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COUPLED TRANSMISSION-LINE MODEL OF MARX GENERATOR WITH PEAKING-CAPACITOR ARMS

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Abstract

The object of this note is to study the wave transport characteristics across a Marx-peaker system. Specifically the Marx generator is situated parallel to a ground plane with a finite number of peaking-capacitor arms surrounding it in an electromagnetically optimal manner [1]. This related past work [1] also defined and computed a parameter δ which is indicative of the effectiveness of the peaking-capacitor arms in shielding the Marx from the ground plane. Given this backdrop of information, a coupled transmission-line model is formulated and wave transport characteristics are investigated for such a Marx-peaker configuration.

CONTENTS

1. Introduction

Consider a two-parallel-plate transmission-line type of EMP simulator which is energized by a Marx generator. If the Marx column is parallel to a ground plane and has the associated N_p number of peaking capacitor arms surrounding it, an electromagnetically optimal way of distributing these peaker arms was addressed by the authors in [1]. The essense of optimal peaker arm distribution lies in the requirements that during both charge and discharge cycles, all the peaker arms be held at the same potential and that they carry equal currents. If the Marx column is modelled by an appropriate line current or charge source that makes the outer periphery of the column an equipotential, then the equipotential on which the peaker arms are to be distributed is easily chosen so as to match the characteristic impedance of the transmission-line simulator. Once such an equipotential is chosen, the peaker arms are distributed by requiring an equal increment of the stream function value between adjacent peaker arms. This was the basis of optimal peaker distributions in [1], and the approximations made in this procedure become increasingly valid as N_p increases. We have also defined and computed a parameter δ which is indicative of how well the Marx column is shielded by the N_p number of peaker arms from the groundplane.

Somewhat related to this subject is another possible way of configuring the pulser components viz., Marx column behind the ground plane at the same angle as the top plate, peaker arms in the ground plane and the output switch on the wave propagation side. It is also possible to compute an optimal way of distributing a given set of peaker arms in the ground plane, so that they carry approximately equal currents [2] and a parameter δ may also be defined and estimated to minimize the coupling of Marx between peakers in the ground plane into the wave launching section during the charge and discharge cycles. Of course, this configuration leads to a somewhat different transmission-line network model, especially if the Marx and peaker arms are of different lengths. Nevertheless, waveform computations, similar to what is performed in this note, are possible.

Returning to the present configuration of the Marx column above and parallel to the ground plane with the peaker arms surrounding it, observe that if N_p is large enough, then the peaker cage resembles a conducting tubular surface and can be modelled as such. We then have an effective conductor representing all of the peakers on an equipotential surface and the Marx conductor above a ground plane that is assumed to be perfectly conducting.

This leads one naturally to a coupled transmission-line model, formulated in this note. Such a coupled transmission-line comprising of two conductors and a third reference conductor, is then useful for studying the wave transport characteristics across the Marx-peaker system. It is noted that it is adequate to impress an ideal step function voltage source at the input end for computing the waveform across the load. The load is the impedance seen into the simulator which is dominated by the characteristic impedance of the TEM mode propagating in the transmission line. So, fortunately, the load into which the pulser delivers transient energy is a quantity that is fairly accurately known and it is mainly purely resistive. Typical values for this load vary in the range of 90Ω to 150Ω. The objective here is to evaluate the output voltage waveform for a step function input waveform. This study has provided insights into the wave transport properties, presence of a fast and slow wave in the system and is useful in optimizing the pulser design and generally to launch a desirable wave onto the simulator plates.

2. Two-Conductor Model of Marx with Peakers

Consider N_p number of peaker arms, each with an effective radius r_{PA}. These peaker arms are optimally distributed [1] around a Marx column of radius r_M, located at a normalized height of h_M. All linear dimensions in the cross section will be normalized with respect to the Marx height. An example case of $N_p = 8$ (figure 2.1) is reproduced here from [1], showing a central Marx column with 8 peakers surrounding it in an optimal way. Recall that all of the N_p peaker arms are located on an equipotential u_{P0} which may be corrected to a slightly different value u_P to account for the finite number of arms emulating a cylindrical surface. The potential (u) and stream function (v) are the result of a line current or charge source above a ground plane. This line source models the Marx conductor by rendering an "average" circular periphery of the physical Marx column enclosure into an equipotential surface. The $N_p = 8$ example of figure 2.1 also ensures equal peaker arm currents avoiding non-uniformities in the electromagnetic fields in the pulser region. We may designate the Marx voltage and Marx current as V_M and I_M. Since all of the peaker arms are at the same potential, this can be designated as V_P and let the total peaker current be I_P with the individual peaker arm currents given simply by $I_{PA} = I_P/N_P$. All of these voltages and currents are in the time domain and are functions of z. The spectral quantities (two-sided Laplace transformed) are designated by a tilde over the symbols.

Implicit in the above characterization of Marx and peaker arms is the realization that all of the N_p arms can be represented by an effective peaker conductor with a voltage V_P and carrying a current I_P. This leads one to a two-conductor (plus reference), coupled transmission-line model for the Marx-peaker assembly as shown in figure 2.2. It is noted that the Marx and effective peaker conductors are tied together at both the input and output ends. This line is excited by a step function driver $V_0u(t)$ at the input and terminated by a purely resistive load Z_L at the output end. Z_L represents the characteristic impedance of a single transmission line. A per-unit-length equivalent circuit of the coupled transmission line showing the impedance and admittance per unit length matrices can also be seen in figure 2.2. These matrices can be evaluated as follows.

Under the lossless assumption, the series impedance matrix $(\tilde{Z}'_{n,m}(s))$, including the L'_M term, and the shunt admittance matrix $(\tilde{Y}'_{n,m}(s))$ may be written down in terms of a geometric factor matrix $(f_{g_{n,m}})$ as

$$(\tilde{Z}'_{n,m}) = s\left[(L'_{n,m}) + \begin{pmatrix} 0 & 0 \\ 0 & L'_m \end{pmatrix}\right] = s\mu_o\left[(f_{g_{n,m}}) + \begin{pmatrix} 0 & 0 \\ 0 & X \end{pmatrix}\right] \tag{2.1}$$

$$(\tilde{Y}'_{n,m}) = s(C'_{n,m}) = s\varepsilon_0 (f_{g_{n,m}})^{-1} \tag{2.2}$$

with

$$(f_{g_{n,m}}) \equiv \begin{pmatrix} f_{g_{1,1}} & f_{g_{1,2}} \\ f_{g_{2,1}} & f_{g_{2,2}} \end{pmatrix} \quad , \quad X \equiv \frac{L'_M}{\mu_0} \tag{2.3}$$

Note that, all of the above matrices are of dimension (2×2) and that

$$\begin{aligned} f_{g_{1,1}} &= f_{g_p} \quad , \quad f_{g_{2,2}} = f_{g_M} \\ f_{g_{1,2}} &= f_{g_{2,1}} = f_{g_{PM}} = f_{g_{MP}} \end{aligned} \tag{2.4}$$

In addition, the parameter δ [1] also obeys the relationship

$$\delta = \left(1 - (f_{g_{PM}}/f_{g_p})\right) \text{ or } \quad f_{g_{PM}} = f_{g_{MP}} = f_{g_p}(1-\delta) \tag{2.5}$$

It is now recognized that we have four independent parameters $(f_{g_p}, f_{g_M}, \delta, X)$, in terms of which the elements of both per-unit-length matrices are computable. These independent parameters are known as follows [1]

$$f_{g_p} = (Z_p/Z_o)$$

$$f_{g_M} \simeq \frac{1}{2\pi} \ln\left[\left(\frac{h_M}{r_M}\right) + \sqrt{\left(\frac{h_M}{r_M}\right)^2 - 1}\right]$$

$$\delta = \left[1 - \frac{u_P}{u_{PO}}\right]$$

$$
\begin{aligned}
u_P &= \text{arccsch}\,(r_P);\; r_P = r_0 + \Delta r \\
r_0 &= \text{csch}\,(u_{P0});\; \Delta r = \left\{\frac{r_0}{N_P} \ln\left(\frac{r_0}{N_P r_{PA}}\right)\right\} \\
u_{P0} &= 2\pi f_{g_p} \\
\chi &= L'_M/\mu_o
\end{aligned}
\tag{2.6}
$$

Z_P = peaker cage impedance to ground

Z_0 = characteristic impedance of the free space medium

The height h_M and the various radii appearing in (2.6) are illustrated in figure 2.1. Also observe that

$$(f_{g_{n,m}}) = \begin{pmatrix} f_{g_p} & f_{g_p}(1-\delta) \\ f_{g_p}(1-\delta) & f_{g_M} \end{pmatrix} \tag{2.7}$$

and therefore

$$(f_{g_{n,m}})^{-1} = \frac{1}{\det((f_{g_{n,m}}))} \begin{pmatrix} f_{g_M} & -f_{g_p}(1-\delta) \\ -f_{g_p}(1-\delta) & f_{g_p} \end{pmatrix} \tag{2.8}$$

with

$$\det((f_{g_{n,m}})) = f_{g_p}[f_{g_M} - f_{g_p}(1-\delta)^2] \simeq f_{g_p}[f_{g_M} - f_{g_p}(1-2\delta)]\text{(for } \delta<<1) \tag{2.9}$$

Knowing the above four independent parameters, we can write down the elements of the per-unit-length impedance and admittance matrices, under lossless assumption, as follows.

$$
\left.
\begin{aligned}
&\tilde{Z}'_{1,1} = sL'_{1,1} = s\mu_o f_{g_p}\,,\; \tilde{Z}'_{2,2} = s(L'_{2,2}+L'_M) = s\mu_o(f_{g_M}+\chi) \\
&\tilde{Z}'_{1,2} = \tilde{Z}'_{2,1} = sL'_{1,2} = s\mu_o f_{g_{PM}} = s\mu_o f_{g_p}(1-\delta) \\
&\tilde{Y}'_{1,1} = sC'_{1,1} = s\varepsilon_o f_{g_M}/\det((f_{g_{n,m}})) \\
&\tilde{Y}'_{2,2} = sC'_{2,2} = s\varepsilon_o f_{g_p}/\det((f_{g_{n,m}})) \\
&\tilde{Y}'_{1,2} = sC'_{1,2} = -s\varepsilon_o f_{g_p}(1-\delta)/\det((f_{g_{n,m}}))
\end{aligned}
\right\}
\tag{2.10}
$$

With a knowledge of $(\tilde{Z}'_{n,m})$ and $(\tilde{Y}'_{n,m})$ matrices, we can also formally write down the characteristic impedance $(\tilde{Z}_{c_{n,m}})$ and admittance matrices $(\tilde{Y}_{c_{n,m}})$, as well as the propagation matrix $(\tilde{\gamma}_{c_{n,m}})$ as follows.

$$(\tilde{Z}_{c_{n,m}}) = (\tilde{\gamma}_{c_{n,m}}) \cdot (\tilde{Y}'_{n,m})^{-1} = (\tilde{\gamma}_{c_{n,m}})^{-1} \cdot (\tilde{Z}'_{n,m}) \tag{2.11}$$

$$(\tilde{Y}_{c_{n,m}}) = (\tilde{Z}_{c_{n,m}})^{-1} \tag{2.12}$$

$$(\tilde{\gamma}_{c_{n,m}}) = \text{principal value of } \left((Z'_{n,m}) \cdot (Y'_{n,m}) \right)^{1/2} \tag{2.13}$$

Thus, all of the matrices characterizing the coupled transmission-line model are known in terms of the four independent parameters f_{g_p}, f_{g_M}, δ and X.

In concluding this section, it is noted that during the peaker charge cycle, the $I_M = I_p = (I_{p_A} N_p)$ and that immediately after the output switch closes, the Marx current is small compared to the peaker current. This characterization of pulser operation indicates the need for two switches closing at different times to actually represent the Marx pulser operation. However, this is not the intent here. The goal being the study of wave transport characteristics across the Marx-peaker assembly, the switches representing the peaker charge cycle and the discharge following the output switch closure are not addressed here. On the other hand, the chief benefit of such modelling lies in being able to study the sensitivity of the output waveforms to parameters like N_p and the additional series inductance L_M of the Marx column. This additional inductance $L_M = (L'_M \ell)$ of the Marx column has a significant effect on the wave transport properties in as much as it leads to two waves of propagation viz., the desirable fast wave traveling at the speed of light c and the undesirable slow wave with a speed less than c. The determination of these velocities and the output voltage waveform $V_L(t)$ are the subjects of later sections. Before we can formulate the expressions for the output waveform, we need to briefly review the BLT equation [3 and 4] and specialize its solution to the transmission-line network problem at hand.

3. Brief Review of General BLT Equation

Consider a generalized transmission line network [3] as exemplified in figure 3.1. This network is a graph formed by junctions (vertices) and tubes (branches). The junctions are described as general (linear) N-port networks which may be distributed as well as lumped. The junctions are also characterized by scattering (or impedance or admittance) matrices which are functions of complex frequency s. The tubes are described as general N-wire (plus reference) transmission lines of various lengths. The numbering scheme of junctions and tubes and the two waves travelling in opposite directions on each tube is also shown in the figure. Various matrices can be defined which exhibit the interconnection of junctions and tubes with themselves and with each other.

