)}{\tilde{V}_1(s)}\tilde{X}^{-1}(s) &= 4\left[1+\tilde{X}(s)\right]^{-2}\end{aligned} \tag{4.8}$$

In this latter case we have a set of second order poles at

$$\tilde{X}(s) = e^{-2st_1} = -1$$
$$st_1 = \left[n + \frac{1}{2}\right]\pi \quad , \quad n = \text{all integers} \tag{4.9}$$

Note that $\zeta_2 = 0$ is impractical and so a small ζ_2 can be used to give some damping to the second order poles and keep the energy bounded.

5. Concluding Remarks

The inclusion of a transformer section after the switched oscillator gives us additional design flexibility for improving performance. By making it a quarter wavelength long at the basic oscillator frequency there is a voltage step up to a high-impedance load. This transformer section can also serve as a region with a high-dielectric-strength medium to withstand the high electric fields near the exit from the switched oscillator. While present considerations have, for simplicity, considered a section of uniform transmission line, a nonuniform section might also prove useful [2].

Appendix A. Time-Domain Representation

One can consult tables or readily find from geometric-series considerations the Laplace-transform pair

$$\begin{aligned}
\left[1-a\,e^{-st_a}\right]^{-1} &= \sum_{n=0}^{\infty} a^n e^{-n\,st_a} \\
&\leftrightarrow \sum_{n=0}^{\infty} a^n \delta(t-nt_a) \\
t_a &\geq 0
\end{aligned} \tag{A.1}$$

For this to apply to a passive system we also require $|a| < 1$ (with a real for real coefficients if the above is the only term).

A more general form considers two such factors with the Laplace-transform pair

$$\begin{aligned}
&\left[1-a\,e^{-st_a}\right]^{-1}\left[1-b\,e^{-st_b}\right]^{-1} \leftrightarrow \\
&\quad \left[\sum_{n=0}^{\infty} a^n \delta(t-nt_a)\right] \circ \left[\sum_{n=0}^{\infty} b^n \delta(t-nt_b)\right] \\
&\quad = \delta(t) + \left[a\delta(t-t_a) + b\delta(t-t_b)\right] \\
&\quad + \left[a^2\delta(t-2t_a) + ab\delta(t-t_a-t_b) + b^2\delta(t-2t_b)\right] \\
&\quad + \cdots
\end{aligned} \tag{A.2}$$

$\circ \equiv$ convolution with respect to time t

$t_a, t_b \geq 0 \quad , \quad a, b$ real for $t_a \neq t_b$

If $t_a = t_b = t_0 \; (=2\,t_1)$ we have the case in Section 4. The Laplace-transform pair is now

$$\begin{aligned}
\left[1-ae^{-st_0}\right]^{-1}\left[1-be^{-st_0}\right]^{-1} &= \left[1-[a+b]e^{-st_0}+abe^{-2st_0}\right]^{-1} \\
&\leftrightarrow \sum_{n=0}^{\infty} c_n \delta(t-nt_0) \\
c_n &= \sum_{m=0}^{\infty} a^m b^{n-m}
\end{aligned}$$

$$b = a^* \text{ (for real coefficients)}$$
$$|a| = |b| < 1 \text{ (for passive system)} \tag{A.3}$$
$$t_0 \geq 0 \text{ (causal)}$$

If, in addition, $a = b$ we have

$$\left[1 - ae^{-st_0}\right]^{-2} \leftrightarrow \sum_{n=0}^{\infty} [n+1]\, a^n \delta(t - nt_0) \tag{A.4}$$
$$a = real, \quad |a| < 1$$

This is like a second order pole, except in the variable e^{-st_0}. This gives an infinite set of second order poles in the left half of the s plane.

The formulae in this appendix can be extended to include an additional factor of s^{-1}, corresponding to step excitation. The time-domain formulae (inverse Laplace transforms) are simply modified by replacing delta functions by step functions.

References

1. C. E. Baum, "Antennas for the Switched-Oscillator Source," Sensor and Simulation Note 455, March 2001.

2. C. E. Baum and J. M. Lehr, "Nonuniform-Transmission-Line Transformers for Fast High-Voltage Transients," Circuit and Electromagnetic System Design Note 44, February 2000.

3. C. E. Baum, "Switched Oscillators," Circuit and Electromagnetic System Design Note 45, September 2000.