As developed in [3] and summarized in [4], the BLT equation has the form

$$\begin{aligned}[((1_{n,m})_{u,v}) - ((\tilde{S}_{n,m}(s))_{u,v}) \odot ((\tilde{\Gamma}_{n,m}(s))_{u,v})] \odot ((\tilde{V}_n(0,s))_u) \\ = ((\tilde{S}_{n,m}(s))_{u,v}) \odot ((\tilde{\Lambda}_{n,m}(x,s;(\cdot)))_{u,v}) \odot ((\tilde{V}'_{s_n}(x,s))_u)\end{aligned} \tag{3.1}$$

In this equation there are

$$((\tilde{\Gamma}_{n,m}(s))_{u,v}) \equiv \text{delay supermatrix}$$

$$= \bigoplus_{u=1}^{N_W} (\Gamma_{n,m}(s))_{u,v} = \bigoplus_{u=1}^{N_W} e^{-(\tilde{\gamma}_{c_{n,m}}(s))_{u,u}L_u}$$

$$(\tilde{\Gamma}_{n,m}(s))_{u,v} = \begin{cases} e^{-(\tilde{\gamma}_{c_{n,m}}(s))_{u,v}L_u} & \text{for } u = v \\ (0_{n,m})_{u,v} & \text{for } u \neq v \end{cases}$$

$$(\tilde{\gamma}_{c_{n,m}}(s))_{u,v} = \text{propagation matrix for u-th N-wave}$$

$$= [(Z'_{n,m}(s))_{u,u} \quad (Y'_{n,m}(s))_{u,u}]^{1/2} \text{ (principal or p.r. value)} \tag{3.2}$$

$$(\tilde{Z}_{c_{n,m}}(s))_{u,u} \equiv \text{characteristic impedance matrix for u-th N-wave}$$

$$= (\tilde{\gamma}_{c_{n,m}}(s))_{u,u} \cdot (\tilde{Y}'_{n,m}(s))_{u,u} = (\tilde{\gamma}_{c_{n,m}}(s))^{-1}_{u,u} \cdot (\tilde{Z}'_{n,m}(s))_{u,u}$$

$$(\tilde{Y}_{c_{n,m}}(s))_{u,u} \equiv \text{characteristic admittance matrix for uth N-wave}$$

$$= (\tilde{Z}_{c_{n,m}}(s))^{-1}_{u,u}$$

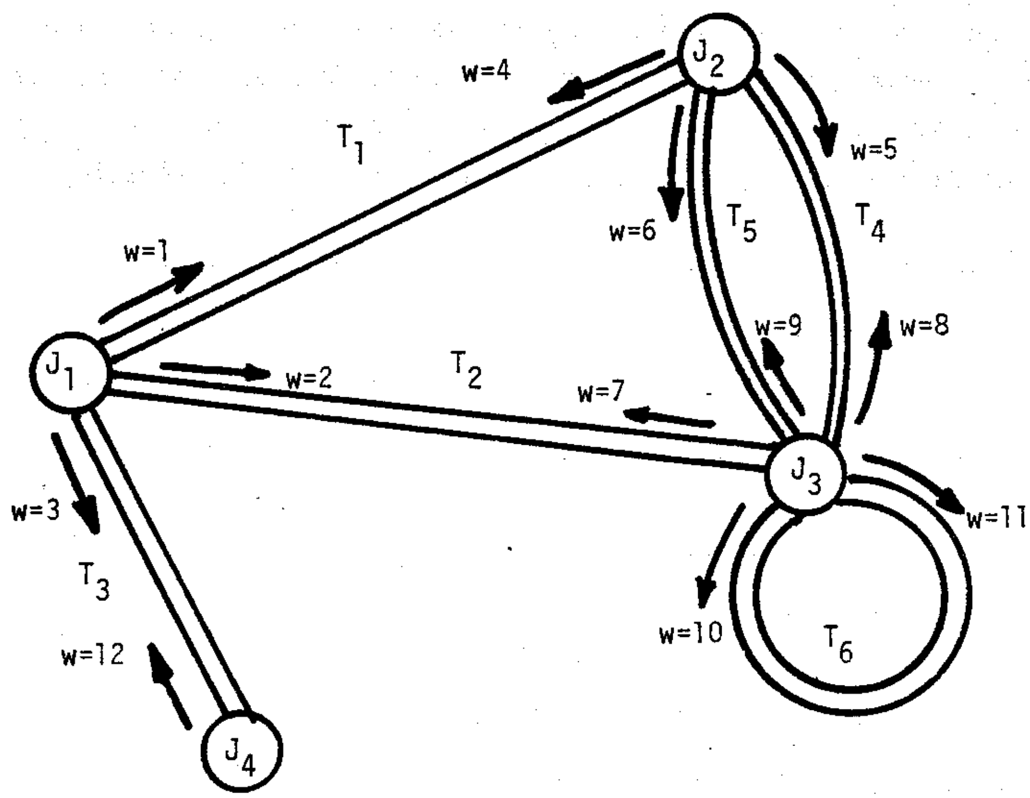


Figure 3.1. A general transmissiun-line graph showing junctions, tubes and waves.

$(\tilde{Z}'_{n,m}(s))_{u,u}$ ≡ longitudinal impedance-per-unit-length matrix for uth N-wave

$(\tilde{Y}'_{n,m}(s))_{u,u}$ ≡ transverse admittance-per-unit-length matrix for uth N-wave

L_u ≡ length of tube on which the uth N-wave propagates

$\oplus$ ≡ direct sum (produces diagonal supermatrix, or block diagonal matrix)

We also have a supermatrix integral operator

$((\tilde{\Lambda}_{n,m}(x,s;(\cdot)))_{u,v})$ ≡ supermatrix integral operator

$$= \bigoplus_{u=1}^{N_W} (\tilde{\Lambda}_{n,m}(x_u,s;(\cdot)))_{u,u} = \bigoplus_{u=1}^{N_W} \int_0^{L_u} e^{-(\tilde{\gamma}_{c_{n,m}}(s))_{u,u}[L_u - x_u]} (\cdot)\, dx_u \tag{3.3}$$

$$(\tilde{\Lambda}_{n,m}(x,s;(\cdot)))_{u,v} = \begin{cases} \int_0^{L_u} e^{-(\tilde{\gamma}_{c_{n,m}}(s))_{u,v}[L_u - x_u]} (\cdot)\, dx_u & \text{for } u = v \\ (0_{n,m})_{u,v} & \text{for } u \neq v \end{cases}$$

x_u = coordinate (position) for uth N-wave along tube $(0 < x_u < L_u)$

Here $(\cdot)$ designates that the terms following the operator are to be inserted at the indicated position in the integral(s) with multiplication in the indicated sense (dot product in this case). Note that this operator operates over the range of each tube.

The response variables in the BLT equation are contained in the combined voltage supervector given by

$((\tilde{V}_n(0,s))_u)$ ≡ combined voltage supervector (for N-waves propagating into tubes (away from junctions)

$$= ((\tilde{V}_n^{(0)}(0,s))_u) + ((\tilde{Z}_{c_{n,m}}(x))_{u,v}) \odot ((\tilde{I}_n^{(0)}(0,s))_u)$$

$((\tilde{Z}_{c_{n,m}}(s))_{u,v})$ ≡ characteristic impedance supermatrix

$$= \bigoplus_{u=1}^{N_W} (\tilde{Z}_{c_{n,m}}(s))_{u,u}$$

$$(\tilde{Z}_{c_{n,m}}(s))_{u,v} = (0_{n,m})_{u,v} \quad \text{for } u \neq v \tag{3.4}$$

$((\tilde{V}_n^{(0)}(x_u,s))_u) \equiv$ voltage supervector for voltages at positions x_u along tubes

$((\tilde{I}_n^{(0)}(x_u,s))_u) \equiv$ current supervector for currents at positions x_u along tubes with positive current in directions of increasing x_u

$((\tilde{Y}_{c_{n,m}}(s))_{u,v}) \equiv$ characteristic admittance supermatrix

$$= ((\tilde{Z}_{c_{n,m}}(s))_{u,v})^{-1} = \bigoplus_{u=1}^{N_W} (\tilde{Y}_{c_{n,m}}(s))_{u,u}$$

$(\tilde{Y}_{c_{n,m}}(s))_{u,u} =$ characteristic impedance matrix for uth N-wave

$$= (\tilde{Z}_{c_{n,m}}(s))_{u,u}^{-1}$$

The source variables in the BLT equation are contained in the combined voltage-per-unit-length supervector given by

$((\tilde{V}'_{s_n}(x_u,s))_u) \equiv$ combined voltage-per-unit-length supervector (a source distributed over the tubes)

$$= ((\tilde{V}'^{(0)}_{s_n}(x_u,s))_u) + ((\tilde{Z}_{c_{n,m}}(s))_{u,v}) \odot ((\tilde{I}'^{(0)}_{s_n}(x_u,x))_u) \tag{3.5}$$

$((\tilde{V}'^{(0)}_{s_n}(x_u,s))_u) \equiv$ voltage-per-unit-length supervector (longitudinal voltage source per unit length)

$((\tilde{I}'^{(0)}_{s_n}(x_u,s))_u) \equiv$ current-per-unit-length supervector (transverse current source per unit length)

The BLT equation then can take a set of assumed sources along the tubes of a transmission-line network and determine the combined voltages leaving the junctions. These combined voltages can be converted back to regular voltages and currents, if desired. First, the combined voltage N-waves entering a tube can be transformed to any position along that tube via

$$(\tilde{V}_n(x_u,s))_u = e^{-(\tilde{\gamma}_{c_{n,m}}(s))_{u,u}x_u} \cdot (\tilde{V}_n(0,s))_u + \int_0^{x_u} e^{-(\tilde{\gamma}_{c_{n,m}}(s))_{u,u}[x_u-x'_u]} \cdot (\tilde{V}'_{s_n}(x'_u,s))_u \, dx'_u \tag{3.6}$$

Now let u and v represent the two N-waves propagating in opposite directions on a given tube. (See fig. 3.1 for an example to see that this is a rather simple correlation.) Then (3.6) applies to both waves on a particular tube with

$$(\tilde{V}_n^{(0)}(x_u,s))_u = (\tilde{V}_n^{(0)}(x_v,s))_v = \frac{1}{2}\left[(\tilde{V}_n(x_u,s))_u + (\tilde{V}_n(x_v,s))_v\right]$$

$$(\tilde{I}_n^{(0)}(x_u,s))_u = -(\tilde{I}_n^{(0)}(x_v,s))_v = \frac{1}{2}(\tilde{Y}_{c_{n,m}}(s))_{u,u} \cdot \left[(\tilde{V}_n(x_u,s))_u - (\tilde{V}_n(x_v,s))_v\right] \tag{3.7}$$

$$x_u + x_v = L_u = L_v$$

This formalism allows for lumped sources as well as distributed sources by the introduction of δ functions at any particular x_u $(= L_u - x_v)$ of interest. By interpreting any sources ascribed to the junctions as sources just inside the tubes (at each $x_u = 0+$), junction equivalent sources are also handled in this formalism.

The above is a statement of the general BLT equation and an identification of the various terms involving the network description, lumped and/or distributed source quantities and the response quantity. This response quantity is a combined voltage supervector for N-waves propagating into tubes and away from junctions. The combined voltages can be transformed back to the usual quantities of response voltages and currents.

The problem at hand, illustrated in figure 2.2 is of course relatively simple with only two junctions and a single tube. For this reason, let us consider the problem of a transmission line network consisting of a sir 'e tube in the following section, before applying its solution to the problem at hand.

4. The Form of Time Domain Solution on a Single Tube

One can think of a multiconductor transmission line as one "tube" in a general network [4]. For the present, let us consider a multiconductor transmission line with per-unit-length model as shown in figure 4.1. This figure represents one incremental section of length dz in such a transmission line.

The telegrapher's equations for the current and voltage response spectra along the multiconductor transmission line are [5]

$$\frac{d}{dz}\left(\tilde{V}_n(z,s)\right) = -\left(\tilde{Z}'_{n,m}(z,s)\right) \cdot \left(\tilde{I}_n(z,s)\right) + \left(\tilde{V}_n^{(s)'}(z,s)\right)$$

$$\frac{d}{dz}\left(\tilde{I}_n(z,s)\right) = -\left(\tilde{Y}'_{n,m}(z,s)\right) \cdot \left(\tilde{V}_n(z,s)\right) + \left(\tilde{I}_n^{(s)'}(z,s)\right) \qquad (4.1)$$

where

$z \equiv$ position along tube

$(\tilde{V}_n(z,s)) \equiv$ voltage vector at z

$(\tilde{I}_n(z,s)) \equiv$ current vector at z

$(\tilde{Z}'_{n,m}(z,s)) \equiv$ per-unit-length series impedance matrix

$(\tilde{Y}'_{n,m}(z,s)) \equiv$ per-unit-length shunt admittance matrix

$(\tilde{V}_n^{(s)'}(z,s)) \equiv$ per-unit-length series voltage source vector

$(\tilde{I}_n^{(s)'}(z,s)) \equiv$ per-unit-length shunt current source vector

$s \equiv$ Laplace transform variable (complex frequency) for transform over time (t)

For our N-wire transmission line all vectors are of dimension N, and all matrices are $N \times N$.

Define the propagation matrix

$$(\tilde{\gamma}_{c_{n,m}}(z,s)) \equiv \{(\tilde{Z}'_{n,m}(z,s)) \cdot (\tilde{Y}'_{n,m}(z,s))\}^{1/2} \qquad (4.2)$$

where the matrix square root is taken in the principal value or positive real (p.r.) sense [3] by diagonalizing $(\tilde{\gamma}_{c_{n,m}})^2$ and then taking p.r. square

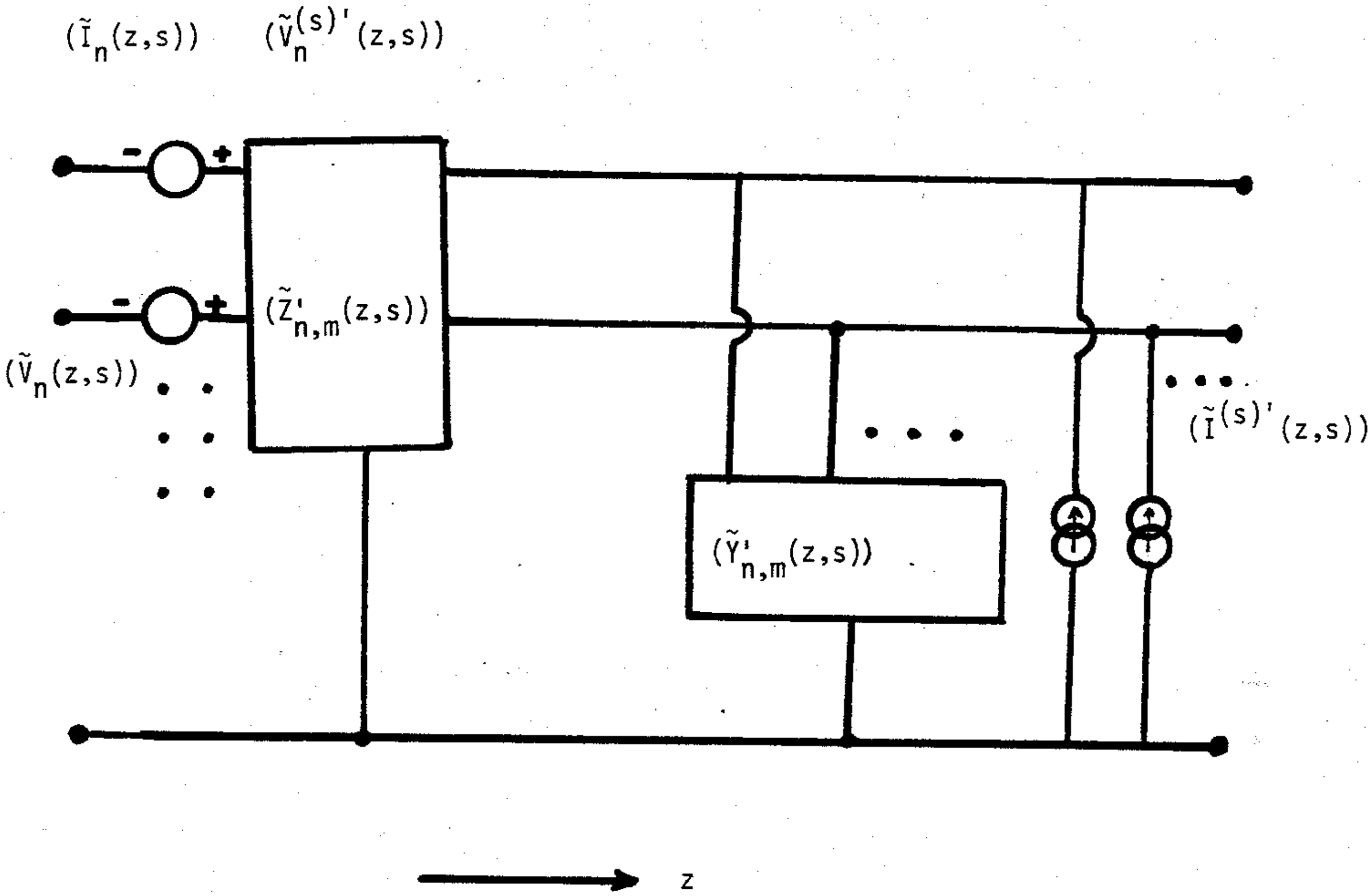


Figure 4.1. Per-unit-length model of a multiconductor transmission line forming a single tube.

roots of the eigenvalues. From this we also derive

$$(\tilde{Z}_{c_{n,m}}(z,s)) = (\tilde{\gamma}_{c_{n,m}}(z,s)) \cdot (\tilde{Y}'_{n,m}(z,s))^{-1} = (\tilde{\gamma}_{c_{n,m}}(z,s))^{-1} \cdot (\tilde{Z}'_{n,m}(z,s))$$

$$\equiv \text{characteristic impedance matrix}$$

$$(4.3)$$

$$(\tilde{Y}_{c_{n,m}}(z,s)) = (\tilde{Y}'_{n,m}(z,s)) \cdot (\tilde{\gamma}_{c_{n,m}}(z,s))^{-1} = (\tilde{Z}_{n,m}(z,s))^{-1} \cdot (\tilde{\gamma}_{c_{n,m}}(z,s))$$

$$= (\tilde{Z}_{c_{n,m}}(z,s))^{-1}$$

$$\equiv \text{characteristic admittance matrix}$$

Combined voltage and source vectors are then defined by

$$(\tilde{V}_n(z,s))_q = (\tilde{V}_n(z,s))_q + q(\tilde{Z}_{c_{n,m}}(z,s)) \cdot (\tilde{I}_n(z,s))$$

$$(\tilde{V}_n^{(s)'}(z,s)_q = (\tilde{V}_n^{(s)'}(z,s))_q + q(\tilde{Z}_{c_{n,m}}(z,s)) \cdot (\tilde{I}_n^{(s)'}(z,s)) \qquad (4.4)$$

$$q = \underline{+}1 \text{ (separation index)}$$

where $\underline{+}$ corresponds to waves propagating in the $\underline{+}z$ directions.

Now in [3] under the assumption that $(\tilde{Z}'_{n,m}(s))$ and $(\tilde{Y}'_{n,m}(s))$ are not functions of z (allowing $(\tilde{Z}_{c_{n,m}}(s))$ to pass through the z derivative) we obtain a combined-voltage differential equation

$$\{(1_{n,m}) \frac{d}{dz} + q(\tilde{\gamma}_{c_{n,m}}(s))\} \cdot (\tilde{V}_n(z,s))_q = (\tilde{V}_n^{(s)'}(z,s))_q \qquad (4.5)$$

where $(\tilde{\gamma}_{c_{n,m}}(s))$ is also not a function of z. This is readily solved [6] to give

$$(\tilde{V}_n(z,s))_q = e^{-q(\tilde{\gamma}_{c_{n,m}}(s))z} \cdot (\tilde{V}_n(0,s))_q$$

$$+ \int_0^z e^{-q(\tilde{\gamma}_{c_{n,m}}(z))[z-z']} \cdot (\tilde{V}_n^{(s)'}(z',s))dz' \qquad (4.6)$$

where without loss of generality we have taken the initial condition as $(\tilde{V}_n(0,s))$ but any other z_0 could be used.

In this paper we consider the case that there are no sources $(\tilde{V}_n^{(s)'})$

and $(I_n^{(s)'})$ along the tube, the only "excitation" coming from conditions at some coordinate we take as z = 0. In this case (4.6) reduces to

$$(\tilde{V}_n(z,s))_q = e^{-q(\tilde{\gamma}_{c_{n,m}}(s))z} \cdot (\tilde{V}_n(0,s))_q \tag{4.7}$$

as the solution of the homogeneous form of (4.5). This is a fairly simple result for propagation along a uniform N-wire transmission line.

The solution in (4.7) above can also be cast in time domain [7 and 8] as follows. In the absence of distributed sources and if the multiconductor line is uniform, we have at a location z_u,

$$(\tilde{V}_n(z_u,s))_u = e^{-(\gamma_{c_{n,m}})z_u} \bullet (\tilde{V}(0,s))_u \tag{4.8}$$

Also, under the assumption that $(\tilde{Y}'_{n,m})$ and $(\tilde{Z}'_{n,m})$ are s times constant $(C'_{n,m})$ and $(L'_{n,m})$, we may define a pair of current and voltage eigenvectors (i_{c_n}) and (v_{c_n}) of the matrix $(L'_{n,m}) \cdot (C'_{n,m})$, resulting in

$$(\tilde{\gamma}_{c_{n,m}}) = \sum_\beta \tilde{\gamma}_\beta (v_{c_n})_\beta \; (i_{c_n})_\beta \tag{4.9}$$

with β representing the eigenvalue or eigenvector index. $\tilde{\gamma}_\beta$ are the eigenvalues, found later for the problem at hand. Substituting (4.9) into (4.8), we have in frequency domain

$$(\tilde{V}_n(z_u,s))_u = \left[\sum_\beta e^{-\tilde{\gamma}_\beta z_u}(v_{c_n})_\beta \; (i_{c_n})_\beta\right] \bullet (\tilde{V}(0,s))_u \tag{4.10}$$

Note that since the modes are eigenmodes of frequency independent matrices, they are themselves frequency independent. The above expression in frequency domain can be Laplace inverted by recognizing that a phase shifted term in frequency domain corresponds to a translation in the time domain to yield

$$(V_n(z_u,t))_u = \sum_\beta (v_{c_n})_\beta \; (i_{c_n})_\beta \cdot \left(V_n(0,\, t - \frac{z_u}{v_\beta})\right) \tag{4.11}$$

Note here that for the modes to be physical time domain quantities, they are real-valued vectors, except possibly in cases of equal eigenspeeds. Here, we have

$$(v_{c_n})_{\beta_1} \cdot (i_{c_n})_{\beta_2} = 1_{\beta_1,\beta_2} \quad \text{(biorthonormalized)} \tag{4.12}$$

$$\tilde{\gamma}_\beta \equiv \text{eigenvalues} = (s/v_\beta), \quad v_\beta = \text{eigenspeeds}, \quad \beta = \text{eigenindex}$$

Note also that the eigenvoltages and eigencurrents are interrelated via the characteristic impedance as given by

$$(v_{c_n})_\beta = (Z_{c_{n,m}})_\beta \cdot (i_{c_n})_\beta \tag{4.13}$$

Furthermore, the characteristic impedance and admittance matrices can also be written in terms of the eigenmodes as follows.

$$\begin{aligned}
(\tilde{Z}_{c_{n,m}}) &= (\tilde{\gamma}_{c_{n,m}})^{-1} \cdot (\tilde{Z}'_{n,m}) \\
&= \{\sum_\beta \tilde{\gamma}_\beta^{-1} (v_{c_n})_\beta (i_{c_n})_\beta\} \cdot (\tilde{Z}'_{n,m}) \\
&= \{\sum_\beta \tilde{\gamma}_\beta^{-1} (v_{c_n})_\beta [(i_{c_n})_\beta \cdot (\tilde{Z}'_{n,m})]\} \\
&= (\tilde{\gamma}_{c_{n,m}}) \cdot (\tilde{Y}'_{n,m})^{-1} \\
&= \{\sum_\beta \tilde{\gamma}_\beta (v_{c_n})_\beta (i_{c_n})_\beta\} \cdot (\tilde{Y}'_{n,m})^{-1}\} \\
&= \{\sum_\beta \tilde{\gamma}_\beta (v_{c_n})_\beta [(i_{c_n})_\beta \cdot (\tilde{Y}'_{n,m})^{-1}]\}
\end{aligned} \tag{4.14}$$

and similarly

$$
\begin{aligned}
(\tilde{Y}_{c_{n,m}}) &= (\tilde{Y}'_{n,m}) \bullet (\tilde{\gamma}_{c_{n,m}})^{-1} \\
&= (\tilde{Y}'_{n,m}) \bullet \{\sum_{\beta} \tilde{\gamma}_{\beta}^{-1} (v_{c_n})_{\beta} (i_{c_n})_{\beta}\} \\
&= \sum_{\beta} \tilde{\gamma}_{\beta}^{-1} (\tilde{Y}'_{n,m}) \cdot (v_{c_n})_{\beta} (i_{c_n})_{\beta} \\
&= (\tilde{Z}'_{n,m})^{-1} \cdot (\tilde{\gamma}_{c_{n,m}}) \\
&= (\tilde{Z}'_{n,m})^{-1} \cdot \{\sum_{\beta} \tilde{\gamma}_{\beta} (v_{c_n})_{\beta} (i_{c_n})_{\beta}\} \\
&= \sum_{\beta} \tilde{\gamma}_{\beta} (\tilde{Z}'_{n,m})^{-1} \cdot (v_{c_n})_{\beta} (i_{c_n})_{\beta}
\end{aligned}
\qquad (4.15)
$$

In this section, we have formally completed a review of obtaining the time-domain responses for the case of a single tube. In concluding, we observe that Sections 3 and 4 present a review of the BLT equation for a generalized transmission line network, and then it is specialized to a network of a single tube in both frequency and time domains. In effect these two review sections present results that can now be applied to the problem at hand. We start applying the results of Section 4, by considering the two modes in Section 5.

5. The Two Modes

The propagation matrix for the coupled transmission-line model, under the lossless assumption is given by,

$$
\begin{aligned}
(\tilde{\gamma}_{c_{n,m}}) &= [(\tilde{Z}'_{n,m}) \cdot (\tilde{Y}'_{n,m})]^{1/2} \\
&= s\left[\left\{(L'_{n,m}) + \begin{pmatrix} 0 & 0 \\ 0 & L'_M \end{pmatrix}\right\} \cdot (C'_{n,m})\right]^{1/2} \\
&= s\left[(1_{n,m})\frac{1}{c^2} + \begin{pmatrix} 0 & 0 \\ 0 & L'_M \end{pmatrix} \cdot (C'_{n,m})\right]^{1/2} \\
&= \frac{s}{c}\left[(1_{n,m}) + \begin{pmatrix} 0 & 0 \\ 0 & L'_M \end{pmatrix} \cdot (L'_{n,m})^{-1}\right]^{1/2} \\
&= \frac{s}{c}\left[(1_{n,m}) + (\varepsilon_{n,m})\right]^{1/2} \equiv \frac{s}{c}(\Lambda_{nm})^{1/2}
\end{aligned}
\tag{5.1}
$$

where $(1_{n,m})$ is an identity matrix and the "error" matrix $(\varepsilon_{n,m})$ due to the additional Marx inductance is given by

$$
\begin{aligned}
(\varepsilon_{n,m}) &= \begin{pmatrix} 0 & 0 \\ 0 & L'_M \end{pmatrix} \cdot (L'_{n,m})^{-1} \\
&= \frac{1}{\det((L'_{n,m}))} \begin{pmatrix} 0 & 0 \\ -L'_M L'_{1,2} & L'_M L'_{1,1} \end{pmatrix}
\end{aligned}
\tag{5.2}
$$

Or in terms of the four normalized parameters,

$$
(\varepsilon_{n,m}) = \left(\begin{array}{c|c} 0 & 0 \\ \hline \dfrac{-\chi(1-\delta)}{f_{g_M} - f_{g_p}(1-\delta)^2} & \dfrac{\chi}{f_{g_M} - f_{g_p}(1-\delta)^2} \end{array}\right)
\tag{5.3}
$$

It is also easily seen that all the elements of $(\varepsilon_{n,m})$ become zero if there is no additional Marx inductance.

Next, the two velocities of propagation are given by $v_\beta = (c/\sqrt{\lambda_\beta})$ for $\beta = 1, 2$ where λ_1 and λ_2 are the eigenvalues of $((1_{n,m}) + (\varepsilon_{n,m}))$ given by

$$\det(1_{n,m}+\varepsilon_{n,m}-\lambda_\beta\, 1_{n,m}) = 0 \tag{5.4}$$

$$\det\begin{pmatrix} 1-\lambda_\beta & 0 \\ \varepsilon_{2,1} & 1+\varepsilon_{2,2}-\lambda_\beta \end{pmatrix} = 0 \tag{5.5}$$

which leads to

$$\lambda_1 = 1 \quad \text{and} \quad \lambda_2 = 1 + \varepsilon_{2,2} = \left[1 + \left(\frac{L'_M\, L'_{1,1}}{\det((L'_{n,m}))}\right)\right] \tag{5.6}$$

Thus, the two velocities are

$$v_1 = c \quad \text{and} \quad v_2 = \frac{c}{\left[1 + \dfrac{L'_M L'_{1,1}}{(L'_{1,1}L'_{2,2}-L'_{1,2}L'_{2,1})}\right]^{1/2}} \tag{5.7}$$

In terms of the previously defined independent parameters (f_{g_p}, f_{g_M}, χ and δ), the two velocities are given by

$$v_1 = c \quad \text{and} \quad v_2 = \frac{c}{\left[1 + \dfrac{\chi}{f_{g_M}-f_{g_p}(1-\delta)^2}\right]^{1/2}} \tag{5.8}$$

knowing the four independent parameters from physical and geometrical considerations, it is now a simple matter to evaluate the two velocities using the above equation.

Furthermore, with ℓ being the length of the Marx column (= length of the peaker), the difference in transit time Δt for the two waves is therefore

$$\Delta t = \left(\frac{\ell}{v_2} - \frac{\ell}{v_1}\right) = \frac{\ell}{c}\left[\left\{1 + \frac{\chi}{f_{g_M}-f_{g_p}(1-\delta)^2}\right\}^{1/2} - 1\right] \tag{5.9}$$

which can also easily be evaluated by knowing the four independent parameters and the length of the Marx generator.

Observe that one of the velocities is c and the second one is less than c. They may be associated respectively with a desirable "fast" wave ($v_1 = c$) and an undesirable "slow" wave ($v_2 < c$). These two waves are propagating on a

2 conductor (plus reference) transmission line illustrated in figure 5.1a. It is cast in an equivalent transmission-line network graph in figure 5.1b. Observe that we basically have a single tube T_1 consisting of two conductors (Marx and peaker). This tube T_1 connects the junctions J_1 and J_2. $w = 1$ and $w = 2$ corresponds respectively to the waves traveling in the +z and -z directions on tube T_1. $w = 3$ is the outward wave at J_2 which is also the output wave of interest. We are interested basically in the voltage waveform associated with this output wave. The input excitation is considered to be a step function voltage applied at J_1 to both the conductors.

Next, we turn our attention to the determination of the eigencurrent and eigenvoltage vectors. We have already seen that under lossless assumption [3]

$$
\begin{aligned}
(\tilde{\gamma}_{c_{n,m}}) &= \{(\tilde{Z}'_{n,m}) \cdot (\tilde{Y}'_{n,m})\}^{1/2} \\
&= s\{(L'_{n,m}) \cdot (C'_{n,m})\}^{1/2} \\
&= \frac{s}{c}\left[(1_{n,m}) + (\varepsilon_{n,m})\right]^{1/2} = \frac{s}{c}(\Lambda_{n,m})^{1/2}
\end{aligned}
\tag{5.10}
$$

Consequently, the eigenvectors are computable from

$$
\begin{aligned}
(\Lambda_{n,m}) \cdot (v_{c_n})_\beta &= \lambda_\beta (v_{c_n})_\beta \\
(i_{c_n})_\beta \cdot (\Lambda_{n,m}) &= \lambda_\beta (i_{c_n})_\beta
\end{aligned}
\tag{5.11}
$$

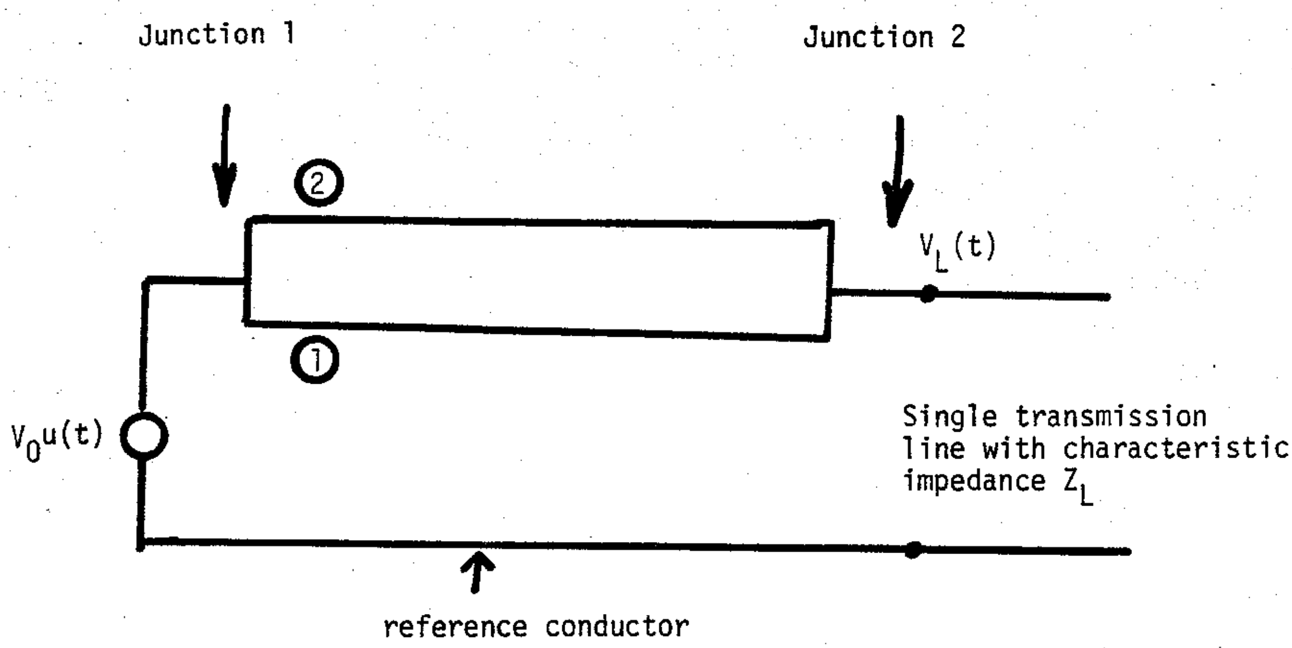
or

$$
\begin{aligned}
\left[(\Lambda_{n,m}) - \lambda_\beta (1_{n,m})\right] \cdot (v_{c_n})_\beta &= 0 \\
(i_{c_n})_\beta \cdot \left[(\Lambda_{n,m}) - \lambda_\beta (1_{n,m})\right] &= 0
\end{aligned}
\tag{5.12}
$$

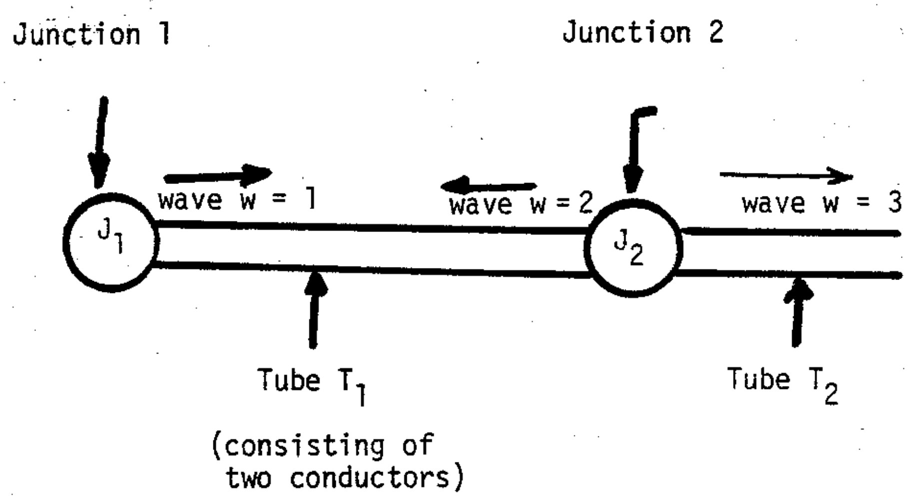
We have already evaluated λ_1 and λ_2 which are given in terms of the four independent parameters by,

$$
\lambda_1 = 1 \quad \text{and} \quad \lambda_2 = \left[1 + \frac{\chi}{f_{g_M} - f_{g_P}(1-\delta)^2}\right]
\tag{5.13}
$$

and



a) Coupled transmission line problem



b) Transmission line network graph

Figure 5.1. The problem at hand cast in the terminology of a transmission-line network.

$$(\Lambda_{n,m}) = \left(\begin{array}{c|c} 0 & 0 \\ \hline \dfrac{-\chi(1-\delta)}{f_{g_M}-f_{g_p}(1-\delta)^2} & 1 + \dfrac{\chi}{f_{g_M}-f_{g_p}(1-\delta)^2} \end{array}\right) \tag{5.14}$$

Using (5.13) and (5.14) in (5.12), we can derive the eigenvectors to be

a) $\beta = 1, \lambda_\beta = \lambda_1$; fast wave

$$\begin{aligned} (i_{c_n})_1 &= (i_{c_n})_f = \zeta_1^{-1/2}(1,0) \\ (v_{c_n})_1 &= (v_{c_n})_f = \zeta_1^{1/2}(1, 1-\delta) \end{aligned} \tag{5.15}$$

b) $\beta = 2, \lambda_\beta = \lambda_2$; slow wave

$$\begin{aligned} (i_{c_n})_2 &= (i_{c_n})_s = \zeta_2^{-1/2}(-1+\delta, 1) \\ (v_{c_n})_2 &= (v_{c_n})_s = \zeta_2^{1/2}(0,1) \end{aligned} \tag{5.16}$$

where ζ_β are the eigenimpedances. Note that ζ_β does not enter the dyadic representation of $(\tilde{\gamma}_{c_{n,m}})$ since

$$\begin{aligned} (v_{c_n})_f (i_{c_n})_f &= (1,1-\delta)\ (1,0) \\ (v_{c_n})_s (i_{c_n})_s &= (0,1)(-1+\delta,1) \\ (\tilde{\gamma}_{c_{n,m}}) &= \frac{s}{c}\{(1,1-\delta)(1,0) + \lambda_2^{1/2}(0,1)(-1+\delta,1)\} \end{aligned} \tag{5.17}$$

with λ_2 as in (5.13). Now; we can find expressions for the eigenimpedances ζ_β as follows

Imposing "Ohm's law" for eigenmodes as in (4.13) for normalization gives

$$(v_{c_n})_\beta = (\tilde{Z}_{c_{n,m}}) \cdot (i_{c_n})_\beta \tag{5.18}$$

$$\zeta_1(1,1-\delta) = (\tilde{Z}_{c_{n,m}}) \cdot (1,0)$$

$$\zeta_2(0,1) = (\tilde{Z}_{c_{n,m}}) \cdot (-1+\delta,1)$$

Using (4.14) we have for the $(v_{c_n})_1$ mode

$$\begin{aligned}\zeta_1(1,1-\delta) &= \sum_\beta \tilde{\gamma}_\beta^{-1}(v_{c_n})_\beta\left((i_{c_n})_\beta \cdot (\tilde{Z}'_{n,m}) \cdot (1,0)\right)\\ &= \frac{c}{s}\Big\{(1,1-\delta)\ \ (1,0) \cdot (\tilde{Z}'_{n,m}) \cdot (1,0)\\ &\qquad + \lambda_2^{-1/2}(0,1)\ (-1+\delta,1) \cdot (\tilde{Z}'_{n,m}) \cdot (1,0)\Big\}\end{aligned} \tag{5.19}$$

From (2.1) through (2.5) we have for the second term

$$\begin{aligned}&(-1+\delta,1) \cdot (Z'_{n,m}) \cdot (1,0)\\ &= s\mu_0(-1+\delta,1) \cdot \begin{pmatrix} f_{g_P} & f_{g_{PM}} \\ f_{g_{PM}} & f_{g_M}+\chi \end{pmatrix} \cdot (1,0)\\ &= s\mu_0(-1+\delta,1) \cdot (f_{g_P}, f_{g_{PM}})\\ &= s\mu_0 f_{g_P}(-1+\delta,1) \cdot (1,1-\delta)\\ &= 0!!!\end{aligned} \tag{5.20}$$

Similarly the first term is

$$(1,0) \cdot (\tilde{Z}'_{n,m}) \cdot (1,0)$$

$$= s\mu_0 (1,0) \cdot \begin{pmatrix} f_{g_P} & f_{g_{PM}} \\ f_{g_{PM}} & f_{g_M}+\chi \end{pmatrix} \cdot (1,0)$$

$$= s\mu_0 (1,0) \cdot (f_{g_P}, f_{g_{PM}}) \tag{5.21}$$

$$= s\mu_0 f_{g_P} (1,0) \cdot (1,1-\delta)$$

$$= s\mu_0 \, f_{g_P}$$

Hence

$$\zeta_1 = Z_0 f_{g_P} = Z_P \tag{5.22}$$

which is the impedance of the peaker arms with respect to the ground plane with no effect of the Marx.

Now consider the slow mode (β=2) in (5.18) for which

$$\zeta_2 (0,1) = (\tilde{Z}_{c_{n,m}}) \cdot (-1+\delta,1) \tag{5.23}$$

Using (4.14) we have for the $(v_{c_n})_2$ mode

$$\zeta_2(0,1) = \sum_\beta \tilde{\gamma}_\beta^{-1} (v_{c_n})_\beta \Big((i_{c_n})_\beta \cdot (\tilde{Z}'_{n,m}) \cdot (-1+\delta,1) \Big)$$

$$= \frac{c}{s} \Big\{ (1,1-\delta) \; (1,0) \cdot (\tilde{Z}'_{n,m}) \cdot (-1+\delta,1) \tag{5.24}$$

$$+ \lambda_2^{-1/2} (0,1) \; (-1+\delta,1) \cdot (\tilde{Z}'_{n,m}) \cdot (-1+\delta,1) \Big\}$$

From (2.1) through (2.5), we have for the first term

$$(1,0) \cdot (\tilde{Z}'_{n,m}) \cdot (-1+\delta,1)$$

$$= s\mu_0 (1,0) \cdot \begin{pmatrix} f_{g_P} & f_{g_{PM}} \\ f_{g_{PM}} & f_{g_M}+\chi \end{pmatrix} \cdot (-1+\delta,1)$$

$$= s\mu_0(1,0) \cdot (0, f_{g_{PM}}(\delta-1) + f_{g_M} + \chi) \tag{5.25}$$

$$= 0!!!$$

Similarly the second term is

$$(-1+\delta,1) \cdot (\tilde{Z}'_{n,m}) \cdot (-1+\delta,1)$$

$$= s\mu_0(-1+\delta,1) \cdot \begin{pmatrix} f_{g_p} & f_{g_{PM}} \\ f_{g_{PM}} & f_{g_M}+\chi \end{pmatrix} \cdot (-1+\delta,1) \tag{5.26}$$

$$= s\mu_0(-1+\delta,1) \cdot (0, f_{g_{PM}}(\delta-1) + f_{g_M} + \chi)$$

$$= s\mu_0\{f_{g_{PM}}(\delta-1) + f_{g_M} + \chi\} = s\mu_0\{f_{g_M} + \chi - f_{g_p}(1-\delta)^2\}$$

Hence

$$\zeta_2 = \lambda_2^{-1/2} Z_0\{f_{g_M} + \chi - f_{g_p}(1-\delta)^2\} \tag{5.27}$$

Substituting for λ_2 from (5.8), we have

$$\zeta_2 = Z_0\left\{f_{g_M} - f_{g_p}(1-\delta)^2\right\}^{1/2}\left\{\chi + f_{g_M} - f_{g_p}(1-\delta)^2\right\}^{1/2} \tag{5.28}$$

For the problem at hand, we have characterized the two modes in this section. The eigenvalues, eigenimpedances, eigenvoltage, and eigencurrent vectors can all be calculated now knowing the four independent parameters f_{g_p}, f_{g_M}, δ and χ. In addition, we require a means of characterizing the two junctions J_1 and J_2 by their scattering matrices. This is the subject of Section 6, at the end of which we will be in a position to compute and optimize the output waveform.

6. Junction Characterization

With reference to figure 5.1b, it is now required to characterize the two junctions J_1 and J_2 by their scattering matrices. The scattering matrices relate waves scattered from a junction to waves that are incident on the junction. For our problem, we have

$$
\begin{aligned}
(V_n)_1 &= (S_{n,m})_{1,2} \cdot (V_n)_2 \quad \text{at } J_1 \\
(V_n)_2 &= (S_{n,m})_{2,1} \cdot (V_n)_1 \quad \text{at } J_2 \\
(V_n)_3 &= (S_{n,m})_{3,1} \cdot (V_n)_1 \quad \text{at } J_2
\end{aligned}
\tag{6.1}
$$

where the left hand sides are the combined voltage waves w = 1, w = 2, and w = 3 (see figure 6.1) given by combining actual voltages and currents as follows.

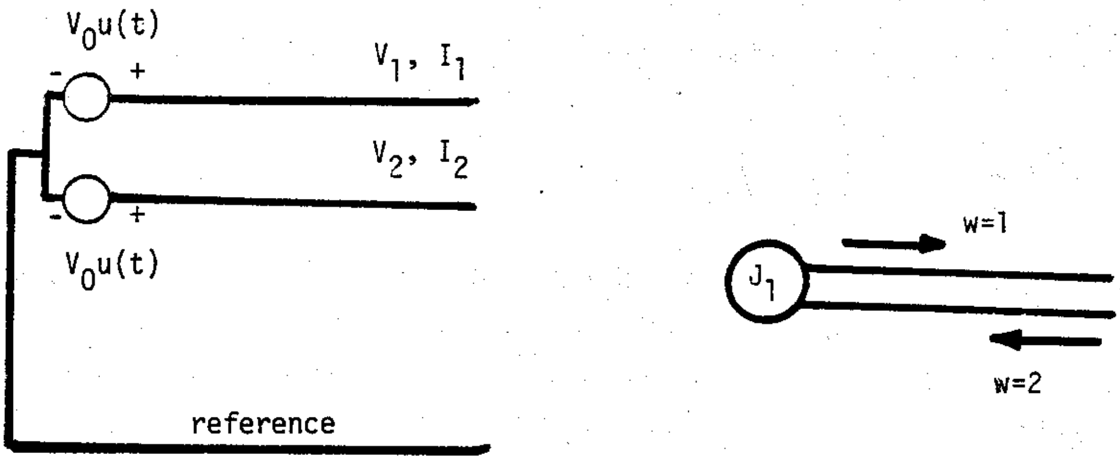
$$
\begin{aligned}
(V_n)_1 &= (V_n^{(0)})_1 + (Z_{c_{n,m}})_{1,1} \cdot (I_n^{(0)})_1 \\
(V_n)_2 &= (V_n^{(0)})_1 - (Z_{c_{n,m}})_{1,1} \cdot (I_n^{(0)})_1 \\
(V_n)_3 &= (V_n^{(0)})_3 + (Z_{c_{n,m}})_{3,3} \cdot (I_n^{(0)})_3
\end{aligned}
\tag{6.2}
$$

Note that in the above notation for combined voltages, e.g., $(V_n)_1$, the inside subscript n refers to the wire or conductor and the outside subscript refers to the wave, w = 1. It is now our object to find the scattering matrices.

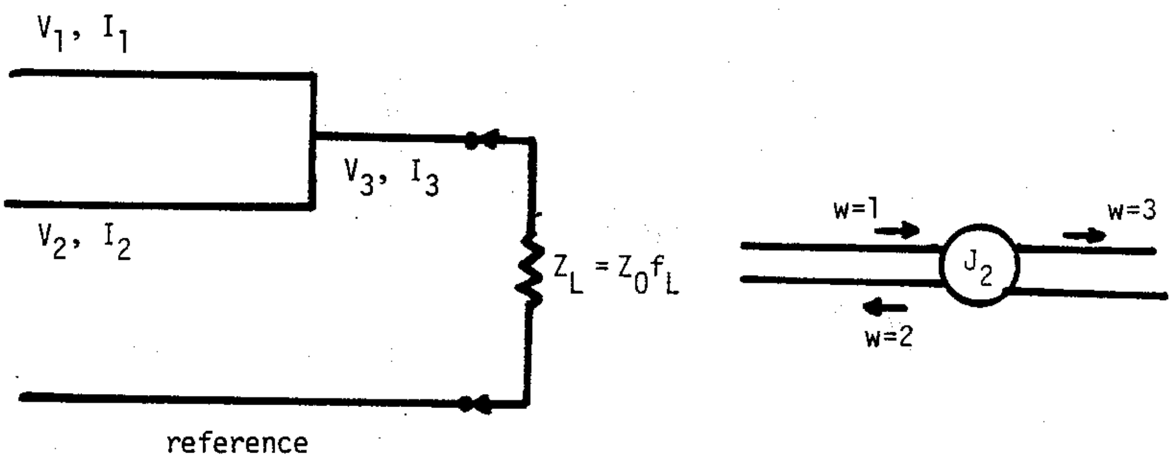
a) Junction J_1

It is observed that this junction is relatively a simple one with an associated scalar problem resulting in a reflection coefficient of -1. With reference to figure 6.1a,

$$
\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = - \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} I_1 \\ I_2 \end{pmatrix}
\tag{6.3}
$$



a) Junction 1



b) Junction 2

Figure 6.1. Characterization of the two junctions for evaluating the associated matrices.

so that the junction impedance and admittance matrices are given by

$$(\tilde{Z}_{n,m}) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}; \; (\tilde{Y}_{n,m}) = \begin{pmatrix} \infty & \infty \\ \infty & \infty \end{pmatrix} \tag{6.4}$$

and the junction scattering matrix is

$$(S_{n,m})_{1,2} = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{1,2}. \tag{6.5}$$

This matrix implies w = 2 wave incident on J_1 scatters into the outgoing w = 1 wave, resulting in the fast and slow wave components of wave w = 2 reflecting back into the same components in w = 1 wave with a minus sign as the only change. So, J_1 can be characterized by

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix}_1 = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{1,2} \cdot \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}_2 = -\begin{pmatrix} V_1 \\ V_2 \end{pmatrix}_2 \tag{6.6}$$

b) Junction J_2

At this junction, w = 1 wave is incident and it scatters into w = 2 wave and w = 3 wave, according as

$$(V_n)_2 = (S_{n,m})_{2,1} \cdot (V_n)_1 = \begin{pmatrix} S_{1,1} & S_{1,2} \\ S_{2,1} & S_{2,2} \end{pmatrix}_{2,1} \cdot (V_n)_1$$

$$(V_n)_3 = (S_{n,m})_{3,1} \cdot (V_n)_1 = (S_{1,1}, S_{1,2})_{3,1} \cdot (V_n)_1 \tag{6.7}$$

and we are now required to find the above scattering matrices in (6.7) to completely characterize this junction J_2. These scattering matrices are obtainable from an application of the junction conditions. One could also derive interrelationships between the elements of $(S_{n,m})_{2,1}$ which simplifies the evaluation of the scattering matrices. The junction conditions at J_2 are

i) voltage on the load = voltage on wire 1
ii) voltage on the load = voltage on wire 2
iii) voltage on the load = Z_L × current through the load
iv) current through the load = sum of currents entering J_2 on wires 1 and 2

The above conditions result in the following equations valid at J_2.

$$V_L(t) = V^{(0)}_{1;3} = \frac{1}{2} V_{1;3} = \frac{1}{2} (V_n)_3 \tag{6.8a}$$

$$V_{1;3} = (1,0)\cdot[(1_{n,m}) + (S_{n,m})_{2,1}] \cdot (V_n)_1 \tag{6.8b}$$

$$V_{1;3} = (0,1)\cdot[(1_{n,m}) + (S_{n,m})_{2,1}] \cdot (V_n)_1 \tag{6.8c}$$

$$V_L = Z_L \; I_L \tag{6.8d}$$

$$Z_L = Z_{c_{1,1;3,3}} \tag{6.8e}$$

$$I_L = I^{(0)}_{1;1} + I^{(0)}_{2;1} \tag{6.8f}$$

$$I_L = (Y_{c_{n,m}})_{1,1} \cdot [(V_n)_1 - (V_n)_2] \tag{6.8g}$$

The above equations representing the conditions on voltages and currents at J_2 are sufficient to determine the scattering matrix elements. For example, using (6.8b) and (6.8c) in (6.7), we observe

$$\begin{aligned}(S_{n,m})_{3,1} &= (1,0)\cdot\left((1_{n,m}) + (S_{n,m})_{2,1}\right) \\ &= (0,1)\cdot\left((1_{n,m}) + (S_{n,m})_{2,1}\right)\end{aligned} \tag{6.9}$$

leading to the following interrelationships between the elements of the scattering matrices.

$$\begin{aligned}\{1 + S_{1,1;2,1}\} &= S_{2,1;2,1} = S_{1,1;3,1} \\ \{1 + S_{2,2;2,1}\} &= S_{1,2;2,1} = S_{1,2;3,1}\end{aligned} \tag{6.10}$$

So, there are only two independent elements of $(S_{n,m})_{2,1}$ (either the diagonal or the two off-diagonal) that are required to characterize J_2 completely. These two independent elements are easily determined by the current conditions at J_2 stated in (6.8g), as follows. Note that

$$\begin{pmatrix} V_{1;3} \\ V_{1;3} \end{pmatrix} = V_{1,3}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \left((1_{n,m}) + (S_{n,m})_{2,1}\right) \cdot (V_n)_1$$

$$= \tilde{Z}_{c_{1,1;3,3}} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \cdot (\tilde{Y}_{c_{n,m}})_{1,1} \cdot \left((V_n)_1 - (V_n)_2\right)$$

$$(1_{n,m}) \cdot \left((V_n)_1 + (V_n)_2\right) = \tilde{Z}_{c_{1,1;3,3}} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \cdot (\tilde{Y}_{c_{n,m}})_{1,1} \cdot \left((V_n)_1 - (V_n)_2\right) \tag{6.11}$$

$$[(1_{n,m}) + \tilde{Z}_{c_{1,1;3,3}} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \cdot (\tilde{Y}_{c_{n,m}})_{2,2}] \cdot (V_n)_2$$

$$= - \left[(1_{n,m}) - \tilde{Z}_{c_{1,1;3,3}} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \cdot (\tilde{Y}_{c_{n,m}})_{1,1}\right] (V_n)_1$$

Using (6.7), we can recognize

$$(S_{n,m})_{2,1} = - \left[(1_{n,m}) + \tilde{Z}_{c_{1,1;3,3}} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \cdot (\tilde{Y}_{c_{n,m}})_{2,2}\right]^{-1}$$

$$\left[(1_{n,m}) - \tilde{Z}_{c_{1,1;3,3}} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \cdot (\tilde{Y}_{c_{n,m}})_{1,1}\right] \tag{6.12}$$

We can compute the scattering matrices from the above equation, if we know $(\tilde{Y}_{c_{n,m}})$, which is evaluated as follows

$$(\tilde{Y}_{c_{n,m}})_{1,1} = (\tilde{Y}_{c_{n,m}})_{2,2} = (\tilde{Y}_{c_{n,m}})$$

$$= \sum_{\beta} (i_{c_n})_{\beta} (i_{c_n})_{\beta} \tag{6.13}$$

$$= \left[\zeta_1^{-1} (1,0)(1,0)\right] + \left[\zeta_2^{-1}(-1+\delta,1)(-1+\delta,1)\right]$$

and

$$(1_{n,m}) \mp Z_L \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \cdot (\tilde{Y}_{c_{n,m}})$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mp Z_L \left\{ \frac{1}{\zeta_1} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + \frac{1}{\zeta_2} \begin{pmatrix} -1+\delta & 1 \\ -1+\delta & 1 \end{pmatrix} \right\}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mp \frac{Z_L}{\zeta_1} \left\{ \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + \alpha \begin{pmatrix} -1+\delta & 1 \\ -1+\delta & 1 \end{pmatrix} \right\}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mp (T_{n,m}) \tag{6.14}$$

where

$$\alpha = \frac{\zeta_1}{\zeta_2} = f_{g_P} \left[\left\{ f_{g_M} - f_{g_P}(1-\delta)^2 \right\} \left\{ \chi + f_{g_M} - f_{g_P}(1-\delta)^2 \right\} \right]^{-1/2} \tag{6.15}$$

$$(T_{n,m}) = \frac{Z_L}{\zeta_1} \left\{ \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + \alpha \begin{pmatrix} -1+\delta & 1 \\ -1+\delta & 1 \end{pmatrix} \right\}$$

$$= \frac{f_L}{f_{g_P}} \left\{ \begin{pmatrix} 1-\alpha+\alpha\delta & \alpha \\ 1-\alpha+\alpha\delta & \alpha \end{pmatrix} \right\} \tag{6.16}$$

(6.12) now becomes

$$(S_{n,m})_{2,1} = -\left[(1_{n,m}) + (T_{n,m}) \right]^{-1} \cdot \left[(1_{n,m}) - (T_{n,m}) \right] \tag{6.17}$$

After some straightforward matrix algebra, we find

$$(S_{n,m})_{2,1} = -\frac{1}{1+T_{1,1}+T_{1,2}} \left(\begin{array}{c|c} 1-T_{1,1}+T_{1,2} & -2T_{1,2} \\ \hline -2T_{1,1} & 1+T_{1,1}-T_{1,2} \end{array} \right) \tag{6.18}$$

which can be verified to obey (6.10). Consequently, the two independent elements of the scattering matrices at J_2 are given by

$$S_{1,2;2,1} = \left(\frac{2T_{1,2}}{1+T_{1,1}+T_{1,2}}\right) = \left[\frac{2\alpha}{1+\left(\frac{f_{g_p}}{f_L}\right)+\alpha\delta}\right] \qquad (6.19)$$

$$S_{2,1;2,1} = \left(\frac{2T_{1,2}}{1+T_{1,1}+T_{1,2}}\right) = \left[\frac{2(1-\alpha+\alpha\delta)}{1+\left(\frac{f_{g_p}}{f_L}\right)+\alpha\delta}\right]$$

In summary, we have determined the three junction matrices as listed below

$$(S_{n,m})_{1,2} = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$(S_{n,m})_{2,1} = \begin{pmatrix} -1+S_{2,1;2,1} & S_{1,2;2,1} \\ S_{2,1;2,1} & -1+S_{1,2;2,1} \end{pmatrix} \qquad (6.20)$$

$$(S_{n,m})_{3,1} = (S_{2,1;2,1}\ ,\ S_{1,2;2,1})$$

We now have completely characterized both the junctions J_1 and J_2 in terms of their scattering matrices. The elements of the scattering matrix at J_1 is simply a negative unity matrix and the elements of the remaining two matrices are known and calculable in terms of the four independent parameters f_{g_p}, f_{g_M}, χ, δ and the normalized load f_L. We are now in a position to proceed with the estimation of the output waveform w = 3 in time domain. The procedure for waveform evaluation is described in the following section.

7. Waveform Evaluation

In the preceding sections, we have reviewed the BLT equation, specialized its solution to the single tube problem at hand and characterized the two junctions. We now have all the mathematical quantities required to evaluate the output waveform. The process of waveform evaluation consist of the following steps. This procedure accounts for the initial fast and slow wave propagation and scattering only. The subject of multiple bounces on the tube T_1 is discussed later.

1. The source $V_0u(t)$ is turned on $t = 0$.
2. Fast and slow waves get launched $t = 0^+$ with different amplitudes ($w = 1$ wave).
3. Fast and slow waves propagate on T_1 with speeds $v_1 = c$ and $v_2 < c$.
4. The fast wave arrives at the load at $t = t_f = (\ell/v_1) = (l/c)$.
5. Part of the fast wave is transmitted to the load ($w = 3$ wave) and the remainder goes back towards the source ($w = 2$ wave) with both fast and slow wave components.
6. The slow wave arrives at the load at $t = t_s = \ell/v_2$.
7. Part of it is transmitted to the output ($w = 3$ wave) and the remainder goes back towards the source ($w = 2$ wave) with both slow and fast wave components.

Observe also that when the fast or slow wave component of $w = 2$ traveling in the -z direction is incident at junction J_1, they turn back with a reflection coefficient of -1. There is no mixing of slow and fast modes at J_1. Fast mode reflects into fast mode and the slow mode reflects into slow mode with reflection coefficient of -1. This is in sharp contrast with the mode transformations at J_2. At J_2, fast wave component of $w = 1$ wave is transmitted in part as a fast wave into the load and also converts to fast and slow wave traveling back ($w=2$). A similar transformation occurs for the slow wave incident on J_2.

Furthermore, the successive incidence of fast and slow waves at J_2 are getting smaller each time by the same amount. The evolution of time domain signals (steps 1 to 7) listed above are pictorially illustrated in figure 7.1. The successive multiple reflections occur at J_1 and consequent incidence at J_2 and transmission to load. The mode launching at successive multiple reflections discussed above can be illustrated as shown in figure 7.2. This figure shows the launching of the initial fast and slow waves at J_1, propagation of these

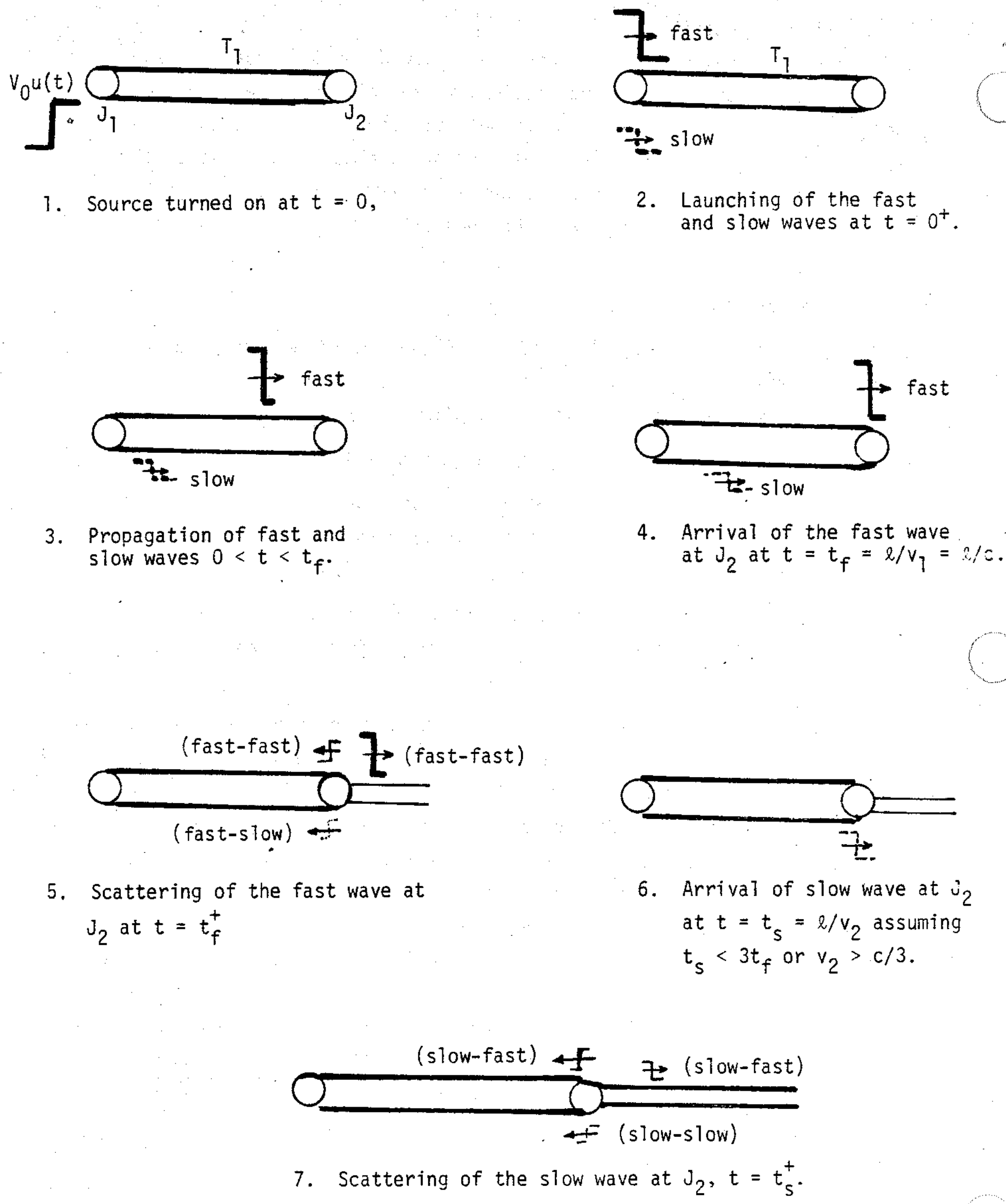


Figure 7.1. Evolution of time domain signals.

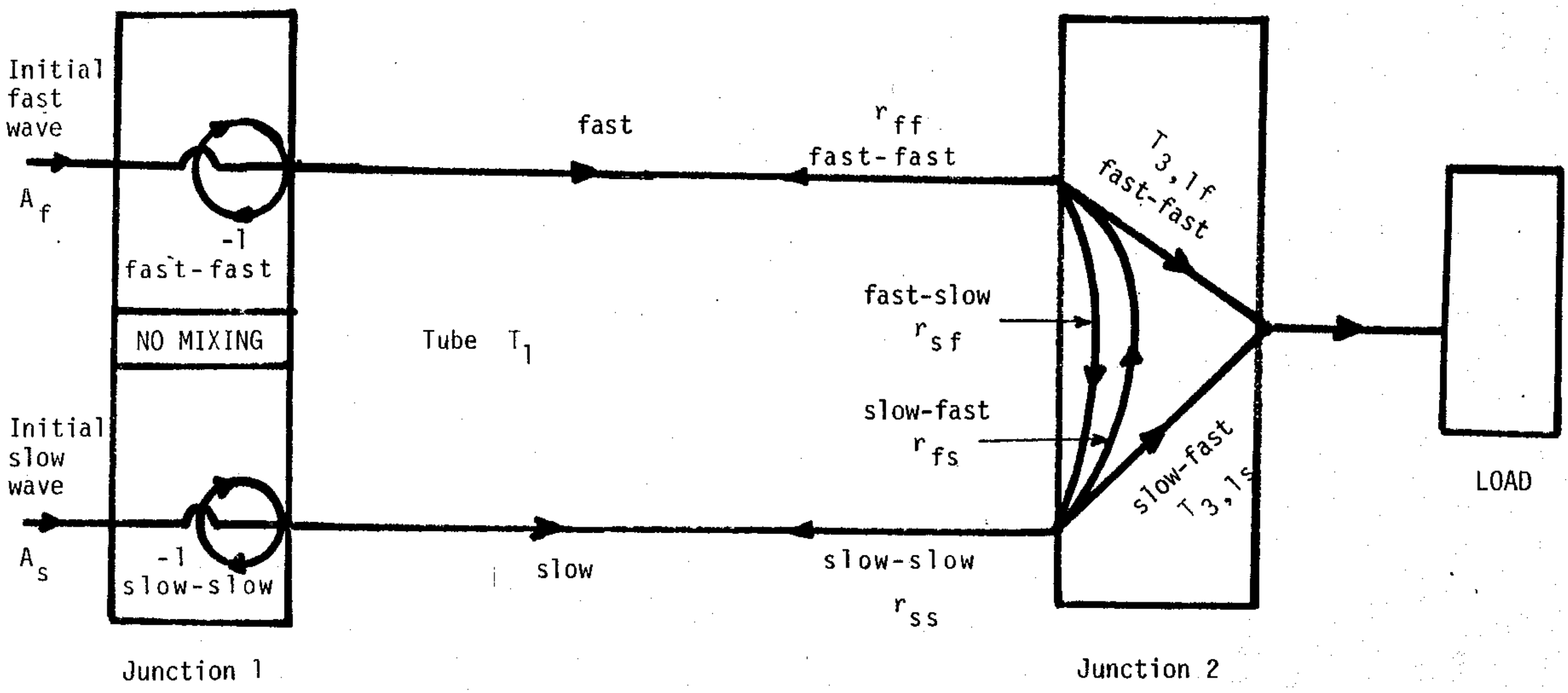


Figure 7.2. Slow and fast wave propagation on tube T_1 between junctions J_1 and J_2 inclusive of all possible transformations at both junctions.

waves on T_1, scattering at J_2, reincidence and reflection at J_1 and multiple incidence and scattering at J_2. In short, this figure illustrates all possible mode transformations on the tube T_1 (Marx-peaker system) and how signals arrive at load. Observe that the signal reaching the load is only the fast wave component, i.e., even when a slow wave scatters at J_2, a fast transmitted component gets to the load. From physical considerations, there is only one speed of propagation = c on tube T_2 representing the transmission line of characteristic impedance, which in turn is represented by the load Z_L.

Multiple reflections are not shown in figure 7.1, but they are indicated in figure 7.2. The initial amplitudes and the various transmission and reflection coefficients for the fast and slow waves at J_1 and J_2 are indicated in figure 7.2. It will be required later to optimize the output waveform, in which case some of the undesirable coefficients (e.g., r_{ff}) can be minimized or even made to vanish. The various symbols in figure 7.2 are described below

$$
\begin{aligned}
A_f &\equiv \text{amplitude of the initial fast wave} \\
A_s &\equiv \text{amplitude of the initial slow wave} \\
r_{ff} &\equiv \text{reflection coefficient of the fast to fast wave at } J_2 \\
r_{sf} &\equiv \text{reflection coefficient of the fast to slow wave at } J_2 \\
r_{fs} &\equiv \text{reflection coefficient of the slow to fast wave at } J_2 \\
r_{ss} &\equiv \text{reflection coefficient of the slow to slow wave at } J_2
\end{aligned}
\tag{7.1}
$$

Note that at J_1, slow and fast waves reflect into slow and fast waves respectively with a minus sign.

$$
\begin{aligned}
\ell &\equiv \text{length of the tube } T_1 \\
c &\equiv \text{speed of the fast wave} \\
v_2 &\equiv \text{speed of the slow wave}
\end{aligned}
\tag{7.2}
$$

$\zeta_1, \zeta_2 \equiv$ eigenimpedances

The procedure of waveform computation outlined above can now be implemented. The various steps listed at the beginning of this section are carried out in the following two sections. The transmission and reflection coefficients at J_2 are also derived and optimized.

8. Initial Waves

Observe that initially at $J_1(z=0)$, the combined voltage is given by

$$(V_n(0,t))_1 = \left[(V_n^{(0)})_1 + (\tilde{Z}_{c_{n,m}})_{1,1} \cdot (I_n^{(0)})_1 \right] \tag{8.1}$$

$$= 2V_0 \; u(t) \; (1,1)$$

and at any general location z, the combined voltage wave w = 1 is given by

$$(V_n(z,t))_1 = \left[(i_{c_n})_f \cdot (1,1) 2V_0 u(t - \tfrac{z}{c}) \right] (v_{c_n})_f + \left[(i_{c_n})_s \cdot (1,1) 2V_0 u(t - \tfrac{z}{v_2}) \right] (v_{c_n})_s \tag{8.2}$$

Furthermore this combined voltage can also be written in terms of its fast and slow components, which are in turn representable by their eigenvoltages as follows

$$(V_n(z,t))_1 = (V_n)_{1f} + (V_n)_{1s} \tag{8.3}$$

with

$$\begin{aligned} (V_n)_{1f} &\equiv \left[A_f \; u(t - \tfrac{z}{c}) \; (v_{c_n})_f \right] = \left[A_f \zeta_1^{1/2} u(t - \tfrac{z}{c})(1, 1-\delta) \right] \\ (V_n)_{1s} &\equiv \left[A_s \; u(t - \tfrac{z}{v_2})(v_{c_n})_s \right] = \left[A_s \, \zeta_2^{1/2} \; u(t - \tfrac{z}{v_2})(0,1) \right] \end{aligned} \tag{8.4}$$

Comparing (8.2) and (8.4) we find

$$\begin{aligned} A_f &= \left[2V_0 (i_{c_n})_f \cdot (1,1) \right] = \left[\frac{2V_0}{\sqrt{\zeta_1}} \right] \\ A_s &= \left[2V_0 (i_{c_n})_s \cdot (1,1) \right] = \left[\frac{2V_0 \delta}{\sqrt{\zeta_2}} \right] \end{aligned} \tag{8.5}$$

It is seen in the above equations that the factors of 2 on the right side are present because we are writing down the combined voltages (true voltage $\pm Z_c \times$ true current). Also observe that the fast mode has the full amplitude and the amplitude of the slow mode is proportional to δ as one may expect. Note that A_f and A_s when squared have the dimensions of power and represent the excitation coefficients of the fast and slow waves. Clearly the first optimization of the output waveform entails making δ as small as possible so that the slow wave amplitude is minimized. We may now proceed to find the scattering of fast and slow waves at J_2. These are considered in the following two sections.

9. Scattering of Fast Wave at J_2

We have already noted (see figure 7.1.4) that the fast wave arrives for the first time at J_2 when $t = t_f = \ell/v_1 = \ell/c$. So at $t = t_f^+$ this fast wave scatters at J_2, converting to fast transmitted wave into the load, fast reflected wave and slow reflected wave. We can now determine the expressions for the various scattered quantities.

A. Transmitted load wave

$$
\begin{aligned}
V_{1;3} &= (S_{n,m})_{3,1} \cdot (V_n)_{1f} \\
&= (S_{1,1}, S_{1,2})_{3,1} \cdot \left[A_f \zeta_1^{1/2}\, u(t - \tfrac{\ell}{c})(1, 1-\delta)\right] \\
&= \left[S_{1,1;3,1} + S_{1,2;3,1}(1-\delta)\right] A_f \zeta_1^{1/2}\, u(t - \tfrac{\ell}{c}) \qquad (9.1)
\end{aligned}
$$

and one may define a transmission coefficient $T_{3,1f}$ indicative of fast-fast scattering into the load at J_2 as follows

$$
T_{3,1f} = \left[\frac{V_{1;3}}{A_f\sqrt{\zeta_1}}\right] = \left[S_{1,1;3,1} + S_{1,2;3,1}\,(1-\delta)\right] = \left[S_{2,1;2,1} + S_{1,2;2,1}\,(1-\delta)\right] \qquad (9.2)
$$

Substituting for the matrix elements from (6.19), we have

$$
\begin{aligned}
T_{3,1f} &= \left(\frac{V_{1;3}}{A_f\sqrt{\zeta_1}}\right) = \left(\frac{2V_L}{2V_0}\right) = \left(\frac{V_L}{V_0}\right)_f \\
&= \left[\frac{2(1-\alpha+\alpha\delta)}{\left(1 + \frac{^f g_p}{f_L} + \alpha\delta\right)} + (1-\delta)\,\frac{2\alpha}{\left(1 + \frac{^f g_p}{f_L} + \alpha\delta\right)}\right] \qquad (9.3) \\
&= \frac{2}{\left(1 + \frac{^f g_p}{f_L} + \alpha\delta\right)} \\
&= \left[1 + \frac{\left(1 - \frac{^f g_p}{f_L} - \alpha\delta\right)}{\left(1 + \frac{^f g_p}{f_L} + \alpha\delta\right)}\right]
\end{aligned}
$$

It is seen that, we now have a second optimization condition given by

$$(f_{g_p}/f_L) = (1-\alpha\delta) \tag{9.4}$$

which makes $T_{3,1f}$ equal to unity which is a highly desirable result. Recall that α is the ratio of eigenimpedances and equals (ζ_1/ζ_2) given in (6.15). We shall return to the optimization condition above after we gather all of the primary optimizations.

B. Reflected wave at $t = t_f$ at J_2

When the fast wave is incident on J_2, it reflects back into w = 2 wave comprising of both fast and slow components which can be computed as follows

$$(V_n)_2 = (V_n)_{2f} + (V_n)_{2s} = (S_{n,m})_{2,1} \cdot (V_n)_{1f} \tag{9.5}$$

The two components of w = 2 wave may be written in terms of eigenvoltage modes as

$$\begin{aligned}(V_n)_{2f} &\equiv \left\{B_f u(t-\ell/c)(v_{c_n})_f\right\} = \left\{B_f\ \zeta_1^{1/2}\ u(t-\ell/c)(1,1-\delta)\right\}\\ (V_n)_{2s} &= \left\{B_s u(t-\ell/c)(v_{c_n})_s\right\} = \left\{B_s\ \zeta_2^{1/2}\ u(t-\ell/c)(0,1)\right\}\end{aligned} \tag{9.6}$$

Using the expression for $(V_n)_{1f}$ from (8.4), we have

$$B_f\zeta_1^{1/2}(1,1-\delta) + B_s\zeta_2^{1/2}(0,1) = A_f\ \zeta_1^{1/2}(S_{n,m})_{2,1}(1,1-\delta) \tag{9.7}$$

We can use the orthogonality property of the eigenmodes to compute the reflection coefficients defined below

$$r_{ff} = B_f/A_f \quad , \quad r_{sf} = \frac{B_s}{A_f} \tag{9.8}$$

First, multiplying (9.7) by $(i_{c_n})_f$ from (5.15) we get

$$B_f = A_f(1,0) \cdot (S_{n,m})_{2,1}(1,1-\delta)$$

leading to

$$r_{ff} = \frac{B_f}{A_f} = (1,0) \cdot (S_{n,m})_{2,1} \cdot (1,1-\delta)$$

$$= (S_{1,1;2,1}, S_{1,2;2,1}) \cdot (1,1-\delta)$$

$$= \left[S_{1,1;2,1} + (1-\delta)S_{1,2;2,1}\right] \tag{9.9}$$

$$= \left[-1 + S_{2,1;2,1} + S_{1,2;2,1} - \delta\, S_{1,2;2,1}\right]$$

Substituting for the scattering matrix elements from (6.19), the fast to fast reflection coefficient r_{ff} at J_2 becomes

$$r_{ff} = \left[\frac{1 - \frac{f_{g_p}}{f_L} - \alpha\delta}{1 + \frac{f_{g_p}}{f_L} + \alpha\delta}\right] \tag{9.10}$$

Similarly, multiplying (9.7) by $(i_{c_n})_s$ from (5.16) one gets

$$r_{sf} = \frac{B_s}{A_f} = \sqrt{\frac{\zeta_1}{\zeta_2}}\,(-1+\delta,1) \cdot (S_{n,m})_{2,1} \cdot (1,1-\delta)$$

$$= \sqrt{\alpha}\left((\delta-1)\, S_{1,1;2,1} + S_{2,1;2,1}\ ,\ -(1-\delta)^2 S_{1,2;2,1} + (1-\delta)S_{2,2;2,1}\right) (1,1-\delta) \tag{9.11}$$

which simplifies to

$$r_{sf} = \sqrt{\alpha}\,\delta\left[S_{2,1;2,1} + S_{1,2;2,1} + \delta\, S_{1,2;2,1}\right]$$

$$= \sqrt{\alpha}\,\delta\ \frac{2}{\left(1 + \frac{f_{g_p}}{f_L} + \alpha\delta\right)} \tag{9.12}$$

$$= \sqrt{\alpha}\,\delta\left[1 + \frac{\left(1 - \frac{f_{g_p}}{f_L} - \alpha\delta\right)}{\left(1 + \frac{f_{g_p}}{f_L} + \alpha\delta\right)}\right]$$

In the interest of waveform optimization we rewrite the transmission and reflection coefficients at J_2 derived above

$$T_{3,1f} = 1 + \left\{ \frac{1 - \frac{f_{g_p}}{f_L} - \alpha\delta}{1 + \frac{f_{g_p}}{f_L} + \alpha\delta} \right\}$$

$$r_{ff} = \left\{ \frac{1 - \frac{f_{g_p}}{f_L} - \alpha\delta}{1 + \frac{f_{g_p}}{f_L} + \alpha\delta} \right\} \qquad (9.13)$$

$$r_{sf} = \sqrt{\alpha}\ \delta \left[1 + \left\{ \frac{1 - \frac{f_{g_p}}{f_L} - \alpha\delta}{1 + \frac{f_{g_p}}{f_L} + \alpha\delta} \right\} \right]$$

We observe from the above that

$$T_{3,1f} = 1 + r_{ff}$$

$$r_{sf} = \sqrt{\frac{\zeta_1}{\zeta_2}}\ \delta\ T_{3,1f} = \sqrt{\frac{\zeta_1}{\zeta_2}}\ \delta \left[1 + r_{ff}\right] \qquad (9.14)$$

$$r_{ff} = \left\{ \frac{1 - \frac{f_{g_p}}{f_L} - \frac{\zeta_1}{\zeta_2}\ \delta}{1 + \frac{f_{g_p}}{f_L} + \frac{\zeta_1}{\zeta_2}\ \delta} \right\}$$

Observe that if we made the reflection coefficient r_{ff} (fast to fast) = 0, then we will have

$$T_{3,1f} = 1, \quad r_{ff} = 0, \quad r_{sf} = \sqrt{\frac{\zeta_1}{\zeta_2}}\ \delta \qquad (9.15)$$

which is highly desirable. $r_{ff} = 0$ requires

$$\frac{f_{g_p}}{f_L} = 1 - \frac{\zeta_1}{\zeta_2}\delta \tag{9.16}$$

which can now be expanded into a polynomial in δ.

$$\frac{f_{g_p}}{f_L} = \left(1 - \frac{f_{g_p}\,\delta}{\left[\{f_{g_M} - f_{g_p}(1-\delta)^2\}\{X + f_{g_M} - f_{g_p}(1-\delta)^2\}\right]^{1/2}}\right)$$

$$= 1 - f_{g_p}\delta\left[\sum_{i=0}^{4} A_i\,\delta^i\right]^{-1/2}$$

$$= \left[1 - f_{g_p}A_0^{-1/2}\,\delta + 0(\delta^2)\right]$$

$$= \left[1 - \frac{f_{g_p}\,\delta}{\{(f_{g_M} - f_{g_p})(X + f_{g_M} - f_{g_p})\}^{1/2}} + 0(\delta^2)\right] \tag{9.17}$$

Since $\delta \ll 1$, to a first order in δ, we require

$$\frac{f_{g_p}}{f_L} = \left[1 - \frac{f_{g_p}\,\delta}{\{(f_{g_M} - f_{g_p})(X + f_{g_M} - f_{g_p})\}^{1/2}}\right] \tag{9.18}$$

Observe that out of the five parameters, f_{g_p}, f_{g_M}, X, δ and f_L, we have control over δ and f_{g_p} and regard the other three f_{g_M} (Marx impedance normalized to Z_0), X (additional Marx inductance normalized to $\mu_0 = L_M'/\mu_0$) and f_L (load impedance normalized to Z_0) as given quantities. In other words, for a given set of X, f_{g_M} and f_L, we need to pick δ and f_{g_p} such that (9.16) or at least (9.18) is satisfied. This will result in an optimized output waveform in the sense that the fast wave transmission and its reflection at J_2 are optimized.

We can now proceed to scatter the slow wave at J_2 and derive a similar set of optimization conditions.

10. Scattering of Slow Wave at J_2

We have already noted (see figure 7.1) that the slow wave arrives for the first time at J_2 when $t = t_s = \ell/v_2$. So at $t = t_s^+$, this slow wave scatters at J_2, converting to a transmitted wave into the load, fast reflected wave and slow reflected wave. We can now determine the expressions for the transmitted and reflected quantities in a manner similar to the scattering of the fast wave considered in Section 9.

A. Transmitted load wave

$$\begin{aligned} V_{1;3} &= (S_{n,m})_{3,1} \cdot (V_n)_{1s} \\ &= (S_{1,1}, S_{1,2})_{3,1} \cdot \left[A_s \zeta_2^{1/2} u\left(t - \frac{\ell}{v_2}\right)(0,1)\right] \\ &= \{S_{1,2;3,1}\} A_s \zeta_2^{1/2} u\left(t - \frac{\ell}{v_2}\right) \end{aligned} \tag{10.1}$$

and one may define a transmission coefficient $T_{3,1s}$ indicative of slow-fast transmission into the load at J_2 as follows

$$T_{3,1s} = \frac{V_{1;3}}{A_s\sqrt{\zeta_2}} = S_{1,2;3,1} = S_{1,2;2,1} \tag{10.2}$$

Substituting for the matrix elements from (6.10) and (6.19), we have

$$T_{3,1s} = \frac{V_{1;3}}{A_s\sqrt{\zeta_2}} = S_{1,2;2,1} = \left[\frac{2\alpha}{\left(1 + \frac{f_{g_p}}{f_L} + \alpha\delta\right)}\right] \tag{10.3}$$

Furthermore the output voltage from slow wave transmission at J_2 is given by

$$\begin{aligned} \left(\frac{V_L}{V_0}\right)_s &= \frac{1}{2}\frac{V_{1;3}}{V_0} = \frac{1}{2}\frac{V_{1;3}}{2V_0\delta}2\delta \\ &= \frac{1}{2}\left(\frac{V_{1;3}}{A_s\sqrt{\zeta_2}}\right)2\delta = \delta T_{3,1s} = \left[\frac{2\alpha\delta}{\left(1 + \frac{f_{g_p}}{f_L} + \alpha\delta\right)}\right] \end{aligned} \tag{10.4}$$

$$= \left[\frac{2\delta f_{g_p}}{\left(1+\frac{f_{g_p}}{f_L}\right)\left[(f_{g_M}-f_{g_p})(\chi+f_{g_M}-f_{g_p})\right]^{1/2}} + O(\delta^2) \right]$$

This will naturally be small from our earlier optimization of $\delta << 1$ when we required that the slow wave excitation be small. We shall return to this optimization after we determine the reflected wave at J_2.

B. Reflected wave at J_2

When the slow wave is incident on J_2, it reflects back into w = 2 wave comprising of both fast and slow components which can be computed as follows

$$(V_n)_2 = (V_n)_{2f} + (V_n)_{2s} = (S_{n,m})_{2,1} \cdot (V_n)_{1s} \qquad (10.5)$$

The two components of w = 2 wave may be written in terms of eigenvoltage modes as

$$(V_n)_{2s} = C_f\, u(t-\frac{\ell}{v_2})(v_{c_n})_f = C_f\, \zeta_1^{1/2}\, u(t-\frac{\ell}{v_2})(1,1-\delta)$$

$$(V_n)_{2s} = C_s\, u(t-\frac{\ell}{v_2})(v_{c_n})_s = C_s\, \zeta_2^{1/2}\, u(t-\frac{\ell}{v_2})(0,1) \qquad (10.6)$$

Using the expression for $(V_n)_{1s}$ from (8.4) we have

$$C_f\, \zeta_1^{1/2}\,(1,1-\delta) + C_s\, \zeta_2^{1/2}\,(0,1) = A_s\, \zeta_2^{1/2}\,(S_{n,m})_{2,1} \bullet (1,1-\delta) \qquad (10.7)$$

We can use the orthogonality property of the eigenmodes to compute the reflection coefficients defined below

$$r_{fs} = C_f/A_s\ , \quad r_{ss} = C_s/A_s \qquad (10.8)$$

First, multiplying (10.7) by $(i_{c_n})_f$ from (5.15) we get

$$C_f = A_s\sqrt{\frac{\zeta_2}{\zeta_1}}\,(1,0) \cdot (S_{n,m})_{2,1} \bullet (1,1-\delta)$$

$$r_{fs} = \frac{C_f}{A_s} = \sqrt{\frac{\zeta_2}{\zeta_1}} \left[\frac{1 - \frac{f_{g_p}}{f_L} - \alpha\delta}{1 + \frac{f_{g_p}}{f_L} - \alpha\delta}\right]. \qquad (10.9)$$

In deriving the above, we have reused the simplification of results associated with (9.9) and (9.10). Once again to minimize r_{fs}, we have the same optimization condition

$$r_{fs} = 0 \text{ iff } \left(\frac{f_{g_p}}{f_L}\right) = \left(1 - \alpha\delta\right) \qquad (10.10)$$

Similarly, multiplying (10.7) by $(i_{c_n})_s$ from (5.16) we get

$$r_{ss} = \frac{C_s}{A_s} = (-1+\delta,1) \cdot (S_{n,m})_{2,1} \cdot (1,1-\delta)$$

$$= \delta\left[1 + \left\{\frac{1 - \frac{f_{g_p}}{f_L} - \alpha\delta}{1 + \frac{f_{g_p}}{f_L} + \alpha\delta}\right\}\right] \qquad (10.11)$$

Once again in deriving above, we have used the simplification of results associated with (9.11) and (9.12). It is noted that the optimization of (10.10) reduces r_{ss} to δ which is a desired result.

We can now summarize the scattering at J_2 of the slow wave by the following three parameters

$$T_{3,1s} = \frac{2\alpha}{\left(1 + \frac{f_{g_p}}{f_L} + \alpha\delta\right)}$$

$$r_{fs} = \sqrt{\alpha}\left[\frac{1 - \frac{f_{g_p}}{f_L} - \alpha\delta}{1 + \frac{f_{g_p}}{f_L} + \alpha\delta}\right] \qquad (10.12)$$

$$r_{ss} = \delta \left[1 + \left\{ \frac{1 - \frac{f_{g_p}}{f_L} - \alpha\delta}{1 + \frac{f_{g_p}}{f_L} + \alpha\delta} \right\} \right]$$

In view of the transmission and reflection coefficients of the fast wave (9.13) and the slow wave (10.12) that have been derived, we see two optimizations that emerge as a matter of consensus. These optimization conditions and the consequent optimized output waveforms are discussed in Section 11.

11. Optimization of the Output Waveform

We have determined the transmission and reflection coefficients at J_2 for the fast and slow waves in the preceding two sections as exhibited in (9.13) for the fast wave and in (10.12) for the slow wave. Examining these equations, one is lead to the following optimization conditions

$$\boxed{\begin{array}{l}\delta << 1 \\ \left(\frac{f_{g_p}}{f_L}\right) = \left[1 - \left(\frac{\zeta_1}{\zeta_2}\right)\delta\right] = (1-\alpha\delta)\end{array}} \qquad (11.1)$$

If the above optimization conditions are satisfied, we achieve an optimized output waveform. The exact and optimized expressions for the various transmission and reflection coefficients are listed in table 11.1 for ease of reference. Satisfying the conditions in (11.1) entails choosing the proper value of N_p such that $\delta << 1$ and then selecting the proper value for $Z_p(=Z_0 f_{g_p})$ such that

$$\begin{aligned}\frac{f_{g_p}}{f_L} &= 1 - \left(\frac{\zeta_1}{\zeta_2}\right)\delta \\ &= 1 - \left[\frac{f_{g_p}\ \delta}{\left\{\left[f_{g_M} - f_{g_p}(1-\delta)^2\right]\left[\chi + f_{g_M} - f_{g_p}(1-\delta)^2\right]\right\}^{1/2}}\right]\end{aligned} \qquad (11.2)$$

The above optimization condition can be cast in terms of a polynomial in $x = \left[1 - (f_{g_p}/ f_L)\right]$

$$\sum_{i=0}^{4} B_i\ x^i = 0 \qquad (11.3)$$

with

$$\begin{aligned}&B_0 = -\delta^2\ , \quad B_1 = -2\delta^2\ , \quad B_2 = \delta^2 - \left[f_{mL} - (1-\delta)^2\right]\left[\chi_L + f_{mL} - (1-\delta)^2\right] \\ &B_3 = -(1-\delta)^2\left[\chi_L + 2f_{mL} - 2(1-\delta)^2\right] \\ &B_4 = (1-\delta)^4\ , \quad f_{mL} = (f_{g_M}/f_L)\ , \quad \chi_L = (\chi/f_L)\end{aligned} \qquad (11.4)$$

Equation(11.3) can be solved numerically for x. However, an approximate solution

Optimization required

$\delta \ll 1$

$$\left(\frac{f_{g_p}}{f_L}\right)_0 = \left[1 - \left(\frac{\zeta_1}{\zeta_2}\right)_0 \delta\right]$$

<u>Note:</u> $\frac{\zeta_1}{\zeta_2} = \sum_{i=0}^{\infty} A_i \delta^i$

$$\left(\frac{\zeta_1}{\zeta_2}\right)_0 = \alpha_0 \equiv \text{optimal value} = \left[1 - \left(\frac{f_{g_p}}{f_L}\right)_0\right] \frac{1}{\delta}$$

	Transmission & reflection coefficients	Exact Expressions	Optimized Expressions
FAST	Excitation	V_0 (initial)	V_0 (initial)
	$T_{3;1f}$	$1 + r_{ff}$	1
	$(V_L/V_0)_f$	$1 + r_{ff}$	1
	r_{ff}	$\left\{1 - \frac{f_{g_p}}{f_L} - \frac{\zeta_1}{\zeta_2}\right\} \left\{1 + \frac{f_{g_p}}{f_L} + \frac{\zeta_1}{\zeta_2}\,\delta\right\}^{-1}$	0
	r_{sf}	$\sqrt{\frac{\zeta_1}{\zeta_2}}\,\delta\,[1+r_{ff}]$	$\delta\sqrt{\frac{\zeta_1}{\zeta_2}} \ll 1$
SLOW	Excitation	$V_0\,\delta$ (initial)	$V_0\,\delta$ (initial)
	$T_{3;1s}$	$\left\{\frac{2(\zeta_1/\zeta_2)}{\left(1 + \frac{f_{g_p}}{f_L} + \left(\frac{\zeta_1}{\zeta_2}\right)\delta\right)}\right\}$	$\sqrt{\frac{\zeta_1}{\zeta_2}}$
	$(V_L/V_0)_s$	$\delta\, T_{3;1s}$	$\delta\sqrt{\frac{\zeta_1}{\zeta_2}} \ll 1$
	r_{ss}	$\delta\,[1 + r_{ff}]$	δ
	r_{fs}	$\sqrt{\frac{\zeta_2}{\zeta_1}}\, r_{ff}$	0

Table 11.1. Transmission and reflection coefficients at J_2 and their exact optimization.

to a first order in δ may be obtained in closed form by substituting $x = c_1\delta$ in (11.3) and retaining only the lowest order terms in δ. This leads to

$$x_{approx.} = \delta \Big/ \left[f_{mL}^2 + f_{mL}\,\chi_L - \chi_L + 1\right]^{1/2}$$
$$\left[\frac{f_{g_p}}{f_L}\right]_0 = 1 - \left[\frac{\delta}{\left[f_{mL}^2 + f_{mL}\chi_L - \chi_L + 1\right]^{1/2}}\right] + O(\delta)^2 \qquad (11.5)$$

Having been given f_{g_M}, χ, f_L and choosing a small enough δ, the coefficients (B_i) are easily computed and then the polynomial can be solved numerically noting that the required root of the polynomial is a small positive real number. Alternatively an approximate solution can be obtained by using (11.5). In either case, it is noted that there exists an optimal value of peaker impedance to ground (slightly smaller than the load impedance) that results in an optimized output waveform.

Let us assume we have chosen an optimum value of (f_{g_p}/f_L) resulting in simplified expressions for the various transmission and reflection coefficients listed in the third column of table 11.1. Noting that r_{ff} and $r_{fs} = 0$ i.e., both the fast and slow w = 1 waves do not produce a reflected fast (w=2) wave, the output waveform simplifies considerably and it is possible to write out the optimized output waveform in terms of the initial fast and slow waves as follows

$$V_{Lf}^{(opt)}(t) = V_0 T_{3,1f}\, u(t-t_f) + \sum_{n=1}^{\infty} V_0 (-1)^n\, r_{sf} r_{ss}^{(n-1)}\, (\delta T_{3,1s})\, u(t-t_f-2nt_s)$$

$$= V_0\, u(t-t_f) + V_0 \delta^2 \alpha_0 \sum_{n=1}^{\infty} (-1)^n\, \delta^{(n-1)}\, u(t-t_f-2nt_s) \qquad (11.6)$$

$$V_{Ls}^{(opt)}(t) = \sum_{m=0}^{\infty} (V_0\delta)(\delta T_{3,1s})(-1)^m\, r_{ss}^m\, u\{t-(2m+1)t_s\}$$

$$= (V_0 \delta^2 \sqrt{\alpha_0}) \sum_{m=0}^{\infty} (-1)^m\, \delta^m\, u\{t-(2m+1)t_s\} \qquad (11.6)$$

The total optimized output voltage is then given by

$$V_L^{(opt)}(t) = V_{Lf}^{(opt)}(t) + V_{Ls}^{(opt)}(t)$$

$$= V_0\, u(t-t_f) + V_0 \delta^2 \alpha_0 \sum_{n=1}^{\infty} (-1)^n\, \delta^{(n-1)}\, u(t-t_f-2nt_s)$$

$$+ V_0\delta^2 \sqrt{\alpha_0} \sum_{m=0}^{\infty} (-1)^m \delta^m u\{t-(2m+1)t_s\} \tag{11.7}$$

Observe that the first term in the above equation is the desired output and the two infinite series contribute small periodic, step-like perturbations since each term in both series is of second or higher order in δ. The above equation is shown plotted in figure 11.1b with the step function source in figure 11.1a. We have only shown the initial full amplitude step at t_f, m = 0 term, n = 1 term and the m = 1 terms, in figure 11.1b. The successive incremental steps get increasingly smaller and the waveform rapidly approches its asymptotic value which is the input itself delayed by t_f. Observe also that there are no incremental steps at $3t_f$ and (t_s+2t_f) in figure 11.1b simply because $r_{ff} = 0$ and $r_{fs} = 0$ under optimal conditions.

In sketching the optimized output waveform in figure 11.1b, we have assumed that $t_s < 3t_f$ or $(v_2 > c/3)$ for purposes of illustration only. This does not restrict the analysis in any way. If the condition $(v_2 > c/3)$ is not valid, it simply means the arrival times of the various waves transmitted into the load are different. The characteristic features of the output waveform remain unchanged.

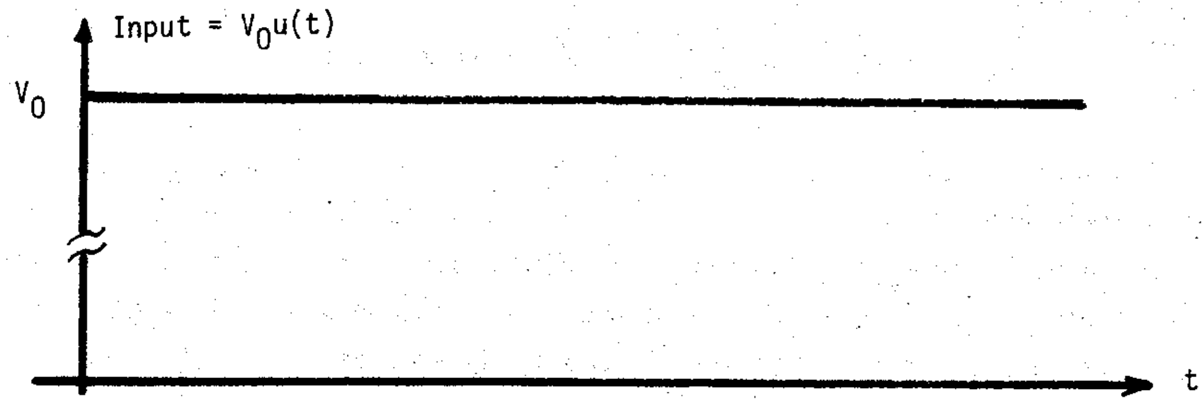
It is also of interest to investigate the optimized output waveform in the frequency domain. Using the linearity and time-shifting property of the two-sided Laplace transformation, we could write the optimized output voltage in freuqency domain as

$$\tilde{V}_L^{(opt)}(s) = \frac{V_0}{s} e^{-st_f} + \frac{V_0}{s}\delta^2\alpha_0 \sum_{n=1}^{\infty} (-1)^n \delta^{(n-1)} e^{-s(t_f+2nt_s)}$$

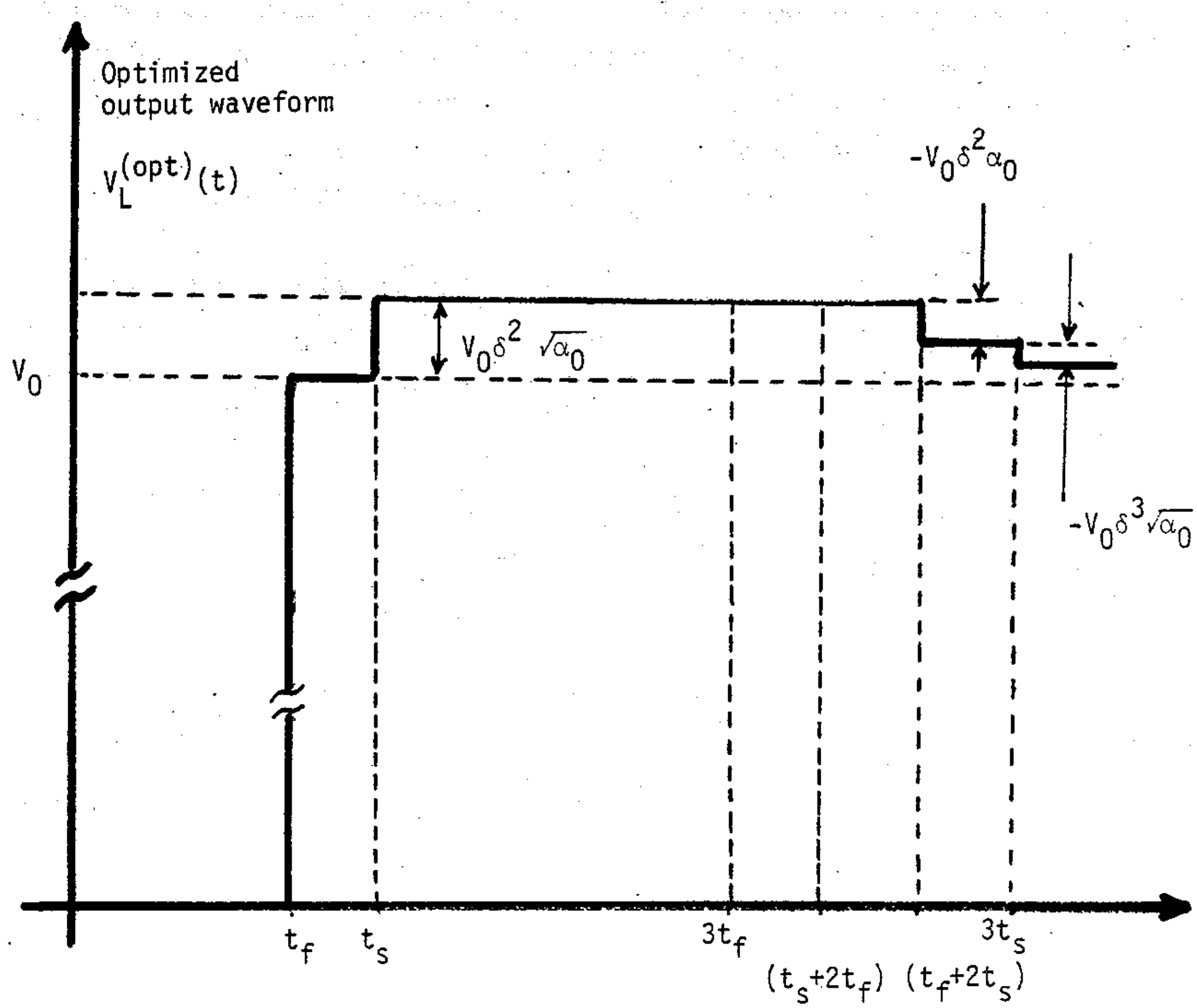
$$+ \frac{V_0}{s}\delta^2\sqrt{\alpha_0} \sum_{m=0}^{\infty} (-1)^m \delta^m e^{-s(2mt_s+t_s)}$$

$$= \frac{V_0}{s} e^{-st_f}\left[1 + \delta\alpha_0 \sum_{n=1}^{\infty} \left(-\delta e^{-2st_s}\right)^n + \delta^2\sqrt{\alpha_0}\, e^{-s(t_s-t_f)} \sum_{m=0}^{\infty} \left(-\delta e^{-2st_s}\right)^m\right] \tag{11.8}$$

The two infinite geometric series are summable giving



a) Idealized input step waveform



b) Optimized output waveform

Figure 11.1. Input and optimized output waveforms.

$$V_L^{(opt)}(s) = \frac{V_0}{s} e^{-st_f}\left[1 + \delta\alpha_0\left\{\frac{1}{1+\delta e^{-2st_s}} - 1\right\} + \delta^2\sqrt{\alpha_0}e^{-s(t_s-t_f)}\left\{\frac{1}{1+\delta e^{-st_s}}\right\}\right]$$

$$= \frac{V_0}{s} e^{-st_f}\left[1 + \frac{\delta\sqrt{\alpha_0}}{1+\delta e^{-2st_s}}\left\{-\delta\sqrt{\alpha_0}e^{-2st_s} + \delta e^{-st_s} e^{st_f}\right\}\right]$$

$$= \frac{V_0}{s} e^{-st_f}\left[1 + \delta^2\sqrt{\alpha_0}\frac{e^{st_f} - \sqrt{\alpha_0}\, e^{-st_s}}{e^{st_s} + \delta e^{-st_s}}\right] \qquad (11.9)$$

So, it is seen that the spectrum of the optimized output wave is a fairly compact expression (11.9), on which the first term is the desired output spectrum and the second term is a relatively small perturbation.

In concluding this section, it is noteworthy that having chosen $\delta << 1$, a single optimization condition (11.1) can yield a remarkably well optimized output.

12. Summary

In this note, we have considered a two-conductor transmission line model for the Marx-peaker assembly. The configuration is such that the Marx column is parallel to the ground plane and a set of N_p peaker arms are optimally distributed around the Marx column [1]. The optimal peaker distribution entails placing them on an equipotential surface and requiring that all peaker arms carry the same current. A small correction may be required to this equipotential to account for the finite number of arms simulating a cylindrical surface. In any case, since the peaker arms are on an equipotential surface, they could be mathematically modelled by a single equivalent peaker conductor. Such an equivalence then leads us to a two conductor (Marx and effective peaker plus reference) model for the entire Marx-peaker assembly.

The coupled two-conductor-transmission line model is then analyzed invoking the BLT equation and its solution for a single tube. The output waveform for an idealized step input, can be written in terms of five parameters namely Marx impedance to ground ($f_{g_M} Z_0$), Marx shielding factor (δ), additional inductance of Marx (L_M), peaker cage impedance to ground ($Z_0 f_{g_p}$) and the load impedance ($Z_L = Z_0 f_L$). Out of these five, f_{g_M}, f_{g_M}, L_M and Z_L are considered as given and we then can optimally choose δ and f_{g_p} to produce desirable characteristics in the output waveform. The optimal value of f_{g_p} can be obtained by solving a fourth degree polynomial and the optimal f_{g_p} is seen to be slightly less than f_L i.e., the optimal peaker cage impedance to ground is seen to be a few percent less than the load impedance.

It is also noted that owing to the additional series inductance in the Marx column, a fast and slow mode get launched at the input. All the characteristics of these two modes i.e., excitation amplitudes, eigenspeeds, eigenvoltages, eigencurrents, eigenimpedances are derived in closed forms. The transmission line network under consideration is a single tube connecting two junctions with the load connected at the output junction. The two junctions are characterized in terms of their scattering matrices and the relevant transmission and reflection coefficients are also derived.

We have been able to eliminate the reflection of the fast wave to fast wave (r_{ff}) and the reflection of the slow to fast wave (r_{fs}) at the output junction by choosing $\delta << 1$, and by employing an optimal $Z_p = Z_0 f_{g_p}$. This choice also makes the transmission coefficient of the fast wave into the load to be unity, which is a highly desirable result. For a given step function source, the optimized output consists of a delayed step function of full amplitude with small incremental steps each time a slow wave is incident on J_2. Optimized output waveform is sketched.

In conclusion, it is seen that there exists an optimal peaker cage impedance (somewhat less than the load impedance) that results in an optimal output waveform. The key lies in having a sufficiently large N_p (such as 8 or 10) to make δ small (.05 or less) and then optimize Z_p ($\equiv$ peaker cage impedance). Precise computations can be made very easily for a given Marx geometry, load impedance and the Marx inductance. It is indeed remarkable that a single optimization condition on Z_p yields an output waveform with many desirable features.

